Variational Iteration Method for A Boundary Value Problem with Left and Right Fractional Derivatives

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Abstract—**This article studies the variational framework for a boundary value problem involving in left and right fractional derivatives. It presents the sufficient conditions and the absolute error between the approximate solutions and exact solutions. The method is easily computable and quite efficient. Several examples are given to verify the reliability and efficiency of it.**

Keywords: ***Variational iteration method, Left fractional derivative, Right fractional derivative, Approximate solution.***

1 Introduction

The purpose of this paper is to study the variational framework for the following differential equation involving in left and right fractional derivatives:

$$y''(t)+q(t)y(t)+\mu({}_0^C D_t^\alpha+{}_t^C D_1^\alpha)y(t)+f(t,y(t))=0 \quad (1.1)$$

subject to

$$y(0)=a, y(1)=b, \quad (1.2)$$

where $0 \le t \le 1$, $0<\alpha<1$, ${}_0^C D_t^\alpha$ and ${}_0^C D_t^\alpha$ are the left and right Caputo fractional derivative of order α separately. a and b are real constants. The differential equation (1.1) is a governing equilibrium equation which comes from the non-local continuum mechanics[1].

Fractional differential equations with right and left derivatives arise naturally as the Euler-Lagrange equation in variational principles. And they have been successfully used in many fields too, such as the optimal control theory for functionals involving in fractional derivatives, see [1, 2, 3, 4, 5, 6, 7] and their references therein. Furthermore, there many papers deal with the type of differential equations, see the papers [8, 9, 10] and their references therein. But the theory of numerical analysis literature is not much available, and it is very difficult to find the analytical solutions for this type of equations. Hence it is interesting and important to discuss the approximate solutions of it.

The variation iteration method (VIM) was proposed to solve the various differential equations by reference [11]. It has been successfully applied to many problems, see [12, 13, 14, 15, 16] and their references therein. Now it is proved to be a valuable tool in numerical analysis. One challenge in solving (1.1) is determining the Lagrange multiplier. And another challenge is that the iteration formula contains two singular terms which cut down the accuracy of the approximate solutions. In the work, the Lagrange multiplier is found by a second differential equation boundary value problem. And the singular terms are reconstructed by applying integration by parts. Several numerical experiments are presented to indicate the variation iteration formula converges well.

The paper is organized as following. Some background material is given in Section 2. The VIM for solving (1.1) is considered in Section 3. We end the paper with several examples of application in Section 4.

2 Preliminary

In this section, we will give some definitions and lemmas which are used further in the article.

Definition 2.1. The left and right Caputo fractional derivative of order $\alpha>0$ of a function $y:(0,1)\to\mathbb{R}$ is defined separately as

$${}_0^C D_t^\alpha y(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{y^{(m)}(s)}{(t-s)^{\alpha-m+1}}ds$$

and

$${}_t^C D_1^\alpha y(t)=\frac{(-1)^m}{\Gamma(m-\alpha)}\int_t^1 \frac{y^{(m)}(s)}{(t-s)^{\alpha-m+1}}ds,$$

where $m-1<\alpha\le m$, $y^{(m)}(t)$ exists.

Similar to Lemma 2.3 in reference [17], the following Lemma can be given.

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Lemma 2.1. *Let u and v be the solutions of*

$$\begin{cases} u''(t)+q(t)u(t)=0,\ t\in(0,1), \\ u(0)=0,\ u(1)=1 \end{cases} \tag{2.1}$$

and

$$\begin{cases} v''(t)+q(t)v(t)=0,\ t\in(0,1), \\ v(0)=1,\ v(1)=0. \end{cases} \tag{2.2}$$

Assume that, $q,h\in L^1[0,1]$, then the following boundary value problem

$$\begin{cases} y''(t)+q(t)y(t)+h(t)=0,\ t\in(0,1), \\ y(0)=0,\ y(1)=0 \end{cases} \tag{2.3}$$

is equivalent to the integral equation

$$y(t)=\int_0^1 G(t,s)h(s)ds,$$

where

$$G(t,s)=\frac{1}{u'(0)}\begin{cases} u(s)v(t),\ 0\le s\le t\le 1, \\ u(t)v(s),\ 0\le t\le s\le 1. \end{cases} \tag{2.4}$$

