Taylor-Series Expansion Methods for Multivariate Hammerstein Integral Equations

Khomsan Neamprem, Apinya Klangrak, and Hideaki Kaneko

Abstract—A new Taylor-series method which was originally developed for the solution of one-dimensional integral equations is extended to solve multivariate nonlinear integral equations. In this paper, a new method is constructed to approximate the solutions of a class of multivariate Hammerstein equations. One of the strength of the new method is that it lends itself to parallel computation. Hence it is a very highly efficient method. Another strength of the proposed method is that it also gives highly accurate approximations for all the derivatives of the solution up to the order of the Taylor-series approximation used in the method. Numerical examples are given to illustrate the efficiency and accuracy of the method.

Index Terms—multivariate Hammerstein integral equation, Taylor-series expansion method, transformation technique.

I. Introduction

MANY science and engineering problems can often be converted to a mathematical model which can be represented as an integral. Once formulated as such, several numerical methods can be used to approximate the solution of an integral equation. Common methods used in application are the method of Galerkin, collocation and Nystrom. Also, an integral reformulation results in linear or nonlinear integral equations. The most well-known linear integral equations are Fredholm and Volterra equations. On the other hand, a well-known nonlinear integral equation is that of Hammerstein. The Hammerstein equation arises naturally as an integral equation reformulation of a class of two-point boundary value problem with nonlinear boundary terms.

As mentioned previously, numerical solutions of the Fredholm and Volterra integral equations can be approximated by the Galerkin method, the collocation method, the quadrature method such as Nystrom (see, e.g., [1], [2] and the references cited within). In 2014, Shekarabi et al. [3] proposed the Petrov-Galerkin method for solving the Stochastic Volterra Integral Equations. In 2015, Khumalo and Mamba [4] presented the numerical method for solving the Volterra integral equations using the collocation and quadrature methods. For these techniques, the integral equations are converted into a system of linear equations. However, the computational cost is usually high since the matrix is large and dense,

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particularly for multidimensional integral equations. In order to reduce the computational cost of numerically solving for integral equations, a large volume of research has been accomplished in the last two decades which apply a class of wavelets to numerical solution of integral equations (see [5], [6], [7], [8] and references cited therein). Alternatively, Taylor-series expansion method introduced by Yuhe Ren et al. (1999) [9] can be used to approximate the solution for a class of second kind integral equation including Fredholm integral equations and Volterra integral equations. In 2010 and 2011, Huabsomboon et al. ([10], [11]) improved Rens technique to approximate both solutions of integral equations and their derivatives. The results were shown to deliver numerical solutions whose accuracies are better than those reported in [9]. We note that the advantage of the Taylor-series expansion method is that approximation can be calculated locally, -i.e., pointwise, as opposed to the global approximation of the traditional methods of Galerkin, collocation and Nystrom. This feature of 'localness' is the key ingredient in enabling this new method to incorporate parallel computation. