A Convolution Theorem for the Polynomial Fourier Transform

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***Abstract*—The polynomial Fourier transform (PFT) is a useful mathematical tool for many areas, including applied mathematics, engineering and signal processing. Some properties and applications for this transform are already known, but an existence of the PFT's convolution theorem is still unknown. The purpose of this paper is to introduce a convolution theorem for the PFT, which has the elegance and simplicity comparable to that of the Fourier Transform (FT). The classical result in the FT domain is shown to be a special case of our achieved theorem.**

***Index Terms*—Convolution theorem, Fourier transform, Minkowski's inequality, polynomial Fourier transform, Young's inequality**

I. Introduction

THE Fourier transform (FT) plays an important part in the theory of applied mathematics, engineering, signal processing and optics [1]–[5]. In the classical signal processing, the FT has developed into a powerful tool for frequency domain-based signal representation. The convolution of the FT, which has been widely used in the theory of linear time-invariant (LTI) systems, is a fundamental and important property in frequency domain-based filter design [1]–[3], [5], [6]. It is a mathematical operation on two functions, and can be expressed by the integral of the pointwise multiplication of the first function and a translated version of the second function. In addition to the conventional FT convolution, convolution theorems of many other types of transformations are currently derived [5], [7]–[16]. These results have found many applications in the fields of computer vision, natural language processing, image and signal processing, statistics and engineering.

It is well-known that the FT is not suitable to deal with time-varying signals that contain frequencies changing with time. To describe such kind of signals the fractional Fourier transform (FrFT) [9], [10], [17]–[19] and local polynomial Fourier transform (LPFT) [20]–[26] which is a generalization of the short-time Fourier transform (STFT) [21], [27] was proposed. A review on recent developments and applications of the LPFT are referred to [21] and references therein.

The polynomial Fourier Transform (PFT) introduces polynomial parameters including the first-order derivative and other higher-order derivatives of the instantaneous frequency (IF) of the analyzed signal [21], [25], [26], [28]. The kernel of the PFT uses extra parameters to approximate the phase of the signal into a polynomial form. With these parameters (polynomial coefficients) the PFT can describe the time-varying frequencies with a better accuracy, and therefore the resolution of signal representation in the time-frequency domain can be significantly improved. The PFT has been developed under different names [18], [20], [29], [30], and its properties including uncertainty principle were published by Li *et al.* in [22]. But an existence of the PFT's convolution theorem is still missing; therefore, the purpose of this paper has been to show convolution theorem for the PFT. This theorem is a generalization of the conventional convolution in the FT domain. The PFT has found many applications, for example, in the ISAR image autofocusing [23], in SAR imaging of moving targets [28], and in reconstruction of compressive sensing signals [25]. Moreover, Zhou *et al.* shows SAR accelerating moving target parameter estimation and imaging based on three-order PFT [26].

This paper is organized as follows: After a review of the PFT and the convolution theory in Section II, a new convolution theorem for the PFT with its properties are derived in Section III and the conclusion is written in the Section IV.

II. Preliminaries

In this section, we provide a brief review of the PFT and convolution theory that will be needed later. The PFT relationships with other well-known transforms, such as FT, FrFT, linear canonical transform (LCT) [7], [8], [11]–[15], [31], [32] and offset linear canonical transform (OLCT) [16], [32], [33] are investigated. Also some fundamental properties of the convolution are given.

A. The PFT and Discrete-time PFT

Definition 1. The form of the PFT of a signal f is as follows [21], [28]

$$F(\varpi) = PFT_f(\varpi) = \int_{-\infty}^{+\infty} f(t) e^{-i\theta_M(t,\varpi)} dt, \tag{1}$$

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where

$$\theta_M(t,\varpi) = \omega t + \frac{\omega_1 t^2}{2} + \ldots + \frac{\omega_{M-1} t^M}{M!},$$
$$\varpi = (\omega, \omega_1, .., \omega_M),$$

M is the order of the polynomial function, $\omega_n = d^n\Omega(\tau)/d\tau^n, n = 1,\ldots, M-1$ are the polynomial coefficients, and $\Omega(\tau)$ is the instantaneous frequency of the signal.

Remark. Difference between PFT and LPFT is the added window function $W(t-\tau)$ of the LPFT

$$LPFT_f(\tau,\varpi) = \int_{-\infty}^{+\infty} f(t) W(t-\tau) e^{-i\theta_M(t,\varpi)} dt.$$

Below we obtain the discrete-time PFT, which will be helpful for interested researchers for utilizing the sampling theorem of band-limited signals in the PFT domain in future.

Firstly, the uniform sampled signal is defined as below

$$\hat{f}(t) = f(t) s_\delta(t) = f(t)\sum_{n=-\infty}^{+\infty}\delta(t-nT) = \sum_{n=-\infty}^{+\infty} f(nT)\delta(t-nT),$$

where $s_\delta(t)$ is the uniform impulse train and T is the sampling period. Using the definition of the PFT, we have

$$\begin{aligned}\hat{F}(\varpi) &= PFT_{\hat{f}}(\varpi) \\ &= \int_{-\infty}^{+\infty} e^{-i\theta_M(t,\varpi)} \times \sum_{n=-\infty}^{+\infty} f(nT)\delta(t-nT) dt \qquad (2) \\ &= \sum_{n=-\infty}^{+\infty} f(nT) e^{-i\theta_M(nT,\varpi)}.\end{aligned}$$

Equation (2) shows how to obtain the PFT of a discrete time signal $f(nT)$. We refer to it as the discrete-time PFT.

B. The PFT Relationships with Other Transforms

Definition 2. For $M = 1$, the FT can be expressed in terms of the first-order PFT as follows

$$FT_f(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt = PFT_f(\omega).$$

Definition 3. For $M = 2, \omega = u\csc(\alpha)$ and $\omega_1 = \cot(\alpha)$ in (1), the FrFT can be expressed in terms of the second-order PFT as

$$FrFT_f(\omega,\omega_1) = \sqrt{\frac{1-i\cot(\alpha)}{2\pi}} e^{i\frac{u^2}{2}\cot(\alpha)} \cdot PFT_f(\omega,\omega_1).$$

Definition 4. For $M = 2, \omega = \frac{1}{b}u$ and $\omega_1 = -\frac{a}{b}$ in (1), the LCT can be expressed in terms of the second-order PFT as follows

$$LCT_f(\omega,\omega_1) = \sqrt{\frac{1}{i2\pi b}} e^{i\frac{d}{2}b\omega^2} \cdot PFT_f(\omega,\omega_1).$$

Definition 5. For $M = 2, \omega = \frac{1}{b}(u-u_0)$ and $\omega_1 = -\frac{a}{b}$ in (1), the OLCT can be expressed in terms of the second-order PFT as

$$OLCT_f(\omega,\omega_1) = \sqrt{\frac{1}{i2\pi b}} e^{i\frac{1}{2b}\left(du_0^2 - 2u(du_0 - b\nu_0) + du^2\right)} \cdot PFT_f(\omega,\omega_1).$$

From Definition 2 we can see that the PFT is a generalization of the FT. By choosing M we can derive FrFT, LCT and OLCT through PFT, as shown in Definitions 3, 4 and 5, respectively.

C. The Convolution Theory

Convolution is a fundamental mathematical operation on two functions. The output of any continuous time LTI system is found via the convolution of the input signal with the system impulse response [1]–[3], [5], [7]–[16].

