Analytical Solution of the Time-fractional Order Black-Scholes Model for Stock Option Valuation on No Dividend Yield Basis

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***Abstract*— In this work, we obtain analytical solutions of the time-fractional Black-Scholes equation for European call option via a proposed relatively new semi-analytic technique hereby referred to as Projected Differential Transform Method (PDTM). This algorithmic technique is a modified version of the classical Differential Transformation Method (DTM). We demonstrate the efficiency and accuracy of the proposed technique by solving some illustrative problems. The results are obtained with ease and less computational work. No linearization or perturbation is required unlike other contemporary techniques. Thus, our results show that the work of Edeki *et al.* [42] is a particular case of this present work. This proposed technique is being reported for the first time in literature for solving time-fractional Black-Scholes equation. It is therefore recommended as an alternative technique for solving linear and nonlinear equations resulting from time-fractional stochastic differential equations (TFSDEs) in financial mathematics, with particular attention to stock option valuation; and fractional equations in applied sciences.**

***Index Terms*— Analytical solution; Black-Scholes model; European option; fractional derivatives; PDTM; stock option; stochastic differential equations**

I. Introduction

PRICING and valuation of options remains a central part with great interest in financial mathematics as regards derivative markets, valuation and financial investment. This problem is both theoretical and practical in nature. As a response to this, Black and Scholes in 1973 derived one of the most famous, effective, and significant models for option pricing and assessment [1]. This model can be used for options of European–type or American–type. The Black–Scholes model equation is a second order partial differential equation (PDE) in parabolic form, governing the valuation of security derivatives.

Generally, models of finance are mainly expressed in terms of stochastic dynamical equations but in [2], it is found that these financial stochastic differential equation (SDE)-based models can also be expressed as linear evolutionary PDEs with variable (non-constant) coefficients based on some certain restrictions.

Thus, the PDE describing the Black–Scholes model for option valuation on a non-dividend paying underlying is:

$$\frac{\partial\Psi}{\partial\tau}+\frac{1}{2}S^2\delta^2\frac{\partial^2\Psi}{\partial S^2}+rS\frac{\partial\Psi}{\partial S}-r\Psi=0 \tag{1}$$

where $\Psi=\Psi(S,\tau)$ represents the worth of S (the contingent claim), based on the time parameter, τ $0\le\tau\le t,\ (S,\tau)\in\mathbb{R}^+\times(0,T),\ \Psi\in C^{2,1}\left[\mathbb{R}\times[0,T]\right]$, for $p_\Psi(S,t)$ a payoff function, with expiration price, E_* such that for European Call Option (ECO) and European Put Option (EPO):

$$p_\Psi(S,t)=\begin{cases}\max(-E_*+S,0), \text{ for ECO}\\ \max(-S+E_*,0), \text{ for EPO}\end{cases} \tag{2}$$

where $\max(0,\ S)$ denotes the maximum value between 0 and S.

All the basic assumptions and shortcomings connected with the Black-Scholes model, and derivation of (1) with details can be found in [3-4] and other SDE standard materials.

In what follows, we will consider a generalization of (1) with regard to fractional order, $\alpha\in\mathbb{R}$ *or* $\mathbb{C}$, not necessarily an integer, to be referred to as time-fractional Black-Scholes model (TFBSM) of the form:

$$\frac{\partial^\alpha\Psi}{\partial\tau^\alpha}+m_1(S,\delta)\frac{\partial^2\Psi}{\partial S^2}+m_2(S,r)\frac{\partial\Psi}{\partial S}=r\Psi\ ,\ \alpha>0 \tag{3}$$

subject to a set of corresponding initial or boundary conditions, $m_i(\cdot,\cdot)$, $i\in\mathbb{N}$, are non-zero functions.

In an attempt to obtain analytical or numerical solution of (1), many researchers have adopted and used various direct and semi-analytical methods.

In recent years, priority has been attributed to the study of fractional differential equations (FDEs) with their applications [5-8]. This is traceable to its wider and important applications in fields not limited to sciences, engineering, management and finance [9]. Fractional calculus appears to be a generalization of the classical calculus. The greatest advantage in using FDEs lies in their nonlocal property since Integer-Order Differential Operators

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(IODOs) are local operators while fractional-order differential operators are nonlocal; meaning that a system next state depends both on its present and all of its historical states [10-12].

It is observed that most FDEs do not have exact roots/solutions in analytical forms; and even if they do, corresponding direct methods seem not available or appear complex in applications. Hence; the involvement of analytical, numerical and semi-analytical methods for approximate and exact solutions.

Many authors and researchers have considered the existence, uniqueness and stability of solutions of IVP and BVP for FDEs (see [13, 14] and the references therein).

Ibis *et al.* in [15], implement a semi-analytical-numerical technique: fractional differential transformation method (FDTM) to fractional-orederd differential-algebracic equations (FDAEs) described in Caputo sense. Kocak and Yildirim introduce a new iterative method (NIM) for solving some nonlinear time-fractional PDEs (NTFPDEs) [16]. Elzaki in [17], proposes a modified version of differential transform method (MVDTM) and applies it to a nonlinear time-fractional biological population model. In [18], Rida *et al.* propose an algorithm of the HAM for solving some time-fractional systems of differential equations of chemical applications.

Hemeda in [19] develop NIM with suitable algorithm for PDEs of fractional order. Herzallah in [20] states some basic features of the Caputo, and the Riemann-Liouville fractional derivatives, and proves some properties of the fractional calculus already in literature to be incorrect, giving some counter examples. In [21], Dhaigude and Birajdar use space discrete ADM for the solution of system of fractional PDEs with initial conditions. Povstenko [22] considers the time-fractional advection diffusion equations based on some generalized boundary conditions. In [23], Yang and Hua propose the local fractional iteration transform method (LFITM) for solving FDEs of local fractional derivatives.

Song *et al.* in [24] study with comparison the difference between fractional VIM and the ADM. Khan *et al*. suggest a fractional form of VIM for fractional-order PDEs [25]. Jafari, Kadem and Baleanu apply the VIM to the fractional-order Brusselator system [26] while Momani and Odibat in [27] apply the ADM and the VIM to FPDEs in fluid mechanics.

In considering solutions of fractional type Black-Scholes equations (FTBSEs) in option pricing; Elbeleze, Kilcman and Taib in [28], apply the homotopy perturbation Sumudu transform (HPSTM) for analytical solution. Kumal *et al.* in [29], apply the homotopy perturbation method coupled with Laplace transform for analytical solution. Ghandehari and Ranjbar in [30] apply an extension of the decomposition method via expansion series. Kumar, Kumar, and Singh in [31] implement the HPM and HAM to the time-fractional Black-Scholes (TFBSE) with boundary conditions. In [32], Ahmed *et al*. employ fractional variation iterative method (FVIM) for analytical solutions of linear fractional Black-Scholes models. Hariharan in [33], employ the Laplace Legendre wavelet method for numerical solutions. Such iterative techniques have wider applications for solutions of nonlinear BVPs of fractional order [34], and models of integro-differential type encountered in finance and actuarial sciences [35].

Here, a modified form of the DTM known as (projected DTM) is implemented for the first time (to the best of our knowledge), for the solution of the well-known Black-Scholes model equation of fractional-order-type for valuation of options.

The remaining sections of this paper are structured as follows: section 2 is for the preliminaries, main notations and definitions with regard to fractional calculus; section 3 is for the basic features and theorems of the PDTM, and the analysis of the FPDTM; section 4 is for the application of the projected DTM to some examples of time-fractional-order-type Black–Scholes equations, while section 5 is on graphical presentation via figures for interpretation of results; and we give concluding remarks in section 6.

II. Fractional Calculus: Notations and Preliminaries

In this section, we present a brief introduction of fractional calculus with regard to its preliminaries, basic definitions and notations [36-38].

A. *A Note on Fractional Calculus: notations and definitions*

Here, we will give a brief introduction to fractional calculus with respect to some definitions, and theorems. In fractional calculus, the power of the differential operator is considered real or complex number. Hence, the following:

Definition 1: {The Gamma function sense of fractional derivative}

Let $D=\frac{d(\cdot)}{dx}$ and J be differential and integration operators respectively, such that, the gamma function of $f(x)$ is defined as:

$$\Gamma(n)=\int_0^\infty e^{-x}t^{n-1}dt,\ \mathrm{Re}(n)\in\mathbb{N} \tag{4}$$

with:

$$n!=\Gamma(n+1),\ \left(\Gamma\left(\frac{1}{2}\right)\right)^2=\pi\ . \tag{5}$$

Suppose $\ddot{f}(x)=x^k$ (a monomial, of degree k, not necessarily a fraction), then:

$$\begin{cases} D\ddot{f}(x)=\dfrac{d\ddot{f}(x)}{dx}=kx^{k-1}, \\ D^2\ddot{f}(x)=\dfrac{d^2\ddot{f}(x)}{dx^2}=k(k-1)x^{k-2}=\dfrac{k!}{(k-2)!}x^{k-2}. \end{cases} \tag{6}$$

In general,

$$\frac{d^m\ddot{f}(x)}{dx^m}=\frac{k!}{(k-m)!}x^{k-m}\ . \tag{7}$$

But in terms of gamma notation, (4) is expressed as:

$$D^{\alpha}\ddot{f}(x)=\frac{d^{\alpha}\ddot{f}(x)}{dx^{\alpha}}=\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}x^{k-\alpha}. \tag{8}$$

We referred to (8) as a fractional derivative of $\ddot{f}(x)$, of order α, if $\alpha\in\mathbb{R}$.

Definition 2: Suppose $\ddot{f}(x)$ is defined for $x>0$, then:

$$\left(J\ddot{f}\right)(x)=\int_{0}^{x}\ddot{f}(s)ds \tag{9}$$

and as such, an arbitrary extension of (9) (i.e. Cauchy formula for repeated integration) yields:

$$\left(J^{n}\ddot{f}\right)(x)=\frac{1}{(n-1)!}\int_{0}^{x}(x-s)^{n-1}\ddot{f}(s)ds. \tag{10}$$

Thus, the gamma sense of (10) is:

$$\left(J^{\alpha}\ddot{f}\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{0}^{x}(x-s)^{\alpha-1}\ddot{f}(s)ds,\ \alpha>0. \tag{11}$$

Equation (11) is an order α-Riemann-Liouville fractional integration.

Definition 3: Fractional derivative (Riemann-Liouville)

$$D^{\alpha}\ddot{f}(x)=\frac{d^{h}\left(J^{h-\alpha}\ddot{f}(x)\right)}{dx^{h}}. \tag{12}$$

Definition 4: Fractional derivative (Caputo)

$$\begin{cases}D^{\alpha}\ddot{f}(x)=\dfrac{J^{h-\alpha}\left(d^{h}\ddot{f}(x)\right)}{dx^{h}},\\ h-1<\alpha<h,\ h\in\mathbb{N}.\end{cases} \tag{13}$$

Note: In (12), Riemann-Liouville compute first, the fractional integral of the function and thereafter, an ordinary derivative of the obtained result but the reverse is the case in Caputo sense of fractional derivatives; this allows the inclusion of the traditional initial conditions (ICs) and the boundary conditions (BCs) in formulating the problem.

Remark: {See Lemma 4 in [24]}: The link between the Caputo fractional differential operator and the Riemann-Liouville operator for $(n-1<\alpha<n,\ n\in\mathbb{N})$ is:

$$\begin{aligned}\left(J^{\alpha}D_{t}^{\alpha}\right)\ddot{f}(t)&=\left(D_{t}^{-\alpha}D_{t}^{\alpha}\right)\ddot{f}(t)\\&=\ddot{f}(t)-\sum_{k=0}^{n-1}\ddot{f}^{k}(0)\frac{t^{k}}{k!}.\end{aligned} \tag{14}$$

As such,

$$\ddot{f}(t)=\left(J^{\alpha}D_{t}^{\alpha}\right)\ddot{f}(t)+\sum_{k=0}^{n-1}\ddot{f}^{k}(0)\frac{t^{k}}{k!}. \tag{15}$$

Definition 5: The Mittag-Leffler function of z, $E_{\alpha}(z)$ holding true in the whole complex plane is denoted by the series representation (see [39] and related references therein) as:

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(1+\alpha k)},\quad \alpha\geq 0,\ z\in\mathbb{C}. \tag{16}$$

Remark: For $\alpha=1$, $E_{\alpha}(z)$ in (16) becomes:

$$E_{\alpha=1}(z)=e^{z}. \tag{17}$$

III. The PDTM and the FPDTM

Here, we present the basic features and theorems of the PDTM, and analyze the FPTM [40-41].

A. *Fundamental Theorems of the Projected DTM*

Consider $z(x,\ t)$ as an analytic function in a domain D_{*}, at a point $(x_{*},\ t_{*})$, so for taking the Taylor series expansion of $z(x,t)$, preference is given to some terms, $s^{v}=t$ unlike the case of the classical DTM where the variables are all considered. Hence, the projected form, the differential transform (DT) of $z(x,\ t)$ in terms to t at t_{*} is defined as:

$$Z(x,\ \dot{h})=\frac{1}{\dot{h}!}\left[\frac{\partial^{\dot{h}}z(x,\ t)}{\partial t^{\dot{h}}}\right]_{t=t_{*}} \tag{18}$$

and as such:

$$z(x,\ \dot{h})=\sum_{\dot{h}=0}^{\infty}Z(x,\ \dot{h})(t-t_{*})^{\dot{h}} \tag{19}$$

where (19) is called inverse projected differential (IPD) transform of $Z(x,\ \dot{h})$ with respect to t.

B. *Some Basic Properties and theorems of the PDTM*

Theorem 1: If $z(x,\ t)=\alpha z_{a}(x,\ t)+\beta z_{b}(x,\ t)$, then $Z(x,\ \dot{h})=\alpha Z_{a}(x,\ \dot{h})+\beta Z_{b}(x,\ \dot{h})$

Theorem 2: If $z(x,\ t)=\alpha\dfrac{\partial^{n}z_{*}(x,\ t)}{\partial t^{n}}$, then

$$\dot{h}!z(x,\ \dot{h})=\alpha(\dot{h}+n)!Z_{*}(x,\ \dot{h}+n).$$

Theorem 3: If $z(x,\ t)=\alpha\dfrac{\partial Z_{*}(x,\ t)}{\partial t}$, then

$$\dot{h}!Z(x,\ \dot{h})=\alpha(\dot{h}+1)!Z_{*}(x,\ \dot{h}+1).$$

Theorem 4: If $z(x,\ t)=q(x)\dfrac{\partial^{n}z_{*}(x,\ t)}{\partial x^{n}}$, then

$$Z(x,\ \dot{h})=q(x)\frac{\partial^{n}Z_{*}(x,\ \dot{h})}{\partial x^{n}}.$$

Theorem 5: If $z(x,\ t)=q(x)z_{*}^{2}(x,\ t)$, then

$$Z(x,\ \dot{h})=q(x)\sum_{\upsilon=0}^{\dot{h}}Z_{*}(x,\ \upsilon)Z_{*}(x,\ \dot{h}-\upsilon).$$

Theorem 6: If $z(x,\ y)=x^{r}y^{r*}$ then

$$Z(k,\ \dot{h})=\delta(k-\upsilon,\ \dot{h}-\upsilon*)$$

$$= \delta(k-\upsilon)\delta(\dot{h}-\upsilon *),$$

where

$$\delta(k-\upsilon) = \begin{cases} 1 \text{ , for } \upsilon = k \\ 0 \text{ , for } \upsilon \neq k \end{cases}$$

$$\delta(k-\upsilon *) = \begin{cases} 1 \text{ , for } \upsilon * = k \\ 0 \text{ , for } \upsilon * \neq k. \end{cases}$$

Theorem 7: **(**PDTM of a fractional derivative**)**
If $\ddot{f}(x,\ t) = D_t^{\alpha} u(x,\ t)$, then,

$$F(x,\ k) = \frac{\Gamma\left(1+\alpha+\frac{k}{q}\right)}{\Gamma\left(1+\frac{k}{q}\right)} U(x,\ k+\alpha q).$$

Consequently, we have:

$$\Gamma\left(1+\alpha+\frac{k}{q}\right) U(x,\ k+\alpha q) = \Gamma\left(1+\frac{k}{q}\right) F(x,\ k). \tag{20}$$

Setting $\alpha q = 1$ in (20) gives:

$$\Rightarrow U(x,\ k+1) = \frac{\Gamma(1+\alpha k)}{\Gamma(1+\alpha(1+k))} F(x,\ k). \tag{21}$$

As such, for $u(x,\ t)$ α - analytic at $x_0 = 0$.

$$u(x,\ t) = \sum_{h=0}^{\infty} U(x,\ h) t^{\frac{h}{q}} = \sum_{h=0}^{\infty} U(x,\ h) t^{\alpha h}. \tag{22}$$

C. Analysis of a Fractional DTM

Consider the general non-linear fractional differential equation (NLFDE):

$$\begin{cases} D_t^{\alpha} u(x,\ t) + L_{[x]} u(x,\ t) + N_{[x]} u(x,\ t) = q(x,\ t), \\ u(x,\ 0) = g(x), t > 0. \end{cases} \tag{23}$$

where $D_t^{\alpha} = \dfrac{\partial^{\alpha}}{\partial t^{\alpha}}$ is the fractional Caputo derivative of $u(x,\ t)$; whose projected differential transform is $U(x,h)$, $L_{[\cdot]}$ and $N_{[\cdot]}$ are linear and nonlinear differential operators with respect to x respectively, while $q = q(x,\ t)$ is the source term.

We rewrite (23) as:

$$\begin{cases} D_t^{\alpha} u(x,\ t) = -L_{[x]} u(x,\ t) - N_{[x]} u(x,\ t) + q(x,\ t), \\ u(x,\ 0) = g(x),\ n-1 < \alpha < n\ ,\ n \in \mathbb{N}. \end{cases} \tag{24}$$

Applying the inverse fractional Caputo derivative, $D_t^{-\alpha}$ to both sides of (24) and with regard to (14) gives:

$$\begin{cases} u(x,\ t) = g(x) + \Theta^*, \\ u(x,\ 0) = g(x), \end{cases} \tag{25}$$

$$\Theta^* = D_t^{-\alpha}\left\{-L_{[x]} u(x,t) - N_{[x]} u(x,t) + q(x,t)\right\}.$$

Thus, expanding $u(x,\ t)$ in terms of fractional power series form, the IPDT of $U(x,h)$ for $u(x,0) = g(x)$ is given as follows:

$$u(x,\ t) = \sum_{h=0}^{\infty} U(x,\ h) t^{\alpha h}$$

$$= u(x,\ 0) + \sum_{h=1}^{\infty} U(x,\ h) t^{\alpha h}. \tag{26}$$

IV. Illustrative examples and Applications

Here, three examples of time-fractional Black-Scholes model equations will be solved with the algorithmic technique as proposed.

Problem 1: Consider the time-fractional Black-Scholes equation (FBSE):

$$\frac{\partial^{\alpha} w}{\partial t^{\alpha}} + x^2 \frac{\partial^2 w}{\partial x^2} + 0.5x\frac{\partial w}{\partial x} - w = 0\ ,\ 0 < \alpha \leq 1 \tag{27}$$

subject to:

$$w(x,0) = \max(x^3, 0) = \begin{cases} x^3, \text{for } x > 0 \\ 0, \text{ for } x \leq 0. \end{cases} \tag{28}$$

Procedure w.r.t Problem 1:

Here, we will consider $x > 0$ and use the following representations for simplicity:

$$\frac{\partial^{\alpha} w}{\partial t^{\alpha}} = w_{x,t}^{\alpha},\ \frac{\partial^2 W(x,h)}{\partial x^2} = W''_{x,h}\ ,$$

$$\frac{\partial W(x,h)}{\partial x^2} = W'_{x,h},\ \text{and } W(x,h+1) = W_{x,h+1}\ . \tag{29}$$

So taking the PDTM of (27) and (28) gives:

$$W_{x,1+h} = \frac{\Gamma(1+\alpha h)}{\Gamma(1+\alpha(1+h))}\left[-x^2 W''_{x,h} - 0.5W'_{x,h} + W_{x,h}\right] \tag{30}$$

subject to:

$$W_{x,0} = x^3. \tag{31}$$

$$\Rightarrow\ W'_{x,0} = 3x^2 \text{ and } W''_{x,0} = 6x, \tag{32}$$

so, when $h = 0$,

$$W_{x,1} = \frac{\Gamma(1)}{\Gamma(1+\alpha)}\left[-x^2 W''_{x,0} - 0.5W'_{x,0} + W_{x,0}\right]$$

$$W_{x,1} = \frac{\Gamma(1)}{\Gamma(1+\alpha)}\left[-6x^3 - 1.5x^3 + x^3\right]$$

$$= \frac{\Gamma(1)}{\Gamma(1+\alpha)}\left[-6.5x^3\right] \tag{33}$$

As such:

$$W'_{x,1} = \frac{-19.5x^2}{\Gamma(1+\alpha)},\ W''_{x,1} = \frac{-39x}{\Gamma(1+\alpha)}. \quad (34)$$

so for $h=1$,

$$W_{x,2} = \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[-x^2W''_{x,1} - 0.5W'_{x,1} + W_{x,1}\right]$$

$$= \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{39x^3 + 9.75x^3 - 6.5x^3}{\Gamma(1+\alpha)}\right)$$

$$= \frac{(6.5)^2 x^3}{\Gamma(1+2\alpha)}. \quad (35)$$

$$\Rightarrow\ W'_{x,2} = \frac{126.75x^2}{\Gamma(1+2\alpha)} \text{ and } W''_{x,2} = \frac{253.5x}{\Gamma(1+2\alpha)}$$

so when $h=2$,

$$W_{x,3} = \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left[-x^2W''_{x,2} - 0.5W'_{x,2} + W_{x,2}\right]$$

$$= \frac{(-6.5)^3 x^3}{\Gamma(1+3\alpha)}. \quad (36)$$

Similarly, we arrive at the recurrence relation below:

$$W_{x,p} = \frac{(-6.5)^p x^3}{\Gamma(1+p\alpha)}\ , p \in \mathbb{Z}^+. \quad (37)$$

