Degree-Based Multiplicative Atom-bond Connectivity Index of Nanostuctures

Wei Gao, Muhammad Kamran Jamil, Waqas Nazeer, and Murad Amin

Abstract—There are a lot of nanomaterials and related chemical substances synthesized in the laboratory every year, which makes the test of their performance has become a hard work. The theory of nanoscience from the perspective of graph theory provides an excellent idea, characteristics of the nanomaterials can be obtained by calculating topological indices in their corresponding molecular graphs, and have attracted the attention of scholars in the field of nanoscience. In this paper, we learn the characteristics of nanostructures from mathematical point of view. Some important nanomaterials are selected and their multiplicative atom-bond connectivity indices are determined by edge set divided trick. These theoretical results can be considered as a guide line in nanoengineering.

Index Terms—Theoretical nanoscience, multiplicative atom-bond connectivity index, nanotubes, nanotori dendrimer, nanostar.

I. INTRODUCTION

OVER the years, the theoretical nanoscience has attracted more and more attention of scholars, the computational results are applied to nanoscience, biological, and pharmaceutical science fields. One of the important research branch of theoretical nanoscience can be stated as follows: the nanostructures related molecular structure is expressed by graphs, by calculating the topological index we can get the properties of the corresponding nanostructures. This technology can obtain effective results in the absence of experimental conditions, which is interested by the scholars from developing countries and regions. Gradually, as the development computing tricks, it has become an important branch in the field of theoretical nanoscience, and concerned by scientists from various fields (see Balaban [1], Munteanu et al. [2], Buscema et al. [3], Gao et al. [5], [4], Sirimulla et al. [6], Bodljaj and Batagelj [7], Nadeem and Shaker [8], Nistor and Troiksky [9], Arockiaraj et al. [10], Khakpoor and Keshe [11], and Ivanciuc [12] for more details).

We only consider simple nanostructure related molecular graph (each vertex represents an atom and each edge expresses as a chemical bond) in our paper. Let \( G \) be a molecular graph with vertex set \( V(G) \) and edge set \( E(G) \). For each vertex \( v \), the degree \( d(v) \) of \( v \) is the number of vertices adjacent to \( v \). A topological index can be regarded as a real function \( f : G \rightarrow \mathbb{R} \) which maps each molecular graph to a real number. In the past four decades, inspired by applications from the chemical engineering, many degree-based, spectral-based or distance-based indices were introduced, such as Zagreb index, atom-bond connectivity index, Wiener index, Harary index, Szeged index, PI index, eccentric connectivity index, harmonic index, Zagreb index and so on. Moreover, there are several advancements on distance-based, spectral-based, degree-based indices of special nanomaterial molecular structures which can be referred to Ramane and Jummannaver [13], Zhao and Wu [14], Sardar [15], Gao and Wang [16], [17], Gao et al. [18], [47], [48], Gao and Siddiqui [21], Abdo et al. [22], Basavanagoud [23], Sunilkumar et al. [24], and Guirao and de Bustos [25]. Estrada and Torres [26] introduced a new topological index called the atom-bond connectivity index (in short, the ABC index) which reflect the properties of alkanes. The atom-bond connectivity index of a molecular graph \( G \) can be stated as

\[
ABC(G) = \sum_{u,v \in E(G)} \sqrt{\frac{d(u) + d(v) - 2}{d(u)d(v)}}.
\]

Dehghan-Zadeh et al. [27] determined the first and second maximum values of the atom-bond connectivity index of tetracyclic graphs with \( n \) vertex. Ashrafi and Dehghan-Zadeh [28] studied the first and the second maximum values of the ABC index of cactus graphs with fixed vertex number. Goubko et al. [29] raised a counterexample for the previous conclusion. Husin et al. [30] researched the ABC index of two families of nanostar dendrimers. Dehghan-Zadeh and Ashrafi [31] derived the ABC index of quasi-tree graphs. Dimitrov [32] proposed an efficient computation approach of trees with the smallest atom-bond connectivity index. The structural characters of trees with a minimal ABC index were considered [33], [34], [35], [36], [37].

As a variant of the ABC index, the first multiplicative atom-bond connectivity index is formulated by

\[
ABC_{II}(G) = \prod_{u,v \in E(G)} \sqrt{\frac{d(u) + d(v) - 2}{d(u)d(v)}},
\]

which was defined in Kulli [38]. In Kulli’s work, he determined the first multiplicative atom-bond connectivity index of \( VC_2C_1[p,q] \) and \( HC_5C_7[p,q] \) nanotubes.