In the general framework of convolution theory, it is known that to every integral transformation $\mathbf{T}$, one can, at least theoretically, associate with it a convolution operation, $\otimes$, such that [12], [16]

$$\mathbf{T}(f\otimes g) = \mathbf{T}(f)\mathbf{T}(g).$$

The convolution of the FT is defined as the integral of the product of the two functions after one is reversed and shifted. As such, it is a particular kind of integral transform

$$(f\otimes g)(t) = \int_{-\infty}^{+\infty} f(\tau) g(t-\tau) d\tau.$$

The convolution theorem of the FT for the signals $f(t)$ and $g(t)$ with their FTs, $FT_f(\omega)$ and $FT_g(\omega)$, respectively is defined as [1]–[3], [5], [13], [14], [16]

$$f(t)\otimes g(t) \overset{FT}{\leftrightarrow} FT_f(\omega) FT_g(\omega),$$

where "$\otimes$" denotes the convolution operation.

The powerful result of this theorem is that the convolution of two signals $f(t)$ and $g(t)$ results in a simple multiplication of their FTs in the FT domain.

Convolution is similar to cross-correlation. If $g(\tau)$ is a symmetrical function $g(-\tau) = g(\tau)$, convolution is equivalent to correlation.

For all f, g and h in Lebesgue space $L^1(\Re)$, the following properties are valid [1]:

Property 1. (Commutativity)

$$f \otimes g = g \otimes f.$$

Property 2. (Associativity)

$$(f \otimes g) \otimes h = f \otimes (g \otimes h).$$

Property 3. (Distributivity)

$$f \otimes (g + h) = f \otimes g + f \otimes h;$$
$$(f + g) \otimes h = f \otimes h + g \otimes h.$$

The above properties are fundamental properties of the convolution.

III. Convolution Theorem for the PFT

In this section, we give the definition and theorem of the newly defined convolution of the PFT. Then we obtain some fundamental properties and basic inequalities of this convolution. We also show that the convolution theorem in the FT domain can be seen as a special case of our achieved result.

A. *New Definition of Convolution for the PFT*

Let us first define a weight function $W(t, \tau)$ by

$$W(t, \tau) = e^{i(-\theta_M(\tau, \varpi) + \theta_M(t, \varpi) - \theta_M(t - \tau, \varpi))}.$$

Definition 6. Let us denote the convolution operation by "$*$", and the convolution of any two functions in time domain is

$$\begin{aligned}(f)(t) * (g)(t) &= (f * g)(t) \\ &= \int_{-\infty}^{+\infty} f(\tau) g(t - \tau) W(t, \tau) d\tau. \end{aligned} \tag{3}$$

If the weight function $W(t, \tau)$ is changed to $W(t, \tau) = 1$, the Equation (3) reduces to the conventional convolution in FT domain, and if $W(t, \tau) = e^{i\tau(\tau - t)a/b}$, the Equation (3) is a conventional convolution in LCT domain [13], [14].

B. *Properties of Convolution for the PFT*

In this subsection we give commutativity, associativity, and distributivity properties of the PFT with its detailed proofs.

For all f, g, h in Lebesgue space $L^1(\Re)$, the following properties are valid:

Property 4. (Commutativity)

$$(f * g) = (g * f).$$

Proof. A simple calculation gives

$$\begin{aligned}(f * g)(t) &= \int_{-\infty}^{+\infty} f(\tau) g(t - \tau) e^{i(-\theta_M(\tau, \varpi) + \theta_M(t, \varpi) - \theta_M(t - \tau, \varpi))} d\tau \\ &= \int_{-\infty}^{+\infty} g(\gamma) f(t - \gamma) e^{i(-\theta_M(t - \gamma, \varpi) + \theta_M(t, \varpi) - \theta_M(\gamma, \varpi))} d\gamma \\ &= (g * f)(t).\end{aligned}$$

This finishes the proof.