3 The VIM for solving (1.1)

In this section, we will apply the VIM to study (1.1) in $C^1[0,1]$ with the corresponding norm

$$\|y\|=\max\{\|y\|_\infty,\|y'\|_\infty\}.$$

Now we give the basic idea of VIM for (1.1). According to VIM, we construct the correction functional of (1.1) in the following form
$y_{m+1}(t)=y_m(t)+\int_0^1\lambda(t,s)[y_m''(s)+q(s)y_m(s) +\mu({}_0^CD_s^\alpha+{}_s^CD_1^\alpha)\tilde{y}_m(s)+f(s,\tilde{y}_m(s))]ds,$
where λ is the Lagrange multiplier. $\tilde{y}_m$ is considered as a restricted variation, that is $\delta\tilde{y}_m=0$. And y_0 is a initial approximation. By $\delta\tilde{y}_m(t)=0$, we get that

$$\delta y_{m+1}(t)=\delta y_m(t)+\delta\int_0^1\lambda(t,s)(y_m''(s)+q(s)y_m(s))ds.$$

Noticing that $\delta y_m(0)=\delta y_m(1)=0$, by lemma 2.1, we have the stationary condition $\lambda(t,s)=G(t,s)$. As a result, we have the following iteration formula

$$y_{m+1}(t)=y_m(t)+\int_0^1 G(t,s)[y_m''(s)+q(s)y_m(s) +\mu({}_0^CD_s^\alpha+{}_s^CD_1^\alpha)y_m(s)+f(s,y_m(s))]ds. \tag{3.1}$$

Now, we show that $\{y_m\}_{m=1}^\infty$ defined by (3.1) with an initial approximation $y_0(t)$ with $y_0(0)=a,\ y_0(1)=b$ converges to the exact solution of (1.1).

Theorem 3.1. *Let $y,y_j\in C^2[0,1]$, $j=0,1,\cdots$, $\theta=\max_{t,s\in[0,1]}|G'(t,s)|$. And $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function which satisfies*

$$|f(t,y_1)-f(t,y_2)|\le r(t)|y_1-y_2| \tag{3.2}$$

for all $t\in[0,1]$, $y_1,y_2\in\mathbb{R}$ with $r\in L^1[0,1]$. Assume that

$$\frac{2|\mu|\theta}{\Gamma(3-\alpha)}+\theta\int_0^1 r(t)dt<1.$$

Then the sequence defined by (3.1) with $y_0(0)=a$, $y_0(1)=b$ converges to the exact solution of (1.1). Furthermore, we have the following error estimate

$$\|E_{m+1}\|<\left(\frac{2|\mu|\theta}{\Gamma(3-\alpha)}+\theta\int_0^1 r(t)dt\right)^{m+1}\|E_0\|,$$

where $E_j(x)=y_j(x)-y(x),\ j=1,2,\cdots$.

Proof. Obviously from (1.1), we have

$$y(t)=y(t)+\int_0^1 G(t,s)[y''(s)+q(s)y(s) +\mu({}_0^CD_s^\alpha+{}_s^CD_1^\alpha)y(s)+f(s,y(s))]ds. \tag{3.3}$$

Now from (3.1) and (3.3), we have
$E_{m+1}(t)=E_m(t)+\int_0^1 G(t,s)\big(E_m''(s)+q(s)E_m(s) +\mu({}_0^CD_s^\alpha+{}_s^CD_1^\alpha)E_m(s)+f(s,y_m(s))-f(s,y(s))\big)ds,$
where $E_j(t)=y_j(t)-y(t),\ j=1,2,\cdots$. By the definition of $G(t,s)$, we know that

$$y_1(0)=y_2(0)\cdots=y_m(0)\cdots=a,$$
$$y_1(1)=y_2(1)\cdots=y_m(1)\cdots=b.$$

Hence

$$E_0(0)=E_1(0)\cdots=E_m(0)\cdots=0,$$
$$E_0(1)=E_1(1)\cdots=E_m(1)\cdots=0.$$

It is from lemma 2.1 that

$$E_m(t)+\int_0^1 G(t,s)(E_m''(s)+q(s)E_m(s))ds=0.$$

Hence
$E_{m+1}(t)=\mu\int_0^1 G(t,s)\big(({}_0^CD_s^\alpha+{}_s^CD_1^\alpha)E_m(s) + f(s,y_m(s))-f(s,y(s))\big)dt.$
Therefore

