Optimizations of Convex and Generalized Convex Fuzzy Mappings in The Quotient Space of Fuzzy Numbers

Wei Zhang, Dong Qiu and Meilin Dong

Abstract—**In this paper we propose the solution concepts for the fuzzy optimization problems in the quotient space of fuzzy numbers and the concepts of convexity, quasiconvexity and pseudoconvexity for fuzzy mappings. Optimizations of convex and generalized convex fuzzy mappings are derived.**

Index Terms—**fuzzy optimization problems, quasiconvex fuzzy mappings, non-dominated solutions, quotient spaces**

I. Introduction

THE uncertainty includes randomness and fuzziness in the real world. Therefore, imposing the uncertainty upon the conventional optimization problems becomes an interesting research topic. The fuzzy set theory was introduced initially in 1965 by Zadeh [29] with a view to reconcile mathematical modeling and human knowledge in the engineering science. In 1992, Nanda and Kar [11] introduced the concept of convexity for fuzzy mappings and proved that a fuzzy mapping is convex if and only if its epigraph is a convex set. Yan and Xu [28] proposed the concepts of epigraph and convexity of the fuzzy mappings and described characteristics of the convex fuzzy mappings and quasi-convex fuzzy mappings by considering the concept of ordering due to Goetschel and Voxman [3]. In addition they discussed the properties of convex fuzzy optimizations. In [21], Syau introduced the concepts of pseudo-convexity and pseudo-invexity for fuzzy mappings of one variable and investigated the relationships among them by using notion of differentiability and the results proposed by Goetschel and Voxman [3]. In [22], Syau defined a differentiable fuzzy mappings of several variables in ways that parallel the definition, proposed by Goetschel and Voxman [3], for a fuzzy mapping of one variable. Wang and Wu [24] proposed the concepts of directional derivative, differential and subdifferential of fuzzy mappings from $\mathbb{R}^n$ into the set of fuzzy numbers and discussed the characterizations of directional derivative and differential.

In [14], Qiu $et\ al.$ intuitively showed a method of finding the inverse operation in the quotient space of fuzzy numbers based on the Mareš equivalence relation [9], [10], which have the desired group properties for the addition operation [7], [13], [27]. As an application of the main results, it is shown that if we identify every fuzzy number with the corresponding equivalence class, there would be more differentiable fuzzy functions than what is found in the literature. In [17], Qiu $et\ al.$ studied the fuzzy differential equations in the quotient space of fuzzy numbers. They dealt with the convergence of successive approximations of the initial value problem of the fuzzy differential equations. In [15], [18] Qiu $et\ al.$ further investigated the differentiability properties of such functions in the quotient space of fuzzy numbers. In this paper, optimizations of convex and generalized convex fuzzy mappings are derived in the quotient space of fuzzy numbers.

Manuscript received May 19, 2017; revised Aug 16, 2017. This work was supported by The National Natural Science Foundations of China (Grant no. 11671001,61472056).

D. Qiu is with College of Mathematics and Physics, Chongqing University of Posts and Telecommunications, Nanan, Chongqing, 400065, P.R. China e-mail: dongqiumath@163.com; qiudong@cqupt.edu.cn

W. Zhang and M. Dong are with the same university as D. Qiu.

II. Preliminaries

We start this section by recalling some pertinent concepts and key lemmas from the function of bounded variation, fuzzy numbers and fuzzy number equivalence classes which will be used later.

Definition 2.1: [8] Let $f : [a, b] \to \mathbb{R}$ be a function. f is said to be of bounded variation if there exists a $C > 0$ such that

$$\sum_{i=1}^{n} |f(x_{i-1}) - f(x_i)| \leq C$$

for every partition $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ on $[a, b]$. The set of all functions of bounded variation on $[a, b]$ is denoted by $BV[a, b]$.

Definition 2.2: [8] Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation. The total variation of f on $[a, b]$, denoted by $V_a^b(f)$, is defined by

$$V_a^b(f) = \sup_p \sum_{i=1}^{n} |f(x_{i-1}) - f(x_i)|,$$

where p represents all partitions of $[a, b]$.

Lemma 2.1: [8] Let $f, g \in BV[a, b]$, then we have

(1) $cf + dg \in BV[a, b]$ and

$$V_a^b(cf + dg) \leq |c|\, V_a^b(f) + |d|\, V_a^b(g)$$

for any contents $c, d \in \mathbb{R}$.

(2) $f \cdot g \in BV[a, b]$ and

$$V_a^b(f \cdot g) \leq V_a^b(f) \sup_{x\in[a,b]} |g(x)| + V_a^b(g) \sup_{x\in[a,b]} |f(x)|.$$

Lemma 2.2: [8] Every monotonic function $f : [a, b] \to \mathbb{R}$ is of bounded variation and

$$V_a^b(f) = |f(a) - f(b)|.$$

A fuzzy set $\widetilde{x}$ in $\mathbb{R}$ is characterized by a membership function $\mu_{\widetilde{x}} : \mathbb{R} \to [0, 1]$. The α-level set of $\widetilde{x}$ is denoted by $[\widetilde{x}]^\alpha = \{x \in \mathbb{R} : \mu_{\widetilde{x}}(x) \geq \alpha\}$ for each $\alpha \in (0,1]$. The 0-level set $[\widetilde{x}]^0$ is defined as the closure of the set $\{x \in \mathbb{R} : \mu_{\widetilde{x}}(x) > 0\}$, i.e., $[\widetilde{x}]^0 = cl(\{x \in \mathbb{R} : \mu_{\widetilde{x}}(x) > 0\})$. A fuzzy set $\widetilde{x}$ is said to be a fuzzy number if it satisfies the following conditions:

1. $\widetilde{x}$ is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $\mu_{\widetilde{x}}(x_0) = 1$;
2. $\widetilde{x}$ is convex, i.e., $\mu_{\widetilde{x}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\widetilde{x}}(x_1), \mu_{\widetilde{x}}(x_2)\}$ for all $x_1, x_2 \in \mathbb{R}$ and $\lambda \in (0,1)$;
3. $\widetilde{x}$ is upper semicontinuous, i.e., the α-level set $[\widetilde{x}]^\alpha$ is a closed subset of $\mathbb{R}$ for all $\alpha \in [0,1]$;
4. The 0-level set $[\widetilde{x}]^0$ is a compact subset of $\mathbb{R}$.