Property 5. (Associativity)

$$(f * g) * h = f * (g * h).$$

Proof. By using Definition 6 and Property 2 of convolution the Property 5 may be written as

$$(f * g)(t) = \int_{-\infty}^{+\infty} f(\tau) g(t - \tau) e^{-i\theta_M(\tau, \varpi)} e^{-i\theta_M(t - \tau, \varpi)} e^{i\theta_M(t, \varpi)} d\tau,$$

by making the changes, $\overline{f}(t) = f(t) e^{-i\theta_M(t, \varpi)}$ and $\overline{g}(t) = g(t - \tau) e^{-i\theta_M(t - \tau, \varpi)}$, we obtain

$$(f * g)(t) = (\overline{f} \otimes \overline{g}) e^{i\theta_M(t, \varpi)}. \tag{4}$$

Therefore,

$$\begin{aligned}(f * g) * h &= \int_{-\infty}^{+\infty} (\overline{f} \otimes \overline{g})(t) e^{i\theta_M(t, \varpi)} \\ &\quad \cdot h(v - t) e^{-i\theta_M(t, \varpi)} e^{-i\theta_M(v - t, \varpi)} e^{i\theta_M(v, \varpi)} dt \\ &= e^{i\theta_M(v, \varpi)} \int_{-\infty}^{+\infty} (\overline{f} \otimes \overline{g})(t) h(v - t) e^{-i\theta_M(v - t, \varpi)} dt.\end{aligned}$$

Similar to the above changes, we let

$$\overline{h}(v) = h(v - t) e^{-i\theta_M(v - t, \varpi)}.$$

Then we have

$$\begin{aligned}(f * g) * h &= e^{i\theta_M(v, \varpi)} (\overline{f} \otimes \overline{g}) \otimes \overline{h} \\ &= e^{i\theta_M(v, \varpi)} \overline{f} \otimes (\overline{g} \otimes \overline{h}) = f * (g * h).\end{aligned}$$

The proof of Property 5 is completed.

Property 6. (Distributivity)

$$f * (g + h) = f * g + f * h;$$
$$(f + g) * h = f * h + g * h.$$

Proof. To prove Property 6 we use the change $\overline{h}(t) = h(t) e^{-i\theta_M(t, \varpi)}$ and get a formula which is similar to (4),

$$(f * h)(t) = \left(\overline{f} \otimes \overline{h}\right) e^{i\theta_M(t,\varpi)}. \tag{5}$$

Thanks to (4) and (5), we obtain the proof of first part of Property 6 as follows

$$\begin{aligned} f * g + f * h &= \left(\overline{f} \otimes \overline{g}\right) e^{i\theta_M(t,\varpi)} + \left(\overline{f} \otimes \overline{h}\right) e^{i\theta_M(t,\varpi)} \\ &= e^{i\theta_M(t,\varpi)} \left(\overline{f} \otimes \overline{g} + \overline{f} \otimes \overline{h}\right) \\ &= e^{i\theta_M(t,\varpi)} \left(\overline{f} \otimes \left(\overline{g} + \overline{f}\right)\right) \\ &= f \otimes (g+h) = f * (g+h). \end{aligned}$$

The second part of Property 6 can be proven in a similar way; so it is omitted.

C. Basic Convolution Inequalities

As pointed out by Grafakos in [1], the most fundamental inequalities involving convolutions are the Minkowski's inequality and Young's inequality.

Theorem 1. (Minkowski's inequality) Let $1 \le p \le \infty$, for $f \in L^p(\Re)$ and $g \in L^1(\Re)$ we have

$$\|g * f\|_{L^p(\Re)} \le \|g\|_{L^1(\Re)} \cdot \|f\|_{L^p(\Re)}. \tag{6}$$

Proof. We may assume that $1 < p < +\infty$, since the cases $p = 1$ and $p = +\infty$ are simple. Clearly, we have

$$\begin{aligned} |(g * f)(t)| &= \left| \int_{-\infty}^{+\infty} g(\tau) f(t-\tau) W(\tau, t) d\tau \right| \\ &\le \int_{-\infty}^{+\infty} |g(\tau)| |f(t-\tau)| d\tau. \end{aligned} \tag{7}$$