$$\begin{aligned}|E_{m+1}'(t)|&\le|\mu|\int_0^1|G_t'(t,s)({}_0^CD_s^\alpha+{}_s^CD_1^\alpha)E_m(s)|ds \\ &\quad+\int_0^1|G_t'(t,s)(f(s,y_m(s))-f(s,y(s)))|ds \\ &\le\tfrac{|\mu|\theta}{\Gamma(1-\alpha)}\|E_m'\|_\infty\int_0^1\Big(\int_0^s\tfrac{1}{(s-\tau)^\alpha}+\int_s^1\tfrac{1}{(\tau-s)^\alpha}\Big)d\tau ds \\ &\quad+\theta\|E_m\|_\infty\int_0^1 r(s)ds \\ &\le\Big(\tfrac{2|\mu|\theta}{\Gamma(3-\alpha)}+\theta\int_0^1 r(s)ds\Big)\|E_m\| \\ &\le\Big(\tfrac{2|\mu|\theta}{\Gamma(3-\alpha)}+\theta\int_0^1 r(s)ds\Big)^{m+1}\|E_0\|.\end{aligned}$$

That is $\|E_{m+1}'\|_\infty<\big(\frac{2|\mu|\theta}{\Gamma(3-\alpha)}+\theta\int_0^1 r(s)ds\big)^{m+1}\|E_0\|$.
By
$E_{m+1}(t)=E_{m+1}(0)+\int_0^t E_{m+1}'(s)ds=\int_0^t E_{m+1}'(s)ds \le\|E_{m+1}'\|_\infty,\ t\in[0,1],$
we have
$\|E_{m+1}\|_\infty\le\|E_{m+1}'\|_\infty <\big(\frac{2|\mu|\theta}{\Gamma(3-\alpha)}+\theta\int_0^1 r(s)ds\big)^{m+1}\|E_0\|.$
Hence
$\|E_{m+1}\|\le\big(\frac{2|\mu|\theta}{\Gamma(3-\alpha)}+\theta\int_0^1 r(s)ds\big)^{m+1}\|E_0\|.$
The proof is completed. □

4 Numerical examples

The iteration formulation (3.1) converges slowly due to the presence of singularity terms $\frac{1}{(s-\tau)^\alpha}$ in ${}_0^C D_s^\alpha y_m(s)$ and $\frac{1}{(\tau-s)^\alpha}$ in ${}_s^C D_1^\alpha y_m(s)$. Now we apply integration by parts transforming (3.1) into a iteration formulation without singularity terms. By computing, we get that
$\int_0^1 G(t,s)(y_m''(s)+q(s)y_m(s))ds$
$= -y_m(t)+av(t)-\frac{bv'(1)}{u'(0)}u(t),$
$\int_0^1 G(t,s){}_0^C D_s^\alpha y_m(s)ds$
$= -\frac{1}{\Gamma(2-\alpha)}\int_0^1(\int_\tau^1 (s-\tau)^{1-\alpha}G_s'(t,s)ds)y_m'(\tau)d\tau,$
$\int_0^1 G(t,s){}_s^C D_1^\alpha y_m(s)ds$
$= -\frac{1}{\Gamma(2-\alpha)}\int_0^1(\int_0^\tau (\tau-s)^{1-\alpha}G_s'(t,s)ds)y_m'(\tau)d\tau.$
Hence (3.1) is equivalent to

$$y_{m+1}(t)=av(t)-\frac{bv'(1)}{u'(0)}u(t)+\int_0^1 G(t,s)f(s)ds$$
$$-\frac{\mu}{\Gamma(2-\alpha)}\int_0^1(\int_0^1|\tau-s|^{1-\alpha}G_s'(t,s)ds)y_m'(\tau)d\tau. \quad (4.1)$$

Since (4.1) does not involve in y_m, we can choose an initial approximation y_0 which does not need satisfying the boundary value conditions.