Furthermore, Kulli [39] introduced the fourth multiplicative atom-bond connectivity index which can be represented as

\[
ABC_{IV}(G) = \prod_{u,v \in E(G)} \sqrt{\frac{S(u) + S(v) - 2}{S(u)S(v)}},
\]

where \( S(v) = \sum_{u \in V(G)} d(u) \) for each \( v \in V(G) \).

Manuscript received May 30, 2017; revised August 08, 2017. The research is partially supported by NSF (no. 11401519).

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(Advance online publication: 17 November 2017)
A. Examples on computing of first multiplicative atom-bond connectivity index

Now, we present some of the examples about the calculating process of the first multiplicative atom bond connectivity index.

Example 1. \( W_n = K_1 \lor C_{n-1} \) is called wheel graph. By simple computation, we have \( \text{ABCP}_{11}(W_n) = \left( \frac{3}{2} \right)^{n-1} \left( \sqrt[n]{\frac{n+1}{n+2}} \right)^{n-1} \).

Example 2. Let \( n \) and \( m \) be two positive integers. The web graph \( W(n, m) \) is constructed from the Cartesian product of cycle \( C_n \) and path \( P_m \) (see Figure 1 as an example).

By analyzing the structure of web graph \( W(n, m) \), if \( m = 2 \), it easy to get

\[
\text{ABCP}_{11}(W(n, m)) = \left( \frac{3 + 3 - 2}{3} \right)^n = \left( \frac{4}{3} \right)^n.
\]

If \( m \geq 2 \), then the first multiplicative atom bond connectivity index of web graph \( W(n, m) \) is

\[
\text{ABCP}_{11}(W(n, m)) = \left( \frac{3 + 3 - 2}{3} \right)^{2n} \left( \frac{4 + 4 - 2}{4} \right)^{n(2m-5)} \times \left( \frac{3 + 4 - 2}{3} \right)^{2n} = \left( \frac{2}{3} \right)^{2n} \left( \frac{6}{4} \right)^{n(2m-5)} \left( \frac{5}{12} \right)^{2n}.
\]

In the following three examples, we show the value of \( \text{ABCP}_{11} \) index for three kinds of vertex gluing graphs.

Example 3. Let \( C_n \times m \) be a graph constructed from two cycles \( C_n \) and \( C_m \) with one common vertex (see Figure 2 as an example).

Using the definition of the first multiplicative atom bond connectivity index, we get

\[
\text{ABCP}_{11}(C_n \times m) = \left( \frac{2 + 2 - 2}{2} \right)^{m+n-4} \left( \frac{2 + 4 - 2}{2} \right)^4 = \left( \frac{1}{2} \right)^{n+m}.
\]

Example 4. Let \( W_{n,m} \) be a graph constructed from two wheel graphs \( W_n \) and \( W_m \) with one common vertex (see Figure 3 as an instance).

According to its graph structure analysis, we obtain

\[
\text{ABCP}_{11}(W_{n,m}) = \left( \frac{3 + 3 - 2}{3} \right)^{m+n-6} \left( \frac{6 + 3 - 2}{6} \right)^4 \times \left( \frac{(n-1) + 3 - 2}{(n-1) \cdot 3} \right)^{n-2} \left( \frac{(m-1) + 3 - 2}{(m-1) \cdot 3} \right)^{m-2} \times \left( \frac{(n-1) + 6 - 2}{(n-1) \cdot 6} \right)^{n-2} \left( \frac{(m-1) + 6 - 2}{(m-1) \cdot 6} \right)^{m-2} = \left( \frac{2}{3} \right)^{m+n-6} \left( \frac{7}{18} \right)^4 \left( \frac{n}{3(n-1)} \right)^{n-2} \times \left( \frac{m}{6(n-1)} \right)^{m-2} \left( \frac{n+3}{6(n-1)} \right) \sqrt[m-2]{\frac{m+3}{6(m-1)}}.
\]

Example 5. Let \( K(n, m) \) be a graph constructed from two complete graphs \( K_n \) and \( K_m \) with one common vertex (see Figure 4 as an instance).
By means of simple calculation, we yield
\[
ABC_{II}(K(n, m)) = \left( \sqrt{\frac{(n-1)+(n-1)-2}{(n-1) \cdot (n-1)}} \right)^{n-2} \cdot \left( \frac{m-1}{m-1} \right)^{n-1} \cdot \left( \frac{m+n-2}{m+n-2} \right)^{m-1} \cdot \left( \frac{n-1}{n-1} \cdot (m+n-2) \right)^{n-1}.
\]

B. Organization of the rest paper

So far, there have been numerous theoretical results about ABC index, but on the degree-based multiplication of ABC index is also very small. As a variable of the original ABC index, but on the degree-based multiplication of ABC index, it is worthy of further study. This motivates us to calculate some important chemical molecular application prospect, it is worthy of further study. The rest of paper is organized as follows: first, we determine the fourth multiplication ABC index of \( V \)-phenylenic nanotubes and nanotori; then, the first multiplication ABC index of \( V \)-phenylenic nanotubes and nanotori is also very small. As a variable of the original ABC index, but on the degree-based multiplication of ABC index, it is worthy of further study. This motivates us to calculate some important chemical molecular structure of ABC multiplication index.

The rest of paper is organized as follows: first, we determine the fourth multiplication ABC index of \( V \)-phenylenic nanotubes and nanotori; then, the first multiplication ABC index of \( V \)-phenylenic nanotubes and nanotori is also very small. As a variable of the original ABC index, but on the degree-based multiplication of ABC index, it is worthy of further study. This motivates us to calculate some important chemical molecular structure of ABC multiplication index.

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II. MAIN RESULTS AND PROOFS

In this section, we present the main results and their detail proofs. The trick to get these conclusions is followed by edge set dividing technology.

A. Fourth multiplication ABC index of \( V \)-phenylenic nanotubes and nanotori

The aim of this section is to determine the fourth multiplication ABC index of \( V \)-phenylenic nanotube and nanotori. The novel phenylenic and naphthylenic lattices consist of a square net embedded on the toroidal surface. As polycyclic conjugated molecules, phenylenes are composed of square nets are considered; at last, the first multiplication ABC index of carbon nanocones \( C_m[n] \) are calculated.

Theorem 1 Let \( n \) and \( m \) be two positive integers. The fourth multiplication ABC index of \( V \)-phenylenic nanotubes and nanotori are
\[
ABC_{II}(VPHX[m, n]) = \left( \frac{1}{2} \right)^m \cdot \left( \frac{7}{32} \right)^m \cdot \left( \frac{5}{24} \right)^m \cdot \left( \frac{4}{9} \right)^{9m-9m}
\]
and
\[
ABC_{II}(VPHY[m, n]) = \left( \frac{4}{9} \right)^{9m}.
\]

Proof. The proof is followed by edge set dividing approach in which the edge set is separated into several subsets according to the value of \( S(u) \) and \( S(v) \).

By analysis the molecular structure of \( V \)-phenylenic nanotubes \( VPHX[m, n] \), we see that its edge set can be divided

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(Archive online publication: 17 November 2017)
into the following four parts:
- \( e = (u, v) : S(u) = 6 \) and \( S(v) = 8 \), and there are \( 4m \) such edges;
- \( e = (u, v) : S(u) = S(v) = 8 \), and there are \( 2m \) such edges;
- \( e = (u, v) : S(u) = 8 \) and \( S(v) = 9 \), and there are \( 2m \) such edges;
- \( e = (u, v) : S(u) = S(v) = 9 \), and there are \( 9mn - 9m \) such edges.

In light of the definition of the fourth multiplication ABC index, we have

\[
ABC_4\Pi(VPHX[m, n]) = \prod_{u \neq v \in E(VPHX[m, n])} \sqrt{\frac{S_u + S_v - 2}{S_u S_v}} = \left(\sqrt{\frac{9 + 9 - 2}{9 \times 9}}\right)^{9mn - 9m}.
\]

For \( V \)-phenylenic nanotori \( VPHY[m, n] \), this is a 3-
regular molecular graph with \( d(v) = 3 \) for each \( v \in V(VPHY[m, n]) \), and thus \( S(v) = 9 \) for vertex \( v \). In view of the definition of the fourth multiplication ABC index, we get

\[
ABC_4\Pi(VPHY[m, n]) = \left(\sqrt{\frac{9 + 9 - 2}{9 \times 9}}\right)^{9mn} = \left(\frac{4}{9}\right)^{9mn}.
\]

Hence, we obtain the desired results.

B. The first multiplication ABC index of nanostructures

The aim of this section is to yield the first multiplication ABC index of \( TUC4C8[p, q] \), where \( q \) is the number of rows and \( p \) is the number of columns. Then we determine this topological index for its nanotubes. At last, the first multiplication ABC index of \( TUC4C8[p, q] \) (can be seen in Figure 7) is yielded. In this subsection, we always assume \( p, q \in \mathbb{N} \).