Hence,

$$w_{x,t} = \sum_{\eta=0}^{\infty} W_{x,\eta} t^{\alpha\eta},\ \ \eta q = 1$$

$$= W_{x,0} + W_{x,1}t^{\alpha} + W_{x,2}t^{2\alpha} + W_{x,3}t^{3\alpha} + \cdots$$

$$= x^3 - \frac{(6.5t^{\alpha})x^3}{\Gamma(1+\alpha)} + \frac{(6.5)^2 x^3 t^{2\alpha}}{\Gamma(1+2\alpha)} + $$

$$- \frac{(6.5)^3 x^3 t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots$$

$$= x^3 \sum_{n=0}^{\infty} \frac{\left(-6.5t^{\alpha}\right)^n}{\Gamma(1+n\alpha)}$$

$$= x^3 E_{\alpha}\left(-6.5t^{\alpha}\right). \quad (38)$$

We therefore remark that $w_{x,t} = x^3 e^{-6.5t}$ is the exact root/solution of *problem 1* when $\alpha = 1$ (a special case).

Problem 2: Consider the (FBSE) ([33, 42] for $\alpha = 1$):

$$\frac{\partial^{\alpha} w}{\partial t^{\alpha}} = \frac{\partial^2 w}{\partial x^2} + (k-1)\frac{\partial w}{\partial x} - kw, \quad 0 < \alpha \le 1 \quad (39)$$

for the initial condition:

$$w(x,0) = \max\left(e^x - 1,\ 0\right) \quad (40)$$

where $w = w(x,t)$.

Problem 2: Solution Procedure:

Taking the PDTM of (39) and (40) gives:

$$\begin{cases} W_{x,\eta+1} = \dfrac{\Gamma(1+\alpha\eta)}{\Gamma(1+\alpha(1+\eta))}\left[\underset{**}{A}\right] \\ \underset{**}{A} = \dfrac{\partial^2 W_{x,\eta}}{\partial x^2} + (k-1)\dfrac{\partial W_{x,\eta}}{\partial x} - kW_{x,\eta}\ . \end{cases} \quad (41)$$

$$W_{x,0} = \max\left(e^x - 1, 0\right) \quad (42)$$

$$\Rightarrow \frac{\partial W_{x,0}}{\partial x} = \frac{\partial^2 W_{x,0}}{\partial x^2} = \max\left(e^x, 0\right). \quad (43)$$

Thus, when $\eta = 0$, we have:

$$W_{x,1} = \frac{\Gamma(1)}{\Gamma(1+\alpha)}\left[\frac{\partial^2 W_{x,0}}{\partial x^2} + (k-1)\frac{\partial W_{x,0}}{\partial x} - kW_{x,0}\right]$$

$$\begin{cases} = \dfrac{1}{\Gamma(1+\alpha)}\left((k-1)\max\left(e^x,\ 0\right) + H^x\right) \\ H^x = \max\left(e^x,\ 0\right) - k\max\left(e^x - 1,\ 0\right) \end{cases}$$

$$= \frac{1}{\Gamma(1+\alpha)}\left(k\left\{\max\left(e^x, 0\right) - \max\left(e^x - 1,\ 0\right)\right\}\right) \quad (44)$$

so, $$\frac{\partial W_{x,1}}{\partial x} = \frac{\partial^2 W_{x,1}}{\partial x^2} = 0, \quad (45)$$

$\therefore$ when $\eta = 1$,

$$W_{x,2} = \frac{-kW_{x,1}\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{\left(-k^2H^x\right)}{\Gamma(1+\alpha)}$$

$$= \frac{-k^2}{\Gamma(1+2\alpha)}\left(\max\left(e^x, 0\right) - \max\left(e^x - 1, 0\right)\right), \quad (46)$$

so,

$$\frac{\partial W_{x,2}}{\partial x} = 0 = \frac{\partial^2 W_{x,2}}{\partial x^2}. \quad (47)$$

When $\eta = 2$,

$$W_{x,3} = \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left[-kW_{x,2}\right]$$

$$= \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left[\frac{k^3 H^x}{\Gamma(1+2\alpha)}\right]$$

$$= \frac{k^3}{\Gamma(1+3\alpha)}\left(\max\left(e^x, 0\right) - \max\left(e^x - 1, 0\right)\right) \quad (48)$$

such that

$$\frac{\partial W_{x,3}}{\partial x} = 0 = \frac{\partial^2 W_{x,3}}{\partial x^2}. \quad (49)$$

When $\eta = 3$,

$$W_{x,4}=\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\left[-kW_{x,3}\right]$$

$$=\frac{-k^4}{\Gamma(1+4\alpha)}\left(\max\left(e^x,0\right)-\max\left(e^x-1,0\right)\right). \quad (50)$$