Let $\mathscr{F}$ be the set of all fuzzy numbers on $\mathbb{R}$. Then for any $\widetilde{x} \in \mathscr{F}$, it is well known that the α-level set $[\widetilde{x}]^\alpha = [\widetilde{x}_L(\alpha), \widetilde{x}_R(\alpha)]$ is a non-empty bounded closed interval in $\mathbb{R}$ for all $\alpha \in [0,1]$, where $\widetilde{x}_L(\alpha)$ denotes the left-hand end point of $[\widetilde{x}]^\alpha$ and the $\widetilde{x}_R(\alpha)$ denotes the right one. For any $\widetilde{x}, \widetilde{y} \in \mathscr{F}$ and $\lambda \in \mathbb{R}$, owing to Zadeh's extension principle, the addition and scalar multiplication can be respectively defined for any $x \in \mathbb{R}$ by

$$\mu_{\widetilde{x}+\widetilde{y}}(x) = \sup_{x_1,x_2:x_1+x_2=x} \min\{\mu_{\widetilde{x}}(x_1), \mu_{\widetilde{y}}(x_2)\}$$

and

$$\mu_{\lambda\times\widetilde{x}}(x) = \mu_{\lambda\widetilde{x}}(x) = \begin{cases} \mu_{\widetilde{x}}(\frac{x}{\lambda}), & \lambda \neq 0, \\ 0, & \lambda = 0. \end{cases}$$

We say that a fuzzy number $\widetilde{s} \in \mathscr{F}$ is symmetric [9], if $\widetilde{s} = -\widetilde{s}$, i.e., $\mu_{\widetilde{s}}(x) = \mu_{-\widetilde{s}}(x) = \mu_{\widetilde{s}}(-x)$ for all $x \in \mathbb{R}$. We denote the set of all symmetric fuzzy numbers by $\mathscr{S}$.

Definition 2.3: [14] Let $\widetilde{x} \in \mathscr{F}$, we define a function $\widetilde{x}_M : [0,1] \to \mathbb{R}$ by assigning the midpoint of each α-level set to $\widetilde{x}_M(\alpha)$ for all $\alpha \in [0,1]$, i.e.,

$$\widetilde{x}_M(\alpha) = \frac{\widetilde{x}_L(\alpha) + \widetilde{x}_R(\alpha)}{2}.$$

Then the function $\widetilde{x}_M : [0,1] \to \mathbb{R}$ will be called the midpoint function of the fuzzy number $\widetilde{x}$.

Lemma 2.3: [14] For any $\widetilde{x} \in \mathscr{F}$, the midpoint function $\widetilde{x}_M$ is continuous from the right at 0 and continuous from the left on $[0,1]$. Furthermore, it is a function of bounded variation on $[0,1]$.

Definition 2.4: [4] Let $\widetilde{x}, \widetilde{y} \in \mathscr{F}$, we say that $\widetilde{x}$ is equivalent to $\widetilde{y}$, if there exist two symmetric fuzzy numbers $\widetilde{s}_1, \widetilde{s}_2 \in \mathscr{S}$ such that $\widetilde{x} + \widetilde{s}_1 = \widetilde{y} + \widetilde{s}_2$ and then we denote this by $\widetilde{x} \sim \widetilde{y}$.

It is easy to verify that the equivalence relation defined above is reflexive, symmetric and transitive[9]. Let $\langle\widetilde{x}\rangle$ denote the fuzzy number equivalence class containing the element $\widetilde{x}$ and denote the set of all fuzzy number equivalence classes by $\mathscr{F}/\mathscr{S}$.

Definition 2.5: [10] Let $\widetilde{x} \in \mathscr{F}$ and let $\widehat{x}$ be a fuzzy number such that $\widetilde{x} = \widehat{x} + \widetilde{s}$ for some $\widetilde{s} \in \mathscr{S}$, if $\widehat{x} = \widetilde{y} + \widetilde{s}_1$ for some $\widetilde{y} \in \mathscr{F}$ and $\widetilde{s}_1 \in \mathscr{S}$, then $\widetilde{s}_1 = \widetilde{0}$. Then the fuzzy number $\widehat{x}$ will be called the Mareš core of the fuzzy number $\widetilde{x}$.

Definition 2.6: [15] Let $\langle\widetilde{x}\rangle \in \mathscr{F}/\mathscr{S}$, we define the midpoint function $M_{\langle\widetilde{x}\rangle} : [0,1] \to \mathbb{R}$ by

$$M_{\langle\widetilde{x}\rangle}(\alpha) = \widehat{x}_M(\alpha)$$

for all $\alpha \in [0,1]$, where $\widehat{x}$ is the Mareš core of $\langle\widetilde{x}\rangle$.

Definition 2.7: [15] Let $\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle \in \mathscr{F}/\mathscr{S}$, we define the sum of this two fuzzy number equivalence classes as a fuzzy equivalence class $\langle\widetilde{z}\rangle \in \mathscr{F}/\mathscr{S}$, which satisfies the condition

$$M_{\langle\widetilde{x}\rangle}(\alpha) + M_{\langle\widetilde{y}\rangle}(\alpha) = M_{\langle\widetilde{z}\rangle}(\alpha)$$

for all $\alpha \in [0,1]$ and we denote this by

$$\langle\widetilde{x}\rangle + \langle\widetilde{y}\rangle = \langle\widetilde{x}+\widetilde{y}\rangle = \langle\widetilde{z}\rangle.$$

Definition 2.8: [14] Let $\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle \in \mathscr{F}/\mathscr{S}$, we say that $\langle\widetilde{z}\rangle \in \mathscr{F}/\mathscr{S}$ is the product of $\langle\widetilde{x}\rangle$ and $\langle\widetilde{y}\rangle$ if their midpoint functions satisfy

$$M_{\langle\widetilde{x}\rangle}(\alpha) \cdot M_{\langle\widetilde{y}\rangle}(\alpha) = M_{\langle\widetilde{z}\rangle}(\alpha)$$

for all $\alpha \in [0,1]$ and we denote this by

$$\langle\widetilde{x}\rangle \cdot \langle\widetilde{y}\rangle = \langle\widetilde{z}\rangle.$$

Definition 2.9: [15] For any $\langle\widetilde{x}\rangle \in \mathscr{F}/\mathscr{S}$ and $\lambda \in \mathbb{R}$, we define $\lambda \cdot \langle\widetilde{x}\rangle = \lambda\langle\widetilde{x}\rangle$ by

$$\lambda\langle\widetilde{x}\rangle = \langle\widetilde{x}\rangle\lambda = \langle\lambda\widetilde{x}\rangle.$$

It is obvious that $M_{\lambda\langle\widetilde{x}\rangle}(\alpha) = M_{\langle\lambda\widetilde{x}\rangle}(\alpha) = \lambda M_{\langle\widetilde{x}\rangle}(\alpha)$ for all $\alpha \in [0,1]$.

Definition 2.10: [14] Let $\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle \in \mathscr{F}/\mathscr{S}$, we define $d_{\sup} : \mathscr{F}/\mathscr{S} \times \mathscr{F}/\mathscr{S} \to \mathbb{R}^+ \cup \{0\}$ by

$$d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle) = \sup_{\alpha\in[0,1]} \left|M_{\langle\widetilde{x}\rangle}(\alpha) - M_{\langle\widetilde{y}\rangle}(\alpha)\right|.$$

It is easy to see that $(\mathscr{F}/\mathscr{S}, d_{\sup})$ is a metric space [14].