**Theorem 2** Let \( G = TUC4C8[p, q] \) be the two dimensional molecular lattice structure depicted in Figure 7. Then,

\[
ABC_1\Pi(G) = \left(\frac{1}{2}\right)^{3p+3q-2} \left(\frac{2}{3}\right)^{12pq-8(p+q)+4}.
\]

**Proof.** By analysis its structure, its edge set can be divided into three subsets:
- \( e = (u, v) : d(u) = 2 \), and there are \( 2p + 2q + 4 \) such edges;
- \( e = (u, v) : d(u) = 2 \) and \( d(v) = 3 \), and there are \( 4p + 4q - 8 \) such edges;
- \( e = (u, v) : d(u) = d(v) = 3 \), and there are \( 12pq - 8(p + q) + 4 \) such edges.

In term of the definition of the first multiplication ABC index, we infer

\[
ABC_1\Pi(G) = \prod_{u \neq v \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} = \left(\sqrt{\frac{2 + 2 - 2}{2 \times 2}}\right)^{2p+2q+4} \times \left(\sqrt{\frac{3 + 3 - 2}{3 \times 3}}\right)^{12pq-8(p+q)+4}.
\]

Hence, the formula in the theorem is correct.

**Theorem 3** Let \( G \) be the two dimensional \( C4C8[p, q] \) nanotube described in Figure 8. Then,

\[
ABC_1\Pi(G) = \left(\frac{1}{2}\right)^{3p} \left(\frac{2}{3}\right)^{12pq-8p}.
\]
Fig. 9. The structure of 2-D graph of $C_4C_8[p, q]$ nanotorus.

**Proof.** Similarly, its edge set can be divided into three subsets:
- $e = (u, v): d(u) = d(v) = 2$, and there are $2p$ such edges;
- $e = (u, v): d(u) = 2$ and $d(v) = 3$, and there are $4p$ such edges;
- $e = (u, v): d(u) = d(v) = 3$, and there are $12pq - 8p$ such edges.

By virtue of the definition of the first multiplication ABC index, we deduce

$$\text{ABC}_1\Pi(G) = (2\sqrt{3})^{2p}(2\sqrt{3})^{4p} \times (12pq - 8p).$$

Therefore, we verify the expected conclusion.

**Theorem 4** Let $G$ be the two dimensional $C_4C_8[p, q]$ nanotori described in Figure 9. Then,

$$\text{ABC}_1\Pi(G) = \left(\frac{2}{3}\right)^{12pq}.$$

**Proof.** Since $C_4C_8[p, q]$ nanotori is a 3-regular molecular graph with $12pq$ edges. We directly get the result.

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**C. The first multiplication ABC index of $NS_1[n]$**

In this part, we raise the first multiplication ABC index of an infinite class of $NS_1[n]$ (as examples, the basic structures of $NS_1[n]$ can be seen in Figure 10, Figure 11 and Figure 12).

**Theorem 5** Let $n \in \mathbb{N}$ be the step number of growth. The first multiplication ABC index of $NS_1[n]$ is given by

$$\text{ABC}_1\Pi(NS_1[n]) = \frac{\sqrt{3}^2}{2} \cdot (\sqrt{2})^{2n} - 9 \cdot (\sqrt{\frac{5}{12}})^3.$$

**Proof.** By analysis $E(NS_1[n])$, we have four separate subsets listed as follows:
- $e = (u, v): d(u) = 1$ and $d(v) = 4$, and there one edge in this subset;
- $e = (u, v): d(u) = d(v) = 2$, and there are $9 \cdot 2^n + 3$ such edges;
- $e = (u, v): d(u) = 2$ and $d(v) = 3$, and there are
18 \cdot 2^3 \cdot 2^6 = 12 \text{ such edges};
• \(e = (u, v); \) \(d(u) = 3\) and \(d(v) = 4\), and there are three such edges.

Hence, using the definition of the first multiplication ABC index, we derive

\[
ABC_1,2(\Pi (NS_1[n])) = \prod_{u \in E(\Pi (NS_1[n]))} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}
\]
\[
= \left( \sqrt{1 + 4 - 2} + \frac{2 + 2}{2} + \frac{2}{2} \right)^{2^3 \cdot 2^6 + 3} \times \left( \sqrt{2 + 3 - 2} + \frac{2 + 3}{2} \times 2 \times 3 \right)^{2 \cdot 2^3 \cdot 2^6 - 2}
\]
\[
= \frac{\sqrt{3}}{2} \left( \frac{1}{2} \right)^{2^3 \cdot 2^6 - 9} \left( \frac{5}{12} \right)^3.
\]

The proof is completed.

D. The first multiplication ABC index of dendrimer nanostars \(D_1[n]\) and \(D_3[n]\)

Here, we discuss the first multiplication ABC index of dendrimer nanostars \(D_1[n]\) and \(D_3[n]\), where these two molecular structures are widely appeared in the chemical compounds, drugs, and nanomaterials.