Let $p' = \frac{p}{p-1}$, so that $\frac{1}{p} + \frac{1}{p'} = 1$. By using the Hölder's inequality, we obtain

$$\begin{aligned} \int_{-\infty}^{+\infty} |g(\tau)| |f(t-\tau)| d\tau &= \int_{-\infty}^{+\infty} \left(|g(\tau)|^{\frac{1}{p}} |f(t-\tau)| \right) |g(\tau)|^{\frac{1}{p'}} d\tau \\ &\le \left(\int_{-\infty}^{+\infty} \left(|g(\tau)|^{\frac{1}{p}} |f(t-\tau)| \right)^p d\tau \right)^{\frac{1}{p}} \left(\int_{-\infty}^{+\infty} \left(|g(\tau)|^{\frac{1}{p'}} \right)^{p'} d\tau \right)^{\frac{1}{p'}} \\ &= \left(\int_{-\infty}^{+\infty} \left(|g(\tau)| |f(t-\tau)|^p \right) d\tau \right)^{\frac{1}{p}} \left(\int_{-\infty}^{+\infty} |g(\tau)| d\tau \right)^{\frac{1}{p'}}. \end{aligned} \tag{8}$$

With (7) and (8), we derive

$$|(g * f)(t)| \le \left(\int_{-\infty}^{+\infty} |g(\tau)| |f(t-\tau)|^p d\tau \right)^{\frac{1}{p}} \left(\int_{-\infty}^{+\infty} |g(\tau)| d\tau \right)^{\frac{1}{p'}}. \tag{9}$$

Taking L^p norms of both sides of (9) and using Fubini's theorem, we have

$$\begin{aligned} \|g * f\|_{L^p(\Re)} &= \left(\int_{-\infty}^{+\infty} |(g * f)(t)|^p dt \right)^{\frac{1}{p}} \\ &\le \left(\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} |g(\tau)| |f(t-\tau)|^p d\tau \right) \left(\int_{-\infty}^{+\infty} |g(\tau)| d\tau \right)^{\frac{p}{p'}} dt \right)^{\frac{1}{p}} \\ &= \left(\int_{-\infty}^{+\infty} |g(\tau)| d\tau \right)^{\frac{1}{p'}} \left(\int_{-\infty}^{+\infty} |g(\tau)| \left(\int_{-\infty}^{+\infty} \left| \tilde{f}(t-\tau) \right|^p dt \right) d\tau \right)^{\frac{1}{p}} \\ &= \left(\int_{-\infty}^{+\infty} |g(\tau)| d\tau \right)^{\frac{1}{p'}} \left(\int_{-\infty}^{+\infty} |g(\tau)| \left(\int_{-\infty}^{+\infty} |f(t)|^p dt \right) d\tau \right)^{\frac{1}{p}} \\ &= \left(\int_{-\infty}^{+\infty} |g(\tau)| d\tau \right)^{\frac{1}{p'}} \left(\int_{-\infty}^{+\infty} |g(\tau)| d\tau \right)^{\frac{1}{p}} \left(\int_{-\infty}^{+\infty} |f(t)|^p dt \right)^{\frac{1}{p}} \\ &= \int_{-\infty}^{+\infty} |g(\tau)| d\tau \left(\int_{-\infty}^{+\infty} |f(t)|^p dt \right)^{\frac{1}{p}} \\ &= \|g\|_{L^1(\Re)} \|f\|_{L^p(\Re)}. \end{aligned}$$

The proof is completed.

Minkowski's inequality (6) is only a special case of Young's inequality in which the function g can be in any space $L^r(\Re)$ for $1 \le r < +\infty$.