Example 4.1. The following boundary value problem is considered

$$\begin{cases} y''(t)+\frac{9}{50}({}_0^C D_t^{\frac{1}{2}}+{}_t^C D_1^{\frac{1}{2}})y(t)=0,\ t\in(0,1),\\ y(0)=0,\ y(1)=0,\end{cases} \quad (4.2)$$

the exact solution of (4.1) is $y=0$. By computing, we get

$$G(t,s)=\begin{cases} s(1-t), 0\le s\le t\le 1,\\ t(1-s), 0\le t\le s\le 1.\end{cases}$$

Hence $\theta=1$, $r=0$, $\frac{2\mu\theta}{\Gamma(2.5)}=0.2708<1$. Therefore the iteration formula for (4.1) can be written as following
$y_{m+1}(t)=-\frac{9}{50\Gamma(1.5)}\int_0^1 y_m'(\tau)((1-t)\int_0^t|\tau-s|^{1-\alpha}$
$\quad -t\int_t^1|\tau-s|^{1-\alpha})dsd\tau$
by (4.1) and Theorem 3.1. Let the initial approximation $y_0=c,(c\in\mathbb{R})$, then we get that $y_1=0$ is the exact solution of (4.1). Let the initial approximation $y_0=-t(t-1)$. By using MATLAB, we have the following approximations: y_1,y_2,y_3,y_4,y_5. The experimental results are presented in Table 1 and Figure 1.

Example 4.2. The following boundary value problem is considered

$$\begin{cases} y''(t)+\frac{9}{50}({}_0^C D_t^{\frac{1}{2}}+{}_t^C D_1^{\frac{1}{2}})y(t)\\ +\frac{9}{50\Gamma(1.5)}((1-t)^{\frac{1}{2}}-t^{\frac{1}{2}})=0,\ t\in(0,1),\\ y(0)=0,\ y(1)=1,\end{cases} \quad (4.3)$$

the exact solution of (4.2) is $y=t$. By computing, we get

$$G(t,s)=\begin{cases} s(1-t), 0\le s\le t\le 1,\\ t(1-s), 0\le t\le s\le 1.\end{cases}$$

Hence $\theta=1$, $r=0$, $\frac{2\mu\theta}{\Gamma(2.5)}=0.2708<1$. We have the following iteration formula
$y_{m+1}(t)=t$
$+\frac{2}{3\Gamma(1.5)}\int_0^t(2t(t-\tau)^{\frac{3}{2}}-t(1-\tau)^{\frac{3}{2}}-(t-\tau)^{\frac{3}{2}})y_m'(\tau)d\tau$
$+\frac{2}{3\Gamma(1.5)}\int_t^1((\tau-t)^{\frac{3}{2}}-(1-t)\tau^{\frac{3}{2}}+t(1-\tau)^{\frac{3}{2}})y_m'(\tau)d\tau$
$+\frac{9(1-t)}{25\Gamma(0.5)}\int_0^t s((1-s)^{\frac{1}{2}}-s^{\frac{1}{2}})ds$
$+\frac{9t}{50\Gamma(1.5)}\int_t^1(1-s)((1-s)^{\frac{1}{2}}-s^{\frac{1}{2}})ds$
by (4.1) and Theorem 3.1. Let the initial approximation $y_0=1$, then by using MATLAB, we have the following approximations: $y_1,y_2,y_3,y_4,y_5,y_6,y_7,y_8,y_9$. The experimental results are presented in Table 2 and Figure 2.

Example 4.3. The following boundary value problem is considered

$$\begin{cases} y''(t)+\frac{9}{50}({}_0^C D_t^{\frac{1}{2}}+{}_t^C D_1^{\frac{1}{2}})y(t)\\ +\frac{9}{50\Gamma(1.5)}(1+t(1-t)^{\frac{1}{2}}-t^{\frac{1}{2}}-y(t))=0,\ t\in(0,1),\\ y(0)=1,\ y(1)=2,\end{cases} \quad (4.4)$$

the exact solution of (4.3) is $y=t+1$. By computing, we get

$$G(t,s)=\begin{cases} s(1-t), 0\le s\le t\le 1,\\ t(1-s), 0\le t\le s\le 1.\end{cases}$$