III. Optimization of Convex and Generalized Convex Fuzzy Mappings

In this paper, we always suppose that the range of fuzzy mappings is the set of all fuzzy number equivalence classes.

Definition 3.1: [15] Let $F : T \to \mathscr{F}/\mathscr{S}$ be a fuzzy mapping, where $T = [a,b] \subseteq \mathbb{R}$. Then F is said to be continuous at $t \in T$ with respect to $d_{\sup}$ if for any $h \neq 0$ with $t+h \in T$ such that

$$\lim_{h\to 0} d_{\sup}(F(t+h), F(t)) = 0.$$

If $t = a$(or b), then we consider only $h \to 0^+$ (or $h \to 0^-$).

Definition 3.2: [15] Let $F : T \to \mathscr{F}/\mathscr{S}$ be a fuzzy mapping, where $T = [a,b] \subseteq \mathbb{R}$. Then F is said to be differentiable at $t \in T$ if there exists an $F'(t) \in \mathscr{F}/\mathscr{S}$ such that

$$\lim_{h\to 0} d_{\sup}\left(\frac{F(t+h) - F(t)}{h}, F'(t)\right) = 0.$$

If $t = a$(or b), then we consider only $h \to 0^+$ (or $h \to 0^-$).

Lemma 3.1: [15] $F : T \to \mathscr{F}/\mathscr{S}$ is differentiable on T if and only if

1. $M_{F(t)}(\alpha)$ is differentiable with respect to $t \in T$ for all $\alpha \in [0,1]$. i.e., $\frac{\partial}{\partial t}M_{F(t)}(\alpha)$ exists and is of bounded variation with respect to $\alpha \in [0,1]$ for all $t \in T$.
2. The mappings $\left\{M_{F(t)}(\alpha)\right\}_{\alpha\in[0,1]}$ are uniformly differentiable with the derivatives $\frac{\partial}{\partial t}M_{F(t)}(\alpha)$. i.e., for each $t \in T$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that
$$\left|\frac{M_{F(t+h)}(\alpha) - M_{F(t)}(\alpha)}{h} - \frac{\partial}{\partial t}M_{F(t)}(\alpha)\right| < \varepsilon$$
for all $|h| \in (0,\delta)$ and $\alpha \in [0,1]$.

Lemma 3.2: [15] If $F : T \to \mathscr{F}/\mathscr{S}$ is differentiable, then it is continuous with respect to $d_{\sup}$.

Definition 3.3: [16] Let $\langle \widetilde{\boldsymbol{a}} \rangle = (\langle \widetilde{a}_1 \rangle, \langle \widetilde{a}_2 \rangle, \cdots, \langle \widetilde{a}_n \rangle)^T \in (\mathscr{F}/\mathscr{S})^n$ and $\boldsymbol{t} = (t_1, t_2, \cdots, t_n)^T \in \mathbb{R}^n$ be an n-dimensional fuzzy number equivalence class vector and n-dimensional real vector respectively. We define theirs product as

$$\langle \widetilde{\boldsymbol{a}} \rangle^T \boldsymbol{t} = \sum_{i=1}^{n} \langle \widetilde{a}_i \rangle t_i = \langle \widetilde{a}_1 \rangle t_1 + \langle \widetilde{a}_2 \rangle t_2 + \cdots + \langle \widetilde{a}_n \rangle t_n,$$

which is a fuzzy number equivalence class.

Definition 3.4: [16] Let $F : \Omega \to \mathscr{F}/\mathscr{S}$ be a fuzzy mapping, where Ω is an open subset in $\mathbb{R}^n$. We say that F has a partial derivative at $\boldsymbol{t} = (t_1, t_2, \cdots, t_n)^{\boldsymbol{t}} \in \Omega$ with respect to the ith variable t_i if there exists an $\frac{\partial}{\partial t_i} F(\boldsymbol{t}) \in \mathscr{F}/\mathscr{S}$ such that

$$\lim_{h \to 0} d_{\sup} \left(\frac{F\left(\boldsymbol{t} + h\boldsymbol{e^i}\right) - F(\boldsymbol{t})}{h}, \frac{\partial}{\partial t_i} F(\boldsymbol{t}) \right) = 0,$$

where $\boldsymbol{e^i}$ stands for the unit vector that the ith component is 1 and the others are 0.

Definition 3.5: [16] Let $F : \Omega \to \mathscr{F}/\mathscr{S}$ be a fuzzy mapping, where Ω is an open subset in $\mathbb{R}^n$. We say that F is differentiable at $\boldsymbol{t} = (t_1, t_2, \cdots, t_n)^T \in \Omega$ if F has continuous partial derivatives $\frac{\partial}{\partial t_i} F(\boldsymbol{t})$ with respect to ith variable $t_i\,(i = 1, 2, \cdots, n)$ and satisfies

$$F(\boldsymbol{t} + \boldsymbol{h}) = F(\boldsymbol{t}) + \widetilde{\nabla} F(\boldsymbol{t})^T \boldsymbol{h} + o(\|\boldsymbol{h}\|),$$

where $\widetilde{\nabla} F(\boldsymbol{t}) \in (\mathscr{F}/\mathscr{S})^n$ is an n-dimensional fuzzy number equivalence class vector defined by

$$\widetilde{\nabla} F(\boldsymbol{t}) = \left(\frac{\partial F(\boldsymbol{t})}{\partial t_1}, \frac{\partial F(\boldsymbol{t})}{\partial t_2}, \cdots, \frac{\partial F(\boldsymbol{t})}{\partial t_n} \right)^T,$$

$\|\boldsymbol{h}\|$ is the usual Euclid norm of $\boldsymbol{h}$ and $o : [0, +\infty) \to \mathscr{F}/\mathscr{S}$ is a fuzzy mapping that satisfies

$$\lim_{t \to 0} d_{\sup} \left(\frac{o(t)}{t}, \langle \widetilde{0} \rangle \right) = 0.$$

Then we call $\widetilde{\nabla} F(\boldsymbol{t})$, the gradient of the fuzzy mappings F at $\boldsymbol{t}$.

Definition 3.6: [16] Let $\langle \widetilde{x} \rangle, \langle \widetilde{y} \rangle \in \mathscr{F}/\mathscr{S}$.

(1) We say that $\langle \widetilde{x} \rangle \preceq \langle \widetilde{y} \rangle$ if $M_{\langle \widetilde{x} \rangle}(\alpha) \le M_{\langle \widetilde{y} \rangle}(\alpha)$ for all $\alpha \in [0, 1]$.
(2) We say that $\langle \widetilde{x} \rangle \prec \langle \widetilde{y} \rangle$ if $\langle \widetilde{x} \rangle \preceq \langle \widetilde{y} \rangle$ and there exists at least one $\alpha_0 \in [0, 1]$ such that $M_{\langle \widetilde{x} \rangle}(\alpha_0) < M_{\langle \widetilde{y} \rangle}(\alpha_0)$.
(3) If $\langle \widetilde{x} \rangle \preceq \langle \widetilde{y} \rangle$ and $\langle \widetilde{y} \rangle \preceq \langle \widetilde{x} \rangle$, then $\langle \widetilde{x} \rangle = \langle \widetilde{y} \rangle$.