Theorem 2. (Young's inequality) Let $1 \le p, q, r \le +\infty$ satisfy

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}.$$

Then for all $f \in L^p(\Re)$ and all $g \in L^r(\Re)$ satisfying $\|g\|_{L^r(\Re)} = \|\tilde{g}\|_{L^r(\Re)}$, we have

$$\|f * g\|_{L^q(\Re)} \le \|g\|_{L^r(\Re)} \|f\|_{L^p(\Re)}.$$

Proof. Observe that when $r < +\infty$, the hypotheses on the indices imply that

$$\frac{1}{r'} + \frac{1}{q} + \frac{1}{p'} = 1, \quad \frac{p}{q} + \frac{p}{r'} = 1, \quad \frac{r}{q} + \frac{r}{p'} = 1.$$

Using Hölder's inequality, we get

$$|(f * g)(t)| \le \int_{-\infty}^{+\infty} |f(\tau)| |g(t-\tau)| d\tau$$

$$
\begin{aligned}
&= \int_{-\infty}^{+\infty} |f(\tau)|^{\frac{p}{r'}} \left(|f(\tau)|^{\frac{p}{q}} |g(t-\tau)|^{\frac{r}{q}} \right) |g(t-\tau)|^{\frac{r}{p'}} d\tau \\
&\le \left(\int_{-\infty}^{+\infty} \left(|f(\tau)|^{\frac{p}{r'}} \right)^{r'} d\tau \right)^{\frac{1}{r'}} \left(\int_{-\infty}^{+\infty} \left(|f(\tau)|^{\frac{p}{q}} |g(t-\tau)|^{\frac{r}{q}} \right)^{q} d\tau \right)^{\frac{1}{q}} \\
&\cdot \left(\int_{-\infty}^{+\infty} \left(|g(t-\tau)|^{\frac{r}{p'}} \right)^{p'} d\tau \right)^{\frac{1}{p'}} \\
&= \|f\|_{L^p(\Re)}^{\frac{p}{r'}} \left(\int_{-\infty}^{+\infty} |f(\tau)|^p |g(t-\tau)|^r d\tau \right)^{\frac{1}{q}} \left(\int_{-\infty}^{+\infty} |\tilde{g}(t-\tau)|^r d\tau \right)^{\frac{1}{p'}} \\
&= \|f\|_{L^p(\Re)}^{\frac{p}{r'}} \left(\int_{-\infty}^{+\infty} |f(\tau)|^p |g(t-\tau)|^r d\tau \right)^{\frac{1}{q}} \left(\int_{-\infty}^{+\infty} |\tilde{g}(\tau)|^r d\tau \right)^{\frac{1}{p'}} \\
&= \left(\int_{-\infty}^{+\infty} |f(\tau)|^p |g(t-\tau)|^r d\tau \right)^{\frac{1}{q}} \|\tilde{g}\|_{L^r(\Re)}^{\frac{r}{p'}} \|f\|_{L^p(\Re)}^{\frac{p}{r'}}.
\end{aligned} \tag{10}
$$

Taking L^q norms of both sides of (10) and using Fubini's theorem, we derive

$$
\|f * g\|_{L^q(\Re)} = \left(\int_{-\infty}^{+\infty} |(f * g)(t)|^q dt \right)^{\frac{1}{q}}
$$

$$
\le \left[\int_{-\infty}^{+\infty} \left(\begin{array}{l} \left(\int_{-\infty}^{+\infty} |f(\tau)|^p |g(t-\tau)|^r d\tau \right)^{\frac{1}{q}} \\ \cdot \|\tilde{g}\|_{L^r(\Re)}^{\frac{r}{p'}} \|f\|_{L^p(\Re)}^{\frac{p}{r'}} \end{array} \right)^{q} dt \right]^{\frac{1}{q}},
$$