Suppose we set

$$M^*=\left(\max\left(e^x,0\right)-\max\left(e^x-1,0\right)\right). \quad (51)$$

Then,

$$\begin{aligned} w_{x,t}&=\sum_{\eta=0}^{\infty}W_{x,h}t^{\alpha\eta},\ \eta q=1\\ &=W_{x,0}+W_{x,1}t^{\alpha}+W_{x,2}t^{2\alpha}+W_{x,3}t^{3\alpha}+\cdots\\ &=\max\left(e^x-1,\ 0\right)+M^*\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(kt^{\alpha})^n}{\Gamma(1+n\alpha)}\\ &=\max\left(e^x-1,0\right)-M^*\sum_{n=1}^{\infty}\frac{(-kt^{\alpha})^n}{\Gamma(1+n\alpha)} \end{aligned}$$

$$\begin{cases}=\max\left(e^x-1,\ 0\right)+\\ -\left(\max\left(e^x,\ 0\right)-\max\left(e^x-1,\ 0\right)\right)D^x,\\ \text{for } D^x=\left(-1+E_{\alpha}(-kt^{\alpha})\right)\end{cases}$$

$$\begin{cases}=\max\left(e^x-1,0\right)E_{\alpha}(-kt^{\alpha})\\ \quad+\max\left(e^x,0\right)\left(1-E_{\alpha}(-kt^{\alpha})\right),\end{cases} \quad (52)$$

where $E_{\alpha}(-kt^{\alpha})$ denotes a one parameter Mittag-Leffler function.

We remark that a special case of problem 2 at $\alpha=1$ has an exact solution:

$$w_{x,t}=\max\left(e^x-1,\ 0\right)$$

$$+\left\{\max\left(e^x,\ 0\right)-\max\left(e^x-1,\ 0\right)\right\}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{(kt)^m}{m!}$$

$$=\max\left(e^x-1,\ 0\right)e^{-kt}+\max\left(e^x,\ 0\right)\left(1-e^{-kt}\right). \quad (53)$$

Problem 3: Consider the FBSE {Ex 2, [31, 42], for $\alpha=1$}:

$$\frac{\partial^{\alpha}w}{\partial t^{\alpha}}+0.08\left(2+Sinx\right)^2x^2\frac{\partial^2 w}{\partial x^2}+0.06x\frac{\partial w}{\partial x}=0.06w \quad (54)$$

for the initial condition:

$$w\left(x,\ 0\right)=\max\left(x-25e^{-0.06},\ 0\right). \quad (55)$$

Problem 3: Solution Procedure:

Taking the projected DTM of (54) and (55) gives:

$$W_{x,\eta+1}=\frac{\Gamma(1+\alpha\eta)}{\Gamma(1+\alpha(1+\eta))}\times$$

$$\left(-\frac{2}{25}\left(2+Sinx\right)^2x^2\frac{\partial^2W_{x,h}}{\partial x^2}-\frac{3x}{50}\frac{\partial W_{x,h}}{\partial x}+\frac{3W_{x,h}}{50}\right) \quad (56)$$

for the initial condition:

$$W_{x,0}=\max\left(x-25e^{-0.06},\ 0\right). \quad (57)$$

$$\Rightarrow\ \frac{\partial W_{x,0}}{\partial x}=1,\ \text{ and }\ \frac{\partial^2W_{x,0}}{\partial x^2}=0. \quad (58)$$

So, when $\eta=0$,

$$W_{x,1}=\frac{0.06\Gamma(1)}{\Gamma(1+\alpha)}\left[-x+\max\left(x-25e^{-0.06},0\right)\right]$$

$$=\frac{-0.06}{\Gamma(1+\alpha)}\left[x-\max\left(x-25\exp\left(-0.06\right),\ 0\right)\right]. \quad (59)$$

As such:

$$\frac{\partial W_{x,1}}{\partial x}=0=\frac{\partial^2W_{x,1}}{\partial x^2}. \quad (60)$$

So for $\eta=1$,

$$W_{x,2}=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[0.06W_{x,1}\right]$$

$$=\frac{-(0.06)^2}{\Gamma(1+2\alpha)}\left(x-\max\left(x-25\exp\left(-0.06\right),\ 0\right)\right) \quad (61)$$

and,

$$\frac{\partial W_{x,2}}{\partial x}=0=\frac{\partial^2W_{x,2}}{\partial x^2}. \quad (62)$$

So when $\eta=2$,

$$W_{x,3}=\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left[0.06W_{x,3}\right]$$

$$=-\frac{(0.06)^3}{\Gamma(1+3\alpha)}\left(x-\max\left(x-25\exp\left(-0.06\right),0\right)\right). \quad (63)$$

Similarly, for $p\in\mathbb{N}$,

$$W_{x,p}=-\frac{(0.06)^p}{\Gamma(1+3p\alpha)}\left(N^*\right) \quad (64)$$

where

$$N^*=\left(x-\max(x-25\exp\left(-0.06\right),\ 0\right). \quad (65)$$

Hence,

$$w_{x,t}=\sum_{\eta=0}^{\infty}W_{x,h}t^{\alpha\eta},\ \eta q=1$$

$$w_{x,t}=\max\left(x-25\exp\left(-0.06\right),\ 0\right)+$$

$$-N^*\left\{\frac{(0.06t^{\alpha})}{\Gamma(1+\alpha)}+\frac{(0.06t^{\alpha})^2}{\Gamma(1+2\alpha)}+\frac{(0.06t^{\alpha})^3}{\Gamma(1+3\alpha)}+\cdots\right\}$$

$$= \max\left(x - 25e^{(-0.06)}, 0\right) - N^{*}\sum_{n=1}^{\infty}\frac{(0.06t^{\alpha})^{n}}{\Gamma(1+n\alpha)}. \qquad (66)$$

Thus, simplifying (66) gives:

$$\begin{cases} w_{x,t} = x\left(1 - E_{\alpha}\left(0.06t^{\alpha}\right)\right) \\ + \max\left(x - 25\exp(-0.06),\ 0\right)E_{\alpha}\left(0.06t^{\alpha}\right). \end{cases} \qquad (67)$$