Hence $\theta=1$, $r=\frac{9}{50\Gamma(1.5)}$, $\frac{2\mu\theta}{\Gamma(2.5)}=0.2708+0.2031<1$. We have the following iteration formula
$y_{m+1}(t)=t+1$
$+\frac{2}{3\Gamma(1.5)}\int_0^t(2t(t-\tau)^{\frac{3}{2}}-t(1-\tau)^{\frac{3}{2}}-(t-\tau)^{\frac{3}{2}})y_m'(\tau)d\tau$
$+\frac{2}{3\Gamma(1.5)}\int_t^1((\tau-t)^{\frac{3}{2}}-(1-t)\tau^{\frac{3}{2}}+t(1-\tau)^{\frac{3}{2}})y_m'(\tau)d\tau$
$+\frac{9(1-t)}{25\Gamma(0.5)}\int_0^t s((1-s)^{\frac{1}{2}}-s^{\frac{1}{2}})ds$
$+\frac{9t}{50\Gamma(1.5)}\int_t^1(1-s)((1-s)^{\frac{1}{2}}-s^{\frac{1}{2}})ds$
$+\frac{9(1-t)}{25\Gamma(0.5)}\int_0^t s(1+s-y(s))ds$
$+\frac{9t}{50\Gamma(1.5)}\int_t^1(1-s)(1+s-y(s))ds$
by (4.1) and Theorem 3.1. Let the initial approximation $y_0=1$, then by using MATLAB, we have the following approximations: $y_1,y_2,y_3,y_4,y_5,y_6,y_7,y_8,y_9$. The experimental results are presented in Table 3 and Figure 3.

Example 4.4. The following boundary value problem is considered

$$\begin{cases} y''(t)+\frac{\pi^2}{4}y+\frac{9}{50}({}_0^C D_t^{\frac{1}{2}}+{}_t^C D_1^{\frac{1}{2}})y(t)\\ +\frac{9}{50\Gamma(1.5)}((1-t)^{\frac{1}{2}}-t^{\frac{1}{2}})-\frac{\pi^2}{4}t=0,\ t\in(0,1),\\ y(0)=0,\ y(1)=1,\end{cases} \quad (4.5)$$

the exact solution of (4.4) is $y=t$. By computing, we get

$$G(t,s)=\frac{2}{\pi}\begin{cases} \sin(\frac{\pi}{2}s)\cos(\frac{\pi}{2}s), 0\le s\le t\le 1,\\ \cos(\frac{\pi}{2}s)\sin(\frac{\pi}{2}s), 0\le t\le s\le 1.\end{cases}$$

Hence $\theta=1$, $r=0$, $\frac{2\mu\theta}{\Gamma(2.5)}=0.2708<1$. We have the following iteration formula

$$y_{m+1}(t) = \sin(\tfrac{\pi}{2}t)$$
$$-\tfrac{9}{50\Gamma(1.5)}\cos(\tfrac{\pi}{2}t)\int_0^t y_m'(\tau)\int_0^1 |\tau - s|^{\frac{1}{2}}\cos(\tfrac{\pi}{2}s)ds d\tau$$
$$+\tfrac{9}{50\Gamma(1.5)}\sin(\tfrac{\pi}{2}t)\int_t^1 y_m'(\tau)\int_0^1 |\tau - s|^{\frac{1}{2}}\sin(\tfrac{\pi}{2}s)ds d\tau$$
$$+\tfrac{2}{\pi}\cos(\tfrac{\pi}{2}t)\int_0^t \sin(\tfrac{\pi}{2}s)(\tfrac{9}{50\Gamma(1.5)}((1-s)^{\frac{1}{2}} - s^{\frac{1}{2}}) - \tfrac{\pi^2}{4}s)ds$$
$$+\tfrac{2}{\pi}\sin(\tfrac{\pi}{2}t)\int_t^1 \cos(\tfrac{\pi}{2}s)(\tfrac{9}{50\Gamma(1.5)}((1-s)^{\frac{1}{2}} - s^{\frac{1}{2}}) - \tfrac{\pi^2}{4}s)ds$$

by (4.1) and Theorem 3.1. Let the initial approximation $y_0 = t^2$, then by using MATLAB, we have the following approximations: $y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9$. The experimental results are presented in Table 4 and Figure 4.

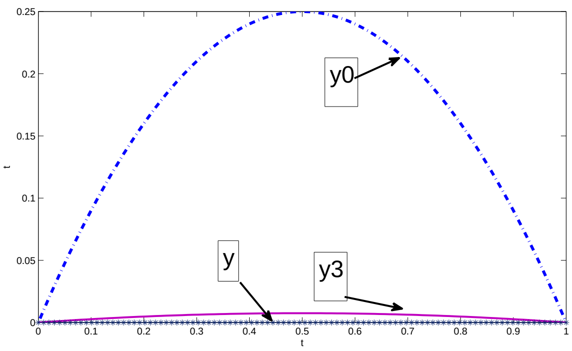

Figure 1: The exact solution and approximate solutions of Example 4.1.