Sometimes we may write $\langle \widetilde{y} \rangle \succeq \langle \widetilde{x} \rangle$ instead of $\langle \widetilde{x} \rangle \preceq \langle \widetilde{y} \rangle$ and write $\langle \widetilde{y} \rangle \succ \langle \widetilde{x} \rangle$ instead of $\langle \widetilde{x} \rangle \prec \langle \widetilde{y} \rangle$. Note that $\preceq$ is a partial order relation on $\mathscr{F}/\mathscr{S}$.

Definition 3.7: Let $\langle \widetilde{a} \rangle \in \mathscr{F}/\mathscr{S}$, we say that $\langle \widetilde{a} \rangle$ is nonnegative if $\langle \widetilde{a} \rangle \succeq \langle \widetilde{0} \rangle$, i.e., $M_{\langle \widetilde{a} \rangle}(\alpha) \ge 0$ for all $\alpha \in [0, 1]$.

Let $F : \mathbb{R}^n \to \mathscr{F}/\mathscr{S}$ be a fuzzy mapping. Consider the following optimization problem

$$\begin{array}{ll} \min & F(\boldsymbol{t}) = F(t_1, t_2, \cdots, t_n), \\ \text{subjec to} & \boldsymbol{t} = (t_1, t_2 \cdots, t_n)^T \in \Omega \subseteq \mathbb{R}^n, \end{array} \tag{1}$$

where the feasible set Ω is assumed to be convex subset of $\mathbb{R}^n$. Since $\preceq$ is a partial order relation on $\mathscr{F}/\mathscr{S}$, we may follow the similar solution concept (the non-dominated solution) used in multi-objective programming problems to interprete the meaning of minimization in problem (1).

Definition 3.8: Let $\boldsymbol{t^*}$ be a feasible solution of problem (1), i.e., $\boldsymbol{t^*} \in \Omega$.

(1) We say that $\boldsymbol{t^*}$ is a local non-dominated solution of problem (1) if there exists an $\varepsilon > 0$ and for no $\boldsymbol{t} \in N_\varepsilon(\boldsymbol{t^*}) \cap \Omega$ such that $F(\boldsymbol{t}) \prec F(\boldsymbol{t^*})$, where $N_\varepsilon(\boldsymbol{t^*})$ is an ε-neighborhood around $\boldsymbol{t^*}$.
(2) We say that $\boldsymbol{t^*}$ is a (global) non-dominated solution of problem (1) if there exists no $\boldsymbol{t} \in \Omega$ such that $F(\boldsymbol{t}) \prec F(\boldsymbol{t^*})$.
(3) We say that $\boldsymbol{t^*}$ is a strongly (global) non-dominated solution of problem (1) if there exists no $\boldsymbol{t}(\neq \boldsymbol{t^*}) \in \Omega$ such that $F(\boldsymbol{t}) \preceq F(\boldsymbol{t^*})$.

Remark 3.1: It is easy to get that if t^* is a strongly non-dominated solution of problem (1), then it is also a non-dominated solution of problem (1).

To present the relationships among above non-dominated solutions, first of all, we provide the concept of convexity and its generalizations for fuzzy mappings.

Definition 3.9: Let $F : \Omega \to \mathscr{F}/\mathscr{S}$ be a fuzzy mapping, where Ω is a non-empty convex subset in $\mathbb{R}^n$. F is said to be convex on Ω if for any $\boldsymbol{s}, \boldsymbol{t} \in \Omega$ and $\lambda \in (0, 1)$, we always have $F(\lambda \boldsymbol{s} + (1 - \lambda)\boldsymbol{t}) \preceq \lambda F(\boldsymbol{s}) + (1 - \lambda) F(\boldsymbol{t})$. F is said to be concave if $-F$ is convex.

Theorem 3.1: Let $F : \Omega \to \mathscr{F}/\mathscr{S}$ be a fuzzy mapping, where Ω is a non-empty convex subset in $\mathbb{R}^n$. Then F is convex on Ω if and only if $M_{F(t)}(\alpha)$ is convex with respect to $t \in \Omega$ for all $\alpha \in [0, 1]$.

Proof. The result follows from Definitions 3.6 and 3.9 immediately. □

Theorem 3.2: Every local non-dominated solution of problem (1) is also a (global) non-dominated solution of problem (1) if the objective function F is convex.

Proof. Let $\boldsymbol{t^*} \in \Omega$ be a local non-dominated solution of problem (1). Thus there exists an $\varepsilon > 0$ and for no $\boldsymbol{t} \in N_\varepsilon(\boldsymbol{t^*}) \cap \Omega$ such that $F(\boldsymbol{t}) \prec F(\boldsymbol{t^*})$. We are going to prove this result by contradiction. Suppose that $\boldsymbol{t^*} \in \Omega$ is not a (global) non-dominated solution of problem (1), then there exists at least one other point $\boldsymbol{t^0} \in \Omega$ such that

$$F\left(\boldsymbol{t^0}\right) \prec F(\boldsymbol{t^*}). \tag{2}$$

Since the feasible set Ω is convex, we have $\lambda \boldsymbol{t^0} + (1 - \lambda)\boldsymbol{t^*} \in \Omega$ for any $\lambda \in (0, 1)$. Considering that the objective function F is convex and using (2), we have

$$\begin{aligned} F\left(\lambda \boldsymbol{t^0} + (1 - \lambda)\boldsymbol{t^*}\right) &\preceq \lambda F\left(\boldsymbol{t^0}\right) + (1 - \lambda) F(\boldsymbol{t^*}) \\ &\prec \lambda F(\boldsymbol{t^*}) + (1 - \lambda) F(\boldsymbol{t^*}) = F(\boldsymbol{t^*}). \end{aligned}$$

We see that $\lambda \in (0, 1)$ can be sufficiently small such that $\lambda \boldsymbol{t^0} + (1 - \lambda)\boldsymbol{t^*} \in N_\varepsilon(\boldsymbol{t^*})$, which contradicts the condition that $\boldsymbol{t^*}$ is a local non-dominated solution of problem (1). Thus, we have that $\boldsymbol{t^*} \in \Omega$ is also a (global) non-dominated solution of problem (1). □

The concept of quasiconvex fuzzy mapping in the space of fuzzy numbers have been introduced by Nanda[11]. Nanda did not discuss the concept for finding the maximum of two fuzzy numbers. Since it may happen that two fuzzy numbers are not comparable. Similarly, since $\preceq$ is a partial order relation on $\mathscr{F}/\mathscr{S}$, to present and modify the definition of quasiconvex fuzzy mapping on $\mathscr{F}/\mathscr{S}$, we shall to present the following definition firstly.