We remark that when $\alpha = 1$, *Problem 3* has a special case whose exact solution is:

$$\begin{aligned} w_{x,t} &= \max\left(x - 25\exp(-0.06), 0\right) \\ &\quad + \left(1 - \exp(0.06t)\right)\times \\ &\quad \left(x - \max\left(x - 25\exp(-0.06),\ 0\right)\right) \\ &= x\left(1 - \exp(0.06t)\right) \\ &\quad + \max\left(x - 25\exp(-0.06),\ 0\right)\exp(0.06t). \end{aligned} \qquad (68)$$

V. Results and Discussion

Here, the graphical views of solutions are presented (see Fig. 1-9). Fig.1 and Fig. 2 are for problem 1, Fig. 3 and Fig. 4 are for problem 2, while Fig. 5 and Fig. 6 are for problem 3. For each case, same interval is used for the x-parameter while different intervals are used for the time-parameter. In Table I, we present the error analysis for different solution methods.

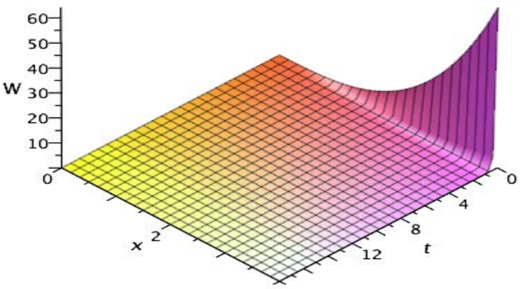

Fig. 1. $w:\ t \in [o,18]\ \&\ x \in [o,4]$

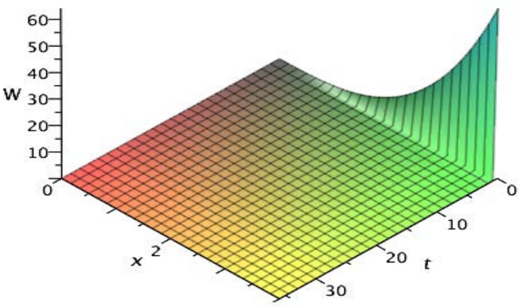

Fig. 2. $w:\ t \in [o,36]\ \&\ x \in [o,4]$

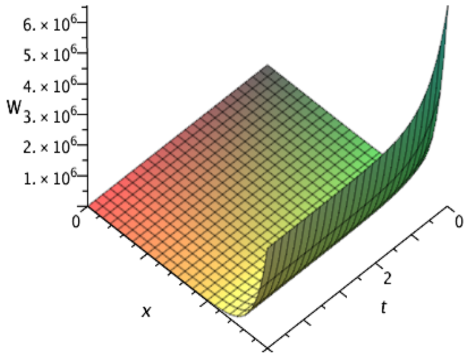

Fig. 3. $w:\ t \in [o,5]\ \&\ x \in [o,15]$

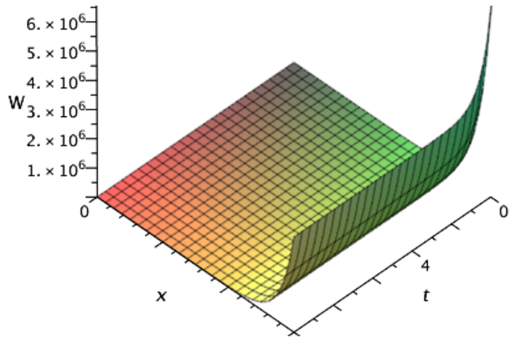

Fig. 4. $w:\ t \in [o,10]\ \&\ x \in [o,15]$

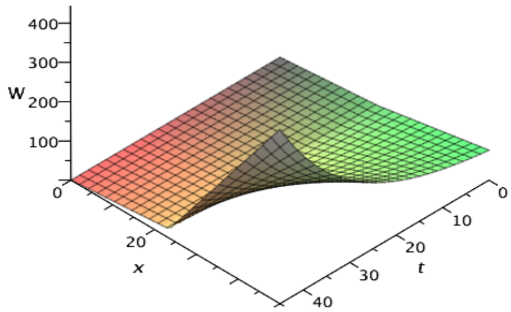

Fig. 5. $w:\ t \in [o,45]\ \&\ x \in [o,50]$

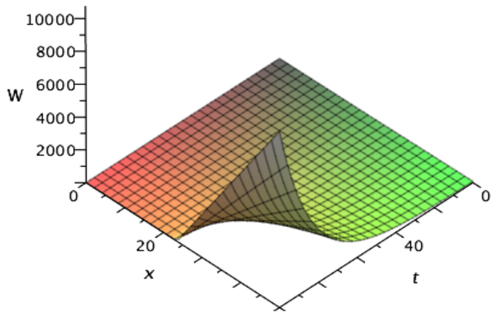

Fig. 6. $w:\ t\in[o,100]\ \&\ x\in[o,50]$

Table I: Solutions analysis of *problem 1* when $\alpha=1$

x	t	$w(\text{ADM})$	$w(\text{HPM})$	$w(\text{PDTM})$
0.1	0.13	9.2646×10^{-3}	7.361×10^{-3}	4.454×10^{-3}
0.2	0.18	9.3652×10^{-3}	7.445×10^{-3}	4.548×10^{-3}
0.3	0.27	9.4713×10^{-3}	7.523×10^{-3}	4.463×10^{-3}
0.4	0.32	9.4747×10^{-3}	7.567×10^{-3}	4.482×10^{-3}
0.5	0.38	9.6129×10^{-3}	7.693×10^{-3}	4.746×10^{-3}