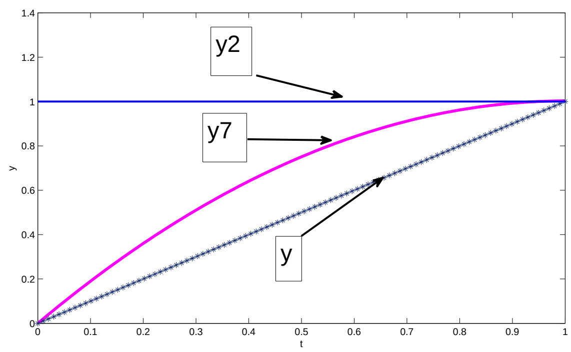

Figure 2: The exact solution and approximate solutions of Example 4.2.

Table 1: The approximate solutions and absolute error for Example 4.1.

t	y_0	y_2	y_3
0.0	0.00	0.0000	0.0000×10^{-4}
0.1	0.09	0.0003	0.2214×10^{-4}
0.2	0.16	0.0006	0.4923×10^{-4}
0.3	0.21	0.0008	0.7286×10^{-4}
0.4	0.24	0.0010	0.8773×10^{-4}
0.5	0.25	0.0010	0.9104×10^{-4}
0.6	0.24	0.0009	0.8290×10^{-4}
0.7	0.21	0.0007	0.6588×10^{-4}
0.8	0.16	0.0005	0.4386×10^{-4}
0.9	0.09	0.0002	0.2084×10^{-4}
1.0	0.00	0.0000	0.0000×10^{-4}
t	y_4	y_5	$\lvert y - y_5\rvert$
0.0	0.0000×10^{-5}	0.0000×10^{-6}	0.0000×10^{-6}
0.1	0.1981×10^{-5}	0.1775×10^{-6}	0.1775×10^{-6}
0.2	0.4412×10^{-5}	0.3954×10^{-6}	0.3954×10^{-6}
0.3	0.6539×10^{-5}	0.5860×10^{-6}	0.5860×10^{-6}
0.4	0.7873×10^{-5}	0.7053×10^{-6}	0.7053×10^{-6}
0.5	0.8154×10^{-5}	0.7296×10^{-6}	0.7296×10^{-6}
0.6	0.7400×10^{-5}	0.6612×10^{-6}	0.6612×10^{-6}
0.7	0.5859×10^{-5}	0.5228×10^{-6}	0.5228×10^{-6}
0.8	0.3890×10^{-5}	0.3468×10^{-6}	0.3468×10^{-6}
0.9	0.1846×10^{-5}	0.1645×10^{-6}	0.1645×10^{-6}
1.0	0.0000×10^{-5}	0.0000×10^{-6}	0.0000×10^{-6}

Table 2: The approximate solutions and absolute error for Example 4.2.

t	y_5	y_6	y_7
0.0	0.00	0.0000	0.0000
0.1	0.01	0.1000	0.1001
0.2	0.04	0.2001	0.2003
0.3	0.09	0.3000	0.3003
0.4	0.16	0.3998	0.4000
0.5	0.25	0.4999	0.5000
0.6	0.36	0.5995	0.5998
0.7	0.49	0.6996	0.6997
0.8	0.64	0.7996	0.7997
0.9	0.81	0.8997	0.8998
1.0	1.00	1.0000	1.0000
t	y_8	y_9	$\lvert y - y_9\rvert$
0.0	0.0000	0.0000	0.0000×10^{-3}
0.1	0.1002	0.1002	0.2067×10^{-3}
0.2	0.2003	0.2003	0.2878×10^{-3}
0.3	0.3003	0.3003	0.2592×10^{-3}
0.4	0.4001	0.4002	0.1506×10^{-3}
0.5	0.5000	0.5000	0.0001×10^{-3}
0.6	0.5998	0.5998	0.1504×10^{-3}
0.7	0.6997	0.6997	0.2590×10^{-3}
0.8	0.7997	0.7997	0.2877×10^{-3}
0.9	0.8998	0.8998	0.2066×10^{-3}
1.0	1.0000	1.0000	0.0000×10^{-3}