Definition 3.10: Let $S \subseteq \mathscr{F}/\mathscr{S}$, then S is said to be bounded above if there exists a fuzzy number equivalence class $\langle\widetilde{a}\rangle \in \mathscr{F}/\mathscr{S}$, called an upper bound of S, such that $\langle\widetilde{x}\rangle \preceq \langle\widetilde{a}\rangle$ for every $\langle\widetilde{x}\rangle \in \mathscr{F}/\mathscr{S}$. Further, a fuzzy number equivalence class $\langle\widetilde{a}_0\rangle \in \mathscr{F}/\mathscr{S}$ is called the least upper bound (sup, in short) for S if the following conditions are hold:

(1) $\langle\widetilde{a}_0\rangle$ is an upper bound of S;
(2) $\langle\widetilde{a}_0\rangle \preceq \langle\widetilde{a}\rangle$ for every upper bound $\langle\widetilde{a}\rangle$ of S.

A lower bound and the greatest lower bound (inf, in short) are defined similarly.

Theorem 3.3: Let any $\langle\widetilde{a}\rangle, \langle\widetilde{c}\rangle \in \mathscr{F}/\mathscr{S}$, then the set $\{\langle\widetilde{a}\rangle, \langle\widetilde{c}\rangle\}$ has the least upper bound and the greatest lower bound.

Proof. For any $\langle\widetilde{a}\rangle, \langle\widetilde{c}\rangle \in \mathscr{F}/\mathscr{S}$, by Lemma 2.3 we have that the midpoint functions $M_{\langle\widetilde{a}\rangle}$ and $M_{\langle\widetilde{c}\rangle}$ are continuous from the right at 0, continuous from the left on [0,1] and are functions of bounded variation on [0,1]. Then we define two functions $M_{\sup} : [0,1] \to \mathbb{R}$ and $M_{\inf} : [0,1] \to \mathbb{R}$ by $M_{\sup}(\alpha) = \max\left\{M_{\langle\widetilde{a}\rangle}(\alpha), M_{\langle\widetilde{c}\rangle}(\alpha)\right\}$ and $M_{\inf}(\alpha) = \min\left\{M_{\langle\widetilde{a}\rangle}(\alpha), M_{\langle\widetilde{c}\rangle}(\alpha)\right\}$ for all $\alpha \in [0,1]$, respectively. It is easy to see that the functions $M_{\sup}$ and $M_{\inf}$ are continuous from the right at 0 and continuous from the left on [0,1]. Furthermore, we have that $V_0^1(M_{\sup}) \leq V_0^1\left(M_{\langle\widetilde{a}\rangle}\right) + V_0^1\left(M_{\langle\widetilde{c}\rangle}\right)$ and $V_0^1(M_{\inf}) \leq V_0^1\left(M_{\langle\widetilde{a}\rangle}\right) + V_0^1\left(M_{\langle\widetilde{c}\rangle}\right)$. Thus $M_{\sup}$ and $M_{\inf}$ are functions of bounded variation on [0,1]. By Theorem 3.10 in [14], the functions $M_{\sup}$ and $M_{\inf}$ can determine two fuzzy number equivalence classes $\langle\widetilde{m}\rangle$ and $\langle\widetilde{n}\rangle$ such that $M_{\langle\widetilde{m}\rangle} = M_{\sup}$ and $M_{\langle\widetilde{n}\rangle} = M_{\inf}$. It is easy to see that $\langle\widetilde{m}\rangle$ and $\langle\widetilde{n}\rangle$ are $\sup\{\langle\widetilde{a}\rangle, \langle\widetilde{c}\rangle\}$ and $\inf\{\langle\widetilde{a}\rangle, \langle\widetilde{c}\rangle\}$, respectively. □

Definition 3.11: Let $F : \Omega \to \mathscr{F}/\mathscr{S}$ be a fuzzy mapping, where Ω is a non-empty convex subset in $\mathbb{R}^n$. F is said to be quasiconvex on Ω if for any $\boldsymbol{s}, \boldsymbol{t} \in \Omega$ and $\lambda \in (0,1)$, we always have $F(\lambda\boldsymbol{s} + (1-\lambda)\boldsymbol{t}) \preceq \sup\{F(\boldsymbol{s}), F(\boldsymbol{t})\}$. The fuzzy mapping F is said to be quasiconcave if $-F$ is quasiconvex.

Theorem 3.4: Let $F : \Omega \to \mathscr{F}/\mathscr{S}$ be a fuzzy mapping, where Ω is a non-empty convex subset in $\mathbb{R}^n$. Then F is quasiconvex on Ω if and only if $M_{F(t)}(\alpha)$ is quasiconvex with respect to $t \in \Omega$ for all $\alpha \in [0,1]$.

Proof. The result follows from Theorem 3.3 and Definition 3.11 immediately. □

Theorem 3.5: Let $F : \Omega \to \mathscr{F}/\mathscr{S}$ be a differentiable fuzzy mapping, where Ω is a non-empty convex subset in $\mathbb{R}^n$. If F is quasiconvex on Ω, then the following statement holds: If $\boldsymbol{s}, \boldsymbol{t} \in \Omega$ such that $F(\boldsymbol{s}) \preceq F(\boldsymbol{t})$, then $\widetilde{\nabla}F(\boldsymbol{t})^T(\boldsymbol{s}-\boldsymbol{t}) \preceq \langle\widetilde{\mathbf{0}}\rangle$. Furthermore, if F is comparable in the sense of Definition 3.1 in [15], then the converse of this theorem is hold.

Proof. Let F be quasiconvex on Ω and $\boldsymbol{s}, \boldsymbol{t} \in \Omega$ such that $F(\boldsymbol{s}) \preceq F(\boldsymbol{t})$. Then we shall show that $\widetilde{\nabla}F(\boldsymbol{t})^T(\boldsymbol{s}-\boldsymbol{t}) \preceq \langle\widetilde{\mathbf{0}}\rangle$. Since F is differentiable at $\boldsymbol{t}$, by Definition 3.5 we have

$$F(\lambda\boldsymbol{s} + (1-\lambda)\boldsymbol{t}) - F(\boldsymbol{t}) = \lambda\widetilde{\nabla}F(\boldsymbol{t})^T(\boldsymbol{s}-\boldsymbol{t}) + o(\lambda\|(\boldsymbol{s}-\boldsymbol{t})\|) \tag{3}$$

for any $\lambda \in (0,1)$, where

$$d_{\sup}\left(\frac{o(\lambda\|(\boldsymbol{s}-\boldsymbol{t})\|)}{\lambda}, \langle\widetilde{0}\rangle\right) \to 0 \tag{4}$$

as $\lambda \to 0$. Since F is quasiconvex on Ω, and as $F(\boldsymbol{s}) \preceq F(\boldsymbol{t})$, we have

$$F(\lambda\boldsymbol{s} + (1-\lambda)\boldsymbol{t}) \preceq \sup\{F(\boldsymbol{s}), F(\boldsymbol{t})\} = F(\boldsymbol{t}) \tag{5}$$

for any $\lambda \in (0,1)$. Now $(3), (4)$ and (5) imply that $\lambda\widetilde{\nabla}F(\boldsymbol{t})^T(\boldsymbol{s}-\boldsymbol{t}) + o(\lambda\|(\boldsymbol{s}-\boldsymbol{t})\|) \preceq \langle\widetilde{0}\rangle$, that is

$$\lambda M_{\widetilde{\nabla}F(\boldsymbol{t})}(\alpha)^T(\boldsymbol{s}-\boldsymbol{t}) + M_{o(\lambda\|(\boldsymbol{s}-\boldsymbol{t})\|)}(\alpha) \leq 0. \tag{6}$$

for all $\alpha \in [0,1]$. Further, we see that

$$d_{\sup}\left(\frac{o(\lambda\|(\boldsymbol{s}-\boldsymbol{t})\|)}{\lambda}, \langle\widetilde{0}\rangle\right) = \sup_{\alpha\in[0,1]}\left|M_{\frac{o(\lambda\|(\boldsymbol{s}-\boldsymbol{t})\|)}{\lambda}}(\alpha) - 0\right| = \sup_{\alpha\in[0,1]}\left|\frac{M_{o(\lambda\|(\boldsymbol{s}-\boldsymbol{t})\|)}(\alpha)}{\lambda}\right| \to 0$$

as $\lambda \to 0$. Since $\lambda \neq 0$, dividing (6) by λ and taking $\lambda \to 0$, we can obtain $M_{\widetilde{\nabla}F(\boldsymbol{t})}(\alpha)^T(\boldsymbol{s}-\boldsymbol{t}) \leq 0$, for all $\alpha \in [0,1]$, that is, $\widetilde{\nabla}F(\boldsymbol{t})^T(\boldsymbol{s}-\boldsymbol{t}) \preceq \langle\widetilde{0}\rangle$.

Conversely, we suppose that $\boldsymbol{s}, \boldsymbol{t} \in \Omega$ such that $F(\boldsymbol{s}) \preceq F(\boldsymbol{t})$ and then we have $\widetilde{\nabla}F(\boldsymbol{t})^T(\boldsymbol{s}-\boldsymbol{t}) \preceq \langle\widetilde{\mathbf{0}}\rangle$. Next, we shall show that $F(\lambda\boldsymbol{s} + (1-\lambda)\boldsymbol{t}) \preceq \sup\{F(\boldsymbol{s}), F(\boldsymbol{t})\} = F(\boldsymbol{t})$ for any $\lambda \in (0,1)$, which implies that F is quasiconvex on Ω. By contradiction, we suppose that there exists a $\lambda^* \in (0,1)$ such that

$$F(\lambda^*\boldsymbol{s} + (1-\lambda^*)\boldsymbol{t}) \npreceq F(\boldsymbol{t}). \tag{7}$$

Further, since we suppose that F is comparable, (7) implies that $F(\lambda^*\boldsymbol{s} + (1-\lambda^*)\boldsymbol{t}) \succ F(\boldsymbol{t})$. Since F is differentiable, by Lemma 3.2 we have that F is continuous with respect to $d_{\sup}$. Denoting $\boldsymbol{w} = \lambda^*\boldsymbol{s} + (1-\lambda^*)\boldsymbol{t}$, then there exists a $\delta \in (0,1)$ such that

$$F(\mu\boldsymbol{w} + (1-\mu)\boldsymbol{t}) \succ F(\boldsymbol{t}) \tag{8}$$

for all $\mu \in (\delta, 1)$ and $F(\boldsymbol{w}) \succ F(\delta\boldsymbol{w} + (1-\delta)\boldsymbol{t})$, which implies that $M_{F(\boldsymbol{w})}(\alpha) > M_{F(\delta\boldsymbol{w}+(1-\delta)\boldsymbol{t})}(\alpha)$ for all $\alpha \in [0,1]$. By Lemma 3.1 we have that $M_{F(\boldsymbol{t})}(\alpha)$ is differentiable with respect to $t \in T$ for all $\alpha \in [0,1]$. Further, by the usually mean value theorem, we have that

$$0 < (1-\delta)M_{\widetilde{\nabla}F(\boldsymbol{v})}(\alpha)^T(\boldsymbol{w}-\boldsymbol{t})$$

for all $\alpha \in [0,1]$, where $\boldsymbol{v} = \mu'\boldsymbol{w} + (1-\mu')\boldsymbol{t}$ for some $\mu' \in (\delta, 1)$. That is

$$0 < M_{\widetilde{\nabla}F(\boldsymbol{v})}(\alpha)^T(\boldsymbol{s}-\boldsymbol{t}) \tag{9}$$

for all $\alpha \in [0,1]$. Furthermore, by (8) we can obtain $F(\boldsymbol{v}) \succ F(\boldsymbol{t})$. Since $F(\boldsymbol{s}) \preceq F(\boldsymbol{t})$, it is easy to see that $F(\boldsymbol{v}) \succ F(\boldsymbol{s})$. Further, we have

$$\boldsymbol{v} = \mu'\boldsymbol{w} + (1-\mu')\boldsymbol{t} = \mu'\lambda^*\boldsymbol{s} + (1-\mu'\lambda^*)\boldsymbol{t},$$

where $\nu^* = \mu'\lambda^* \in (0,1)$. Then by the assumption of the theorem, we have that

$$\langle\widetilde{\mathbf{0}}\rangle \succeq \widetilde{\nabla}F(\boldsymbol{v})^T(\boldsymbol{s}-\boldsymbol{v}) = (1-\nu^*)\widetilde{\nabla}F(\boldsymbol{v})^T(\boldsymbol{s}-\boldsymbol{t}). \tag{10}$$

Since $1-\nu^* > 0$, the inequality (10) implies that $M_{\widetilde{\nabla}F(\boldsymbol{v})}(\alpha)^T(\boldsymbol{s}-\boldsymbol{t}) \leq 0$ for all $\alpha \in [0,1]$, which contradicts the inequality (9). Hence, we have that F is quasiconvex on Ω. □

Definition 3.12: Let $F : \Omega \to \mathscr{F}/\mathscr{S}$ be a fuzzy mapping, where Ω is a non-empty convex subset in $\mathbb{R}^n$. F is said to be strictly quasiconvex on Ω if for any $\boldsymbol{s}, \boldsymbol{t} \in \Omega$ with $F(\boldsymbol{s}) \neq$

$F(\boldsymbol{t})$ and $\lambda \in (0,1)$, we always have $F(\lambda \boldsymbol{s} + (1-\lambda)\boldsymbol{t}) \prec \sup\{F(\boldsymbol{s}), F(\boldsymbol{t})\}$. The fuzzy mapping F is said to be strictly quasiconcave if $-F$ is strictly quasiconvex.

Theorem 3.6: Let $F : \Omega \to \mathscr{F}/\mathscr{S}$ be a strictly quasiconvex fuzzy mapping, where Ω is a non-empty convex subset in $\mathbb{R}^n$. If $\boldsymbol{t}^* \in \Omega$ is a local non-dominated solution of problem (1), then $\boldsymbol{t}^*$ is also a global non-dominated solution of problem (1).

Proof. If $\boldsymbol{t}^* \in \Omega$ is a local non-dominated solution of problem (1), then there exists an $\varepsilon > 0$ and for no $\boldsymbol{t} \in N_\varepsilon(\boldsymbol{t}^*) \cap \Omega$ such that $F(\boldsymbol{t}) \prec F(\boldsymbol{t}^*)$. We are going to prove this result by contradiction. Suppose that $\boldsymbol{t}^* \in \Omega$ is not a (global) non-dominated solution of problem (1), then there exists at least one point $\boldsymbol{t^0} \in \Omega$ such that

$$F\left(\boldsymbol{t^0}\right) \prec F\left(\boldsymbol{t}^*\right). \tag{11}$$

We define $\bar{\boldsymbol{t}} = \lambda \boldsymbol{t^0} + (1-\lambda)\boldsymbol{t}^*$, where $\lambda \in (0,1)$ is selected such that $\bar{\boldsymbol{t}} \in N_\varepsilon(\boldsymbol{t}^*)$. Since the feasible set Ω is convex, we have $\bar{\boldsymbol{t}} \in \Omega$. Considering that the objective function F is strictly quasiconvex and using (11), we have $F\left(\bar{\boldsymbol{t}}\right) \prec \sup\left\{F\left(\boldsymbol{t}^*\right), F\left(\boldsymbol{t^0}\right)\right\} = F\left(\boldsymbol{t}^*\right)$, which contradicts the condition that $\boldsymbol{t}^*$ is a local non-dominated solution of problem (1). Hence, we have that $\boldsymbol{t}^* \in \Omega$ is also a (global) non-dominated solution of problem (1). □

Definition 3.13: Let $F : \Omega \to \mathscr{F}/\mathscr{S}$ be a fuzzy mapping, where Ω is a non-empty convex subset in $\mathbb{R}^n$. F is said to be strongly quasiconvex on Ω if for any $\boldsymbol{s}, \boldsymbol{t} \in \Omega$ with $\boldsymbol{s} \neq \boldsymbol{t}$ and $\lambda \in (0,1)$, we always have $F(\lambda \boldsymbol{s} + (1-\lambda)\boldsymbol{t}) \prec \sup\{F(\boldsymbol{s}), F(\boldsymbol{t})\}$. The fuzzy mapping F is said to be strongly quasiconcave if $-F$ is strongly quasiconvex.

Theorem 3.7: Let $F : \Omega \to \mathscr{F}/\mathscr{S}$ be a strongly quasiconvex fuzzy mapping, where Ω is a non-empty convex subset in $\mathbb{R}^n$. If $\boldsymbol{t}^* \in \Omega$ is a local non-dominated solution of problem (1), then $\boldsymbol{t}^*$ is also a strongly (global) non-dominated solution of problem (1).

Proof. If $\boldsymbol{t}^* \in \Omega$ is a local non-dominated solution of problem (1), then there exists an $\varepsilon > 0$ and for no $\boldsymbol{t} \in N_\varepsilon(\boldsymbol{t}^*) \cap \Omega$ such that $F(\boldsymbol{t}) \prec F(\boldsymbol{t}^*)$. We are going to prove this result by contradiction. Suppose that $\boldsymbol{t}^*$ is not a strongly (global) non-dominated solution of problem (1), then there exists at least one point $\boldsymbol{t^0} \in \Omega$ with $\boldsymbol{t^0} \neq \boldsymbol{t}^*$ such that

$$F\left(\boldsymbol{t^0}\right) \preceq F\left(\boldsymbol{t}^*\right). \tag{12}$$

We define $\bar{\boldsymbol{t}} = \lambda \boldsymbol{t^0} + (1-\lambda)\boldsymbol{t}^*$, where $\lambda \in (0,1)$ is selected such that $\bar{\boldsymbol{t}} \in N_\varepsilon(\boldsymbol{t}^*)$. Since the feasible set Ω is convex, we have $\bar{\boldsymbol{t}} \in \Omega$. Considering that the objective function F is strongly quasiconvex and using (12), we have $F\left(\bar{\boldsymbol{t}}\right) \prec \sup\left\{F\left(\boldsymbol{t}^*\right), F\left(\boldsymbol{t^0}\right)\right\} = F\left(\boldsymbol{t}^*\right)$, which contradicts the condition that $\boldsymbol{t}^*$ is a local non-dominated solution of problem (1). Hence, we have that $\boldsymbol{t}^* \in \Omega$ is also a strongly (global) non-dominated solution of problem (1). □

Definition 3.14: Let $F : \Omega \to \mathscr{F}/\mathscr{S}$ be a differentiable fuzzy mapping, where Ω is a non-empty convex subset in $\mathbb{R}^n$. F is said to be pseudoconvex on Ω if for any $\boldsymbol{s}, \boldsymbol{t} \in \Omega$ such that $F(\boldsymbol{s}) \prec F(\boldsymbol{t})$, we always have $\widetilde{\nabla} F(\boldsymbol{t})^T (\boldsymbol{s} - \boldsymbol{t}) \prec \langle \widetilde{0} \rangle$. The fuzzy mapping F is said to be pseudoconcave if $-F$ is pseudoconvex. Similarly, F is said to be strictly pseudoconvex on Ω if for any $\boldsymbol{s}, \boldsymbol{t} \in \Omega$ with $\boldsymbol{s} \neq \boldsymbol{t}$ satisfying $F(\boldsymbol{s}) \preceq F(\boldsymbol{t})$, we always have $\widetilde{\nabla} F(\boldsymbol{t})^T (\boldsymbol{s} - \boldsymbol{t}) \prec \langle \widetilde{0} \rangle$.

Theorem 3.8: Let $F : \Omega \to \mathscr{F}/\mathscr{S}$ be a pseudoconvex fuzzy mapping, where Ω is a non-empty open convex subset in $\mathbb{R}^n$. If $\widetilde{\nabla} F(\boldsymbol{t}^*) = \langle \widetilde{\boldsymbol{0}} \rangle$, then $\boldsymbol{t}^*$ is a non-dominated solution of problem (1).

Proof. If $\widetilde{\nabla} F(\boldsymbol{t}^*) = \langle \widetilde{\boldsymbol{0}} \rangle$. Then we have $\widetilde{\nabla} F(\boldsymbol{t}^*)^T (\boldsymbol{t} - \boldsymbol{t}^*) = \langle \widetilde{0} \rangle$ for any $\boldsymbol{t} \in \Omega$. Since F is pseudoconvex, we have $F(\boldsymbol{t}) \nprec F(\boldsymbol{t}^*)$ for all $\boldsymbol{t} \in \Omega$, which implies that there exists no $\boldsymbol{t} \in \Omega$ such that $F(\boldsymbol{t}) \prec F(\boldsymbol{t}^*)$. Hence, we get that $\boldsymbol{t}^*$ is a non-dominated solution of problem (1). □

IV. Conclusion

In this present investigation, by considering an ordering relation on the quotient space of fuzzy numbers, we have presented the concepts of convexity, quasiconvexity and pseudoconvexity for fuzzy mappings [19]. Further, the solutions concepts proposed in this paper will follow from the similar solution concept, called non-dominated solution, in the conventional multiobjective programming problems [26], [30], [31]. We hope that our results may provide a background to ongoing work in related fields [12], [20], [25].

References

[1] M. S. Bazaraa, H. D. Sherali and C. M. Shetty, *Nonlinear programming: theory and algorithms*, Harlow, England: John Wiley and Sons, 1993.

[2] R. E. Bellman and L. A. Zadeh, "Decision-making in a fuzzy environment", *Management science*, vol. 17, no. 2, pp. 141-164, 1970.

[3] R. Goestschel and W. Voxman, "Elementary fuzzy calculus", *Fuzzy Sets and Systems*, vol. 18, no. 1, pp. 31-43, 1986.

[4] D. Hong and H. Do, "Additive decomposition of fuzzy quantities", *Information sciences*, vol. 88, no. 3, pp. 201-207, 1996.

[5] R. Horst, *Introduction to global optimization*, Singapore: Springer Science and Business Media, 2000.

[6] R. Jain, "Decision-making in the presence of fuzzy variables", *IEEE transactions on systems, man, and cybernetics*, vol.6, no. 1, pp. 698-703, 1976.

[7] K. D. Jamison, *A normed space of fuzzy equivalence classes*, UCD/CCM Report No. 112, 1997.

[8] A. N. Kolmogorov and S. V. Fomin, *Introductory Real Analysis*, New York, USA: Dover Publications, 1975.

[9] M. Mareš, "Addition of fuzzy quantities: Disjunction-conjunction apporach", *Kybernetika*, vol. 25, no.1, pp. 104-116, 1989.

[10] M. Mareš, "Additive decomposition of fuzzy quantities", *Fuzzy Sets and Systems*, vol. 47, no. 1 pp. 341-346, 1992.

[11] S. Nanda and K. Kar, "Convex fuzzy mappings", *Fuzzy Sets and Systems*, vol. 48, no. 1, pp. 129-132, 1992.

[12] Y. Pan, and Y. Shi, "A Grey Neural Network Model Optimized by Fruit Fly Optimization Algorithm for Short-term Traffic Forecasting," *Engineering Letters*, vol. 25, no.2, pp 198-204, 2017.

[13] G. Panda, M. Panigrahi and S. Nanda, "Equivalence class in the set of fuzzy numbers and its application in decision-making problems", *International journal of mathematics and mathematical sciences*, vol. 19, no. 1, pp. 741-765, 2006.

[14] D. Qiu, C. X. Lu, W. Zhang and Y. Y. Lan, "Algebreic properties and topological properties of the quotient space of fuzzy numbers based on Mareš equivalence relation", *Fuzzy Sets and Systems*, vol. 245, no. 2, pp. 63-82, 2014.

[15] D. Qiu, W. Zhang and C. X. Lu, "On fuzzy differential equations in the quotient space of fuzzy numbers", *Fuzzy Sets and Systems*, vol. 295, no. 2, pp. 72-98, 2016.

[16] D. Qiu and H. Li, "On convexity of fuzzy mappings and fuzzy optimization", *Italian Journal of Pure and Applied Mathematics*, vol. 35, no. 2, pp. 293-304, 2015.

[17] D. Qiu, C. Lu, and C. Mu, "Convergence of Successive Approximations for Fuzzy Differential Equations in The Quotient Space of Fuzzy Numbers," *IAENG International Journal of Applied Mathematics*, vol. 46, no.4, pp 512-517, 2016.

[18] D. Qiu, Y. Xing and L. Zhang, "Asymptotically stability of solutions of fuzzy differential equations in the quotient space of fuzzy numbers", *Journal of Computational Analysis and Applications*, vol. 23, no. 6, pp. 1242-1251, 2017.

[19] F. Renno and S. Papa, "Direct Modeling Approach to Improve Virtual Prototyping and FEM Analyses of Bicycle Frames," *Engineering Letters*, vol. 23, no.4, pp 333-341, 2015.

[20] M. Supriya, K. Sangeeta, and G.K. Patra, "Trustworthy Cloud Service Provider Selection using Multi Criteria Decision Making Methods," *Engineering Letters*, vol. 24, no.1, pp 1-10, 2016.
[21] Y. R. Syau, "On convex and concave fuzzy mappings", *Fuzzy Sets and Systems*, vol. 103, no. 1,pp. 163-168, 1999.
[22] Y. R. Syau, "Differentiability and convexity of fuzzy mappings", *Computer and Mathematics with Applications*, vol. 41, no. 1, pp. 73-81, 2001.
[23] F. Y. Vincent, H. T. X. Chi and L. Q. Dat, "Ranking generalized fuzzy numbers in fuzzy decision making based on the left and right transfer coefficients and areas", *Applied Mathematical Modelling*, vol. 37, no. 2, pp. 8106-8117, 2013.
[24] G. X. Wang and C. X. Wu, "Directional derivatives and subdifferential of convex fuzzy mappingsand application in convex fuzzy programming", *Fuzzy Sets and Systems*, vol. 138, no. 1, pp. 559-591, 2003.
[25] J. Wang and J. Song, "Chaotic Biogeography-based Optimization Algorithm," *IAENG International Journal of Computer Science*, vol. 44, no.2, pp 122-134, 2017.
[26] S. Wang, "A Manufacturer Stackelberg Game in Price Competition Supply Chain under a Fuzzy Decision Environment," IAENG International Journal of Applied Mathematics, vol. 47, no.1, pp 49-55, 2017.
[27] C. X. Wu and Z. Zhao, "Some notes on the characterization of compact sets of fuzzy sets with L_p metric", *Fuzzy Sets and Systems*, vol. 159, no. 2, pp. 2104-2115, 2008.
[28] H. Yan and J. Xu, "A class convex fuzzy mappings", *Fuzzy Sets and Systems*, vol. 129, no. 2, pp. 47-56, 2002.
[29] L. A. Zadeh, "Fuzzy sets", *Information and control*, vol. 8, no. 3, pp. 338-353, 1965.
[30] X. Zhang and C. Wu, "Energy Cost Minimization of a Compressor Station by Modified Genetic Algorithms," *Engineering Letters*, vol. 23, no.4, pp 258-268, 2015.
[31] L. Zhu, Y. Pan, M. Farahani and W. Gao, "Theoretical Characteristics on Scoring Function in Multi-dividing Setting," *IAENG International Journal of Applied Mathematics*, vol. 47, no.1, pp 28-36, 2017.