**Theorem 6** Let \(n \in \mathbb{N}\) be the number of steps of growth, then the first multiplication ABC index of dendrimer nanostars \(D_1[n]\) is stated as

\[
ABC_1,2(D_1[n]) = \sqrt{\frac{1}{3} \left( \frac{2}{2} \right)^{2^3 \cdot 2^6 - 6}}.
\]

**Proof.** Its edge set can be divided into three subsets:
• \(e = (u, v); d(u) = 1\), and \(d(v) = 3\), and there one edge in this subset;
• \(e = (u, v); d(u) = d(v) = 2\), and there are \(6 \cdot 2^3 - 2\) such edges;
• \(e = (u, v); d(u) = 2\), and \(d(v) = 3\), and there are \(12 \cdot 2^3\) - 10 such edges.

Hence, according to the definition of the first multiplication ABC index, we get

\[
ABC_1,2(D_1[n]) = \prod_{u \in E(D_1[n])} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}
\]
\[
= \left( \sqrt{1 + 3 - 2} + \frac{2 + 2}{2} \times 2 \times 3 \right)^{2^3 \cdot 2^6 - 2} \times \left( \frac{2 + 3}{2} \times 2 \times 3 \right)^{2 \cdot 2^3 \cdot 2^6 - 2}
\]
\[
= \frac{\sqrt{2}}{2} \left( \frac{1}{2} \right)^{2^3 \cdot 2^6 - 10}.
\]

The desired result is obtained.

As an instance, the molecular structure of \(D_3[3]\) is presented in Figure 13.

**Theorem 7** Let \(n \in \mathbb{N}\) be the number of steps of growth, then the first multiplication ABC index of dendrimer nanostars \(D_3[n]\) is stated as

\[
ABC_1,2(D_3[n]) = \sqrt{\frac{1}{3} \left( \frac{2}{2} \right)^{2^3 \cdot 2^6 - 9} \left( \frac{2}{3} \right)^{2^3 \cdot 2^6 - 6}}.
\]

**Proof.** The set \(E(NS_3[n])\) can be divided into following four parts:
• \(e = (u, v); d(u) = 1\) and \(d(v) = 3\), and there \(3 \cdot 2^4\) edges in this subset;
• \(e = (u, v); d(u) = d(v) = 2\), and there are \(12 \cdot 2^3 - 6\) such edges;
• \(e = (u, v); d(u) = 2\), and \(d(v) = 3\), and there are \(24 \cdot 2^3 - 12\) such edges;
• \(e = (u, v); d(u) = d(v) = 3\), and there are \(9 \cdot 2^3 - 6\) such edges.

Hence, using the definition of the first multiplication ABC index, we derive

\[
ABC_1,2(D_3[n]) = \prod_{u \in E(D_3[n])} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}
\]
\[
= \left( \frac{1 + 3 - 2}{1 \times 3} \right)^{2^3 \cdot 2^6 - 6} \times \left( \frac{2 + 3 - 2}{2 \times 2 \times 3} \right)^{2 \cdot 2^3 \cdot 2^6 - 2}
\]
\[
= \frac{\sqrt{2}}{2} \left( \frac{1}{2} \right)^{2^3 \cdot 2^6 - 10} \times \left( \frac{2}{3} \right)^{2^3 \cdot 2^6 - 10}.
\]

We complete the proof.

E. The first multiplication ABC index of carbon nanocones

Now, we compute the first multiplication ABC index of carbon nanocones \(CNC_m[n] = C_m[n]\) (see Figure 14 as an example of carbon nanocone).

**Theorem 8** Let \(m \geq 3\) and \(n \geq 1\) be two positive integers, then the first multiplication ABC index of carbon nanocones \(C_m[n]\) can be expressed as

\[
ABC_1,2(C_m[n]) = \sqrt{\frac{1}{2} \left( \frac{2^m}{3} \right)^{\frac{m(m+3)}{2}} + m}.
\]

**Proof.** By analysis the molecular structure of carbon nanocones \(C_m[n]\), we found that its edge set can be divided into three parts:
• \(e = (u, v); d(u) = d(v) = 2\), and there are \(m\) such edges;
• \(e = (u, v); d(u) = 2\), and \(d(v) = 3\), and there are \(2mn\) such edges;
\( e = (u, v): d(u) = d(v) = 3, \) and there are \( \frac{m(3n^2+n)}{2} \) such edges.