Table 3: The approximate solutions and absolute error for Example 4.3.

t	y_0	y_6	y_7
0.0	1.0	1.0000	1.0000
0.1	1.0	1.0999	1.1000
0.2	1.0	1.2000	1.2002
0.3	1.0	1.2998	1.3002
0.4	1.0	1.3997	1.4001
0.5	1.0	1.4999	1.5001
0.6	1.0	1.5996	1.5999
0.7	1.0	1.6997	1.6999
0.8	1.0	1.7998	1.7998
0.9	1.0	1.8999	1.8999
1.0	1.0	2.0000	2.0000
t	y_8	y_9	$\|y - y_9\|$
0.0	1.0000	1.0000	0.0000×10^{-3}
0.1	1.1001	1.1002	0.2066×10^{-3}
0.2	1.2003	1.2003	1.2876×10^{-3}
0.3	1.3003	1.3003	0.2590×10^{-3}
0.4	1.4001	1.4002	0.1505×10^{-3}
0.5	1.5000	1.5000	0.0001×10^{-3}
0.6	1.5999	1.5998	0.1503×10^{-3}
0.7	1.6998	1.6997	0.2589×10^{-3}
0.8	1.7999	1.7997	0.2875×10^{-3}
0.9	1.8998	1.8998	0.2065×10^{-3}
1.0	2.0000	2.0000	0.0000×10^{-3}

Table 4: The approximate solutions and absolute error for Example 4.4.

t	y_0	y_6	y_7
0.0	0.00	0.0000	0.0000
0.1	0.1564	0.1004	0.1005
0.2	0.3090	0.2002	0.2003
0.3	0.4540	0.3001	0.3003
0.4	0.5878	0.3999	0.4000
0.5	0.7071	0.5001	0.5000
0.6	0.8090	0.5999	0.5998
0.7	0.8910	0.6998	0.6997
0.8	0.9511	0.7999	0.7997
0.9	0.9877	0.8999	0.8998
1.0	1.0000	1.0000	1.0000
t	y_8	y_9	$\|y - y_9\|$
0.0	0.0000	0.0000	0.0000×10^{-3}
0.1	0.1004	0.1005	0.4548×10^{-3}
0.2	0.2002	0.2006	0.6121×10^{-3}
0.3	0.3004	0.3005	0.6121×10^{-3}
0.4	0.4001	0.4003	0.3119×10^{-3}
0.5	0.5001	0.5000	0.0046×10^{-3}
0.6	0.5998	0.5997	0.3031×10^{-3}
0.7	0.6996	0.6995	0.5324×10^{-3}
0.8	0.7995	0.7994	0.6064×10^{-3}
0.9	0.8996	0.8995	0.4516×10^{-3}
1.0	1.0000	1.0000	0.0000×10^{-3}

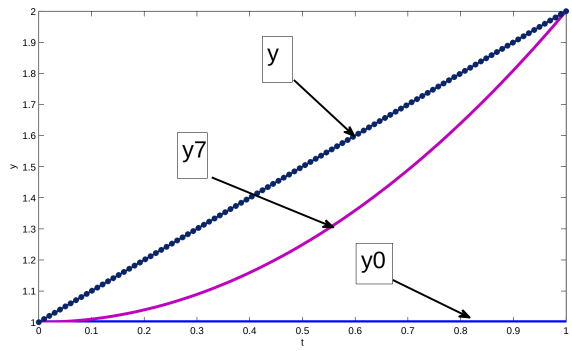

Figure 3: The exact solution and approximate solutions of Example 4.3.

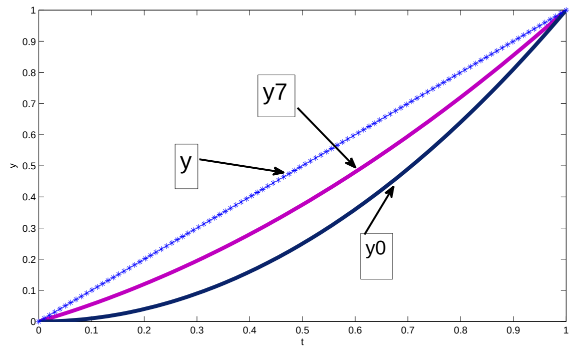

Figure 4: The exact solution and approximate solutions of Example 4.4.

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