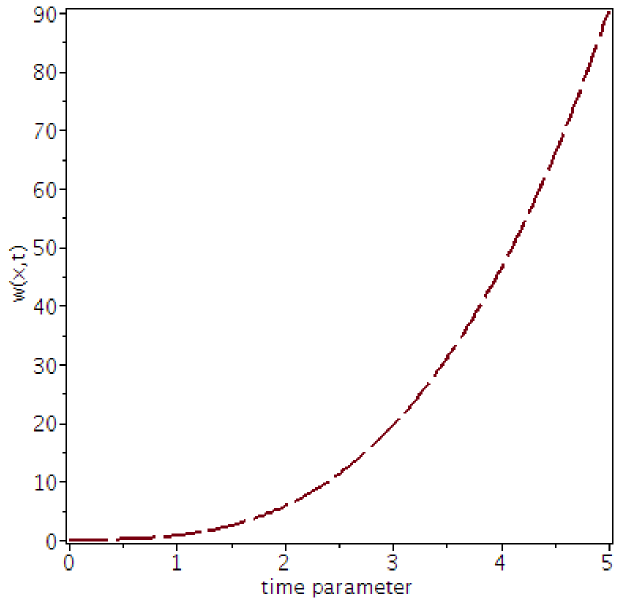

Fig. 7. Exact solutions of *problem 1* when $\alpha=1$

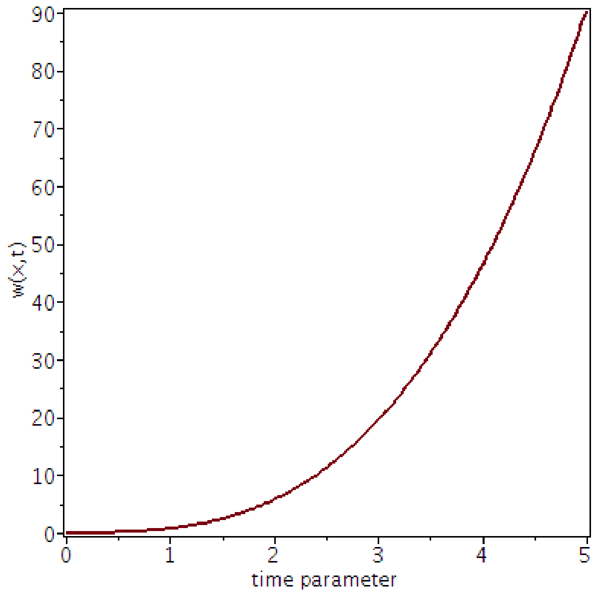

Fig. 8. PDTM solutions of *problem 1* when $\alpha=1$

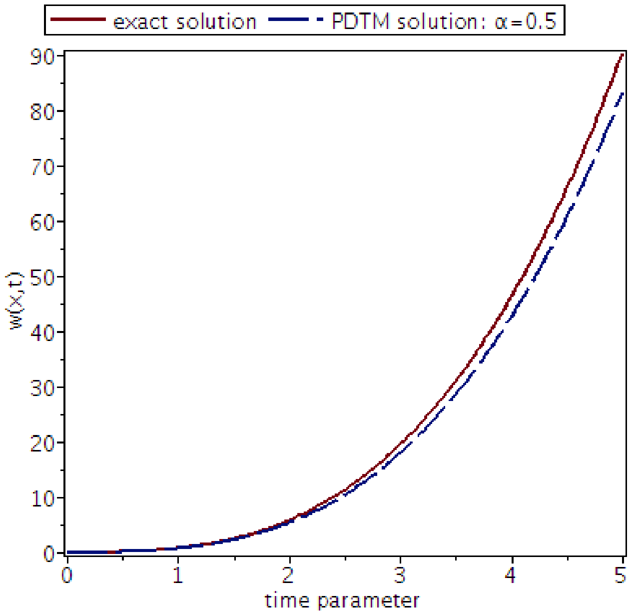

Fig. 9. Solutions of *problem 1* when $\alpha=0.5$

In Table I above, we present the analysis of the solutions by comparing those obtained via the proposed method with those obtained using other semi-analytical methods: the ADM, and the HPM. The comparison is in terms of the errors as related to the exact solution of *problem 1* when $\alpha=1$. We denote by $w(\text{ADM})$ [43], $w(\text{HPM})$ and $w(\text{PDTM})$, the solutions via ADM, HPM [44] and PDTM respectively.

Emphasizing on the robustness and efficiency of the PDTM with regard to the link between integer-type and fractional-type differential equations, we considered problem 1 in terms of integer order $(\alpha=1)$ and time-fractional order $(\alpha=0.5)$. The results are graphically displayed in Fig. 7 Fig. 8, and Fig. 9 respectively. Conclusion is drawn in the following section.

VI. Concluding Remarks

In this work, we have successfully applied a proposed relatively new computational algorithm referred to as projected DTM (PDTM) to the time-fractional Black-Scholes equations in terms of European call option valuation for analytical solutions. The same approach can be employed for the valuation of put options of European type. For efficiency of the proposed method, some examples were solved numerically. The convergence rate of the obtained results is very fast when compared with their exact form of solutions (even without ignoring the accuracy), and they are in strong agreement with those already in literature. Though, the technique is advantageous, reliable and efficient since less computational work is required, no linearization or perturbation is needed. This proposed technique has not been reported in literature for solving time-fractional Black-Scholes equations. This therefore, shows that the work of Edeki *et al.* [42] is a special case of this present work for $\alpha = 1$. Therefore, we note here that the time-fractional B-S model equation for option valuation is a generalization of the classical Black-Scholes model equation in its equivalent form.

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