then, we get the following result

$$
\begin{aligned}
&\|\tilde{g}\|_{L^r(\Re)}^{\frac{r}{p'}} \|f\|_{L^p(\Re)}^{\frac{p}{r'}} \left(\int_{-\infty}^{+\infty} |f(\tau)|^p d\tau \right)^{\frac{1}{q}} \left(\int_{-\infty}^{+\infty} |g(t)|^r dt \right)^{\frac{1}{q}} \\
&= \|\tilde{g}\|_{L^r(\Re)}^{\frac{r}{p'}} \|f\|_{L^p(\Re)}^{\frac{p}{r'}} \|f\|_{L^p(\Re)}^{\frac{p}{q}} \|g\|_{L^r(\Re)}^{\frac{r}{q}} \\
&= \|g\|_{L^r(\Re)} \|f\|_{L^p(\Re)},
\end{aligned}
$$

which completes the proof.

D. The Convolution Theorem for the PFT

The following theorem introduces a new convolution structure for the PFT.

Theorem 3. The $PFT_{f*g}(\varpi)$ is PFT of $f * g$ and PFT_f and PFT_g denote the PFT of f and g, respectively. Then

$$
PFT_{f*g}(\varpi) = PFT_f(\varpi) \cdot PFT_g(\varpi). \tag{11}
$$

Proof. The weight function $W(t,\tau)$ can be rewritten in the following form

$$
W(t,\tau) = e^{i\theta_M(t,\varpi)} e^{-i\theta_M(t-\tau,\varpi)} e^{-i\theta_M(\tau,\varpi)}.
$$

After using the formula (11), it becomes as follows

$$
PFT_{f*g}(\varpi) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\tau) g(t-\tau) W(t,\tau) e^{-i\theta_M(t,\varpi)} d\tau dt. \tag{12}
$$

Formula (12) can be expressed as

$$
\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\tau) g(t-\tau) e^{i\theta_M(t,\varpi)} e^{-i\theta_M(t-\tau,\varpi)} e^{-i\theta_M(\tau,\varpi)} e^{-i\theta_M(t,\varpi)} d\tau dt \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\tau) e^{-i\theta_M(\tau,\varpi)} g(t-\tau) e^{-i\theta_M(t-\tau,\varpi)} d\tau dt.
\end{aligned}
$$

By making a integral transform

$$
\begin{cases} v = \tau \\ u = t - \tau, \end{cases}
$$

here its Jacobi determinant can be calculated as

$$
\frac{D(u,v)}{D(t,\tau)} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1.
$$

We get

$$
PFT_{f*g}(\varpi) = \int_{-\infty}^{+\infty} f(v) e^{-i\theta_M(v,\varpi)} dv \int_{-\infty}^{+\infty} g(u) e^{-i\theta_M(u,\varpi)} du.
$$

Further

$$
PFT_{f*g}(\varpi) = PFT_f(\varpi) \cdot PFT_g(\varpi)
$$

or

$$
f(t) * g(t) \xrightarrow{PFT} PFT_f(\varpi) \cdot PFT_g(\varpi).
$$

The proof is completed.

The result of this theorem is that the convolution of signals f and g in the time domain results in a simple multiplication of their PFTs in the PFT domain. The convolution theorem of the PFT is a generalization of that of the FT. The convolution theorem provides a filtering perspective to how a LTI system operates on an input signal, so that this theorem can be useful in practical analog filtering in PFT domain.

IV. Conclusion

The most fundamental and important property of the FT is convolution. Convolution is mathematical operation which has widely been used in the theory of LTI systems. In this paper, we have proposed an expression for the PFT of convolution integral, from which the convolution theorem for PFT was obtained. The derived theorem is a generalization of convolution theorem of FT. Our results can be applied in parameter estimation [26], [34], [35] and in filter design [6], [7] in the PFT domain, including two research directions: a) Designing of multiplicative filters through the product in the PFT domain, and b) Designing of multiplicative filters through the new convolution in the time domain. The reconstruction of band-limited signals [8] in the PFT domain is one of the potential applications of the PFT's convolution theorem, which will be shown in our future works.

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