Then, by means of the definition of the first multiplication ABC index, we derive

\[
\text{ABC}_1 \Pi(G) = \prod_{u \in V(G)} \sqrt{d_u + d_v - 2}
\]

\[
= \left( \sqrt{\frac{2+2-2}{2 \times 2}} \right)^m \left( \sqrt{\frac{2+3-2}{2 \times 3}} \right)^{2mn}
\times \frac{2m+2}{3 \times 3}
\]

\[
= \left( \sqrt{\frac{2}{3}} \right)^{2mn+m} \left( \frac{2m+2}{3} \right)^{m(3n^2+n)}.
\]

Hence, we check the desired result. \( \Box \)

F. The fifth multiplication ABC index of \( \mathcal{NA}_m^n \) nanotubes

The eccentricity \( ec(u) \) of vertex \( u \in V(G) \) is defined as the maximum distance between \( u \) and any other vertex in \( G \). Gao et al. [51] introduced the fifth multiplicative atom bond connectivity index which was stated as

\[
	ext{ABC}_5 \Pi(G) = \prod_{u \in V(G)} \sqrt{ec(u) + ec(v) - 2}
\]

\[
\frac{ec(u)ec(v)}{ec(u)ec(v)}.
\]

The structure of \( \mathcal{NA}_m^n \) nanotubes was discussed by Bac et al. [52] as follows: consider the \( m \times n \) quadrilateral section \( P_{mn}^n \) with \( m \geq 2 \) hexagons on the top and bottom sides and \( n \geq 2 \) hexagons on the lateral sides cut from the regular hexagonal lattice \( L \). If we identify two lateral sides of \( P_{mn}^n \) such that we identify the vertices \( u_0^j \) and \( u_m^j \), for \( j = 0, 1, 2, \ldots, n \), then the \( \mathcal{NA}_m^n \) nanotubes are obtained. The detailed structure of \( \mathcal{NA}_m^n \) nanotubes nanotube can refer to Figure 15. In this part, we study \( \mathcal{NA}_m^n \) nanotube with \( n = m \) and its fifth multiplication ABC index is determined.

\textbf{Theorem 9} The fifth multiplication ABC index of \( \mathcal{NA}_m^n \) nanotubes can be expressed as follows:

- If \( n \equiv 0 \text{(mod2)} \), then

\[
\text{ABC}_5 \Pi(\mathcal{NP}_m^n) = \prod_{i=1}^{2n-1} \prod_{p=2}^{\frac{2p-1}{2}} \left( \frac{2p-1}{p^2+p} \right)^{(i-3)}
\]

\[
\times \prod_{p=2}^{\frac{2p-1}{2}} \prod_{i=\frac{3n-3}{2}+p}^{\frac{3n-1}{2}} \left( \frac{2p-1}{p^2+p} \right)^{(i-16)}
\]

\[
\times \prod_{i=3n+6}^{3n+1} \prod_{p=\frac{2p-1}{2}}^{\frac{2p-1}{2}} \left( \frac{2p-1}{p^2+p} \right)^{6n-1}.
\]

- If \( n \equiv 1 \text{(mod4)} \), then

\[
\text{ABC}_5 \Pi(\mathcal{NP}_m^n) = \sqrt{\frac{2p-2}{p}} \prod_{p=\frac{2p-1}{2}}^{\frac{2p-1}{2}} \left( \frac{2p-1}{p^2+p} \right)^{(i-1)} \prod_{i=1}^{\frac{2p-1}{2}} \prod_{p=\frac{2p-1}{2}}^{\frac{2p-1}{2}} \left( \frac{2p-1}{p^2+p} \right)^{(i-3)}
\]

\[
\times \prod_{i=\frac{2p-1}{2}}^{\frac{2p-1}{2}} \prod_{p=\frac{2p-1}{2}}^{\frac{2p-1}{2}} \left( \frac{2p-1}{p^2+p} \right)^{(i-16)}
\]

\[
\times \prod_{p=2}^{\frac{2p-1}{2}} \prod_{i=\frac{2p-1}{2}}^{\frac{2p-1}{2}} \left( \frac{2p-1}{p^2+p} \right)^{(i-1)}
\]

\[
\times \prod_{i=3n+6}^{3n+1} \prod_{p=\frac{2p-1}{2}}^{\frac{2p-1}{2}} \left( \frac{2p-1}{p^2+p} \right)^{6n-1}.
\]
If $n \equiv 3 \pmod{4}$, then

$$\text{ABC}_5 \Pi (N \mathcal{A}_m^n) = \prod_{p=2n+1}^{\frac{n+3}{2}} \prod_{i=1}^{\frac{n-1}{2} \cdot \lfloor \frac{n}{2} \rfloor} \left( \prod_{p=2n+1}^{\frac{n+3}{2}} \left( \frac{2p-1}{p^2+p} \right) \right)^{3(n+1)-1(n+3)-2}.$$ 

Proof. The whole proof can be divided into three parts according to the value of $n$.

Case 1. $n \equiv 0 \pmod{2}$. 

In this case, the edge set of $N \mathcal{A}_m^n$ nanotubes can be divided into eight parts according to the value of $\text{ev}(u)$ and $\text{ev}(v)$:

- $\text{ev}(u) = p$ and $\text{ev}(v) = p + 1$ where $p \in \{2n - \frac{n}{2}, \ldots, 2n - 1\}$, and there are $6n - 3$ such edges with $i \in \{1, \ldots, \frac{n-1}{2}\}$;
- $\text{ev}(u) = 2p$ and $\text{ev}(v) = 2p + 1$ where $p = n \equiv 0 \pmod{2}$, and there are $3n$ such edges with $i \equiv 0 \pmod{2}$;
- $\text{ev}(u) = \text{ev}(v) = 2p + 1$ where $p = n \equiv 0 \pmod{2}$, and there are $2s$ such edges;
- $\text{ev}(u) = p$ and $\text{ev}(v) = p + 1$ where $p \in \{2n + 1, \ldots, \frac{n+1}{2} \}$, and there are $(\frac{n}{2} + 1)(3n + 2)$ such edges;
- $\text{ev}(u) = p$ and $\text{ev}(v) = p + 1$ where $p \in \{2n + 1, \ldots, \frac{n+1}{2} \}$, and there are $6n - i$ such edges with $i = 3n + 6j$ and $j \in \{0, \ldots, \frac{n}{2} - 1\}$.

In terms of the definition of the fifth multiplication ABC index, we have

$$\text{ABC}_5 \Pi (N \mathcal{A}_m^n) = \prod_{u \in N \mathcal{A}_m^n} \left( \frac{\text{ev}(u) + \text{ev}(v) - 2}{\text{ev}(u) \text{ev}(v)} \right)^{2(n+1) \cdot \lfloor \frac{n}{2} \rfloor}.$$ 

Case 2. $n \equiv 1 \pmod{2}$. 

In this case, the edge set of $N \mathcal{A}_m^n$ nanotubes can be divided into eight parts according to the value of $\text{ev}(u)$ and $\text{ev}(v)$:

- $\text{ev}(u) = \text{ev}(v) = p$ where $p = n \equiv 0 \pmod{2}$, and there is only one such edge;
- $\text{ev}(u) = \text{ev}(v) = p$ where $\frac{3n+1}{2} < p \leq \frac{5n-1}{2}$ and $p \equiv 0 \pmod{2}$, and there are $n - 1$ such edges;
- $\text{ev}(u) = p$ and $\text{ev}(v) = p + 1$ where $\frac{3n+1}{2} \leq p \leq 2n - 2$ and $p \equiv 0 \pmod{2}$, and there are $4i$ such edges with $i \in \{1, \ldots, \frac{n-3}{2}\}$ and $i \equiv 1 \pmod{2}$;
- $\text{ev}(u) = p$ and $\text{ev}(v) = p + 1$ where $\frac{3n+3}{2} \leq p \leq 2n - 2$ and $p \equiv 0 \pmod{2}$, and there are $8i - 2$ such edges with $i \in \{2, \ldots, \frac{n-1}{2}\}$ and $i \equiv 0 \pmod{2}$;
- $\text{ev}(u) = p$ and $\text{ev}(v) = p + 1$ where $2n \leq p \leq \frac{5n-1}{2}$ and $p \equiv 0 \pmod{2}$, and there are $\frac{n+3}{2}(\frac{n+1}{2})$ such edges;
- $\text{ev}(u) = p$ and $\text{ev}(v) = p + 1$ where $2n + 1 \leq p \leq \frac{5n-3}{2}$ and $p \equiv 1 \pmod{2}$, and there are $n^2 - n$ such edges;
- $\text{ev}(u) = p$ and $\text{ev}(v) = p + 1$ where $\frac{5n+1}{2} \leq p \leq 3n - 2$ and $p \equiv 1 \pmod{2}$, and there are $16i$ such edges with $i \in \{1, \ldots, \frac{n-3}{2}\}$.

In view of the definition of the fifth multiplication ABC index, we get

$$\text{ABC}_5 \Pi (N \mathcal{A}_m^n) = \prod_{u \in N \mathcal{A}_m^n} \left( \frac{\text{ev}(u) + \text{ev}(v) - 2}{\text{ev}(u) \text{ev}(v)} \right)^{2(n+1) \cdot \lfloor \frac{n}{2} \rfloor}.$$ 

($\text{Advance online publication: 17 November 2017}$)
\[
\sum_{i=1}^{3n-2} p \equiv 1 \pmod{2} \frac{2p - 1}{p^2 + p}^{3n-4}.
\]

Case 3. \( n \equiv 3 \pmod{4} \)

In this case, the edge set of \( NA^m_n \) nanotubes can be divided into seven parts according to the value of \( ec(u) \) and \( ec(v) \):

- \( ec(u) = ec(v) = p \) where \( p \in \left\{ \frac{3n+3}{2}, \ldots, \frac{5n+1}{2} \right\} \), and there is one such edge;
- \( ec(u) = p + 1 \) and \( ec(v) = p + 1 \) where \( \frac{3n+3}{2} \leq p \leq 2n - 1 \) and \( p \equiv 1 \pmod{2} \), and there are \( 16i - 10 \) such edges with \( i \in \{1, \ldots, \frac{n-1}{2} \} \);
- \( ec(u) = p + 1 \) and \( ec(v) = p + 1 \) where \( \frac{3n+3}{2} \leq p \leq 2n \) and \( p \equiv 0 \pmod{2} \), and there are \( 8i \) such edges with \( i \in \{1, \ldots, \frac{n-1}{2} \} \);
- \( ec(u) = p \) and \( ec(v) = p + 1 \) where \( 2n + 2 \leq p \leq 2n + 1 \), and there are \( n(n+1) \) such edges;
- \( ec(u) = p \) and \( ec(v) = p + 1 \) where \( 2n + 2 \leq p \leq 3n - 2 \), and \( p \equiv 0 \pmod{2} \), and there are \( 8i - 4 \) such edges with \( i \in \{1, \ldots, \frac{n-1}{2} \} \).

In light of the definition of the fifth multiplication ABC index, we infer

\[
\text{ABC}_5 \Pi(NA^m_n) = \prod_{u \in E(NA^m_n)} \left( \frac{ec(u) + ec(v) - 2}{ec(u) ec(v)} \right)
\]

\[
= \prod_{p \equiv 1 \pmod{2}} \prod_{i=1}^{\frac{3n+3}{2}} \prod_{p \equiv 0 \pmod{2}} \left( \frac{2p - 1}{p^2 + p} \right)^{n+1}
\]

\[
= \prod_{i=1}^{\frac{3n+1}{2}} \prod_{p \equiv 1 \pmod{2}} \prod_{i=1}^{\frac{3n+2}{2}} \prod_{p \equiv 0 \pmod{2}} \left( \frac{2p - 1}{p^2 + p} \right)^{16i-10}
\]

\[
= \prod_{i=1}^{\frac{3n+1}{2}} \prod_{p \equiv 1 \pmod{2}} \prod_{i=1}^{\frac{3n+2}{2}} \prod_{p \equiv 0 \pmod{2}} \left( \frac{2p - 1}{p^2 + p} \right)^{8i}
\]

\[
= \prod_{i=1}^{\frac{3n+1}{2}} \prod_{p \equiv 1 \pmod{2}} \prod_{i=1}^{\frac{3n+2}{2}} \prod_{p \equiv 0 \pmod{2}} \left( \frac{2p - 1}{p^2 + p} \right)^{n(n+1)}
\]

\[
= \prod_{i=1}^{\frac{3n+1}{2}} \prod_{p \equiv 1 \pmod{2}} \prod_{i=1}^{\frac{3n+2}{2}} \prod_{p \equiv 0 \pmod{2}} \left( \frac{2p - 1}{p^2 + p} \right)^{n(n+1)/2}
\]

\[
= \prod_{i=1}^{\frac{3n+1}{2}} \prod_{p \equiv 1 \pmod{2}} \prod_{i=1}^{\frac{3n+2}{2}} \prod_{p \equiv 0 \pmod{2}} \left( \frac{2p - 1}{p^2 + p} \right)^{16i}
\]

\[
= \prod_{i=1}^{\frac{3n+1}{2}} \prod_{p \equiv 1 \pmod{2}} \prod_{i=1}^{\frac{3n+2}{2}} \prod_{p \equiv 0 \pmod{2}} \left( \frac{2p - 1}{p^2 + p} \right)^{3i-4}.
\]

From what we have discussed in above three cases, we deduce the desired result. \( \square \)

III. Conclusion

In this paper, we study the degree-based indices multiplication ABC of several common appeared nanostructures. The exact computational formulas are presented by means of edge set dividing approach. These theoretical results in nanoscience, biology, pharmacy and other fields have a wide application prospect.

Conflict of Interests

The authors hereby declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors thank the reviewers for their constructive comments in improving the quality of this paper.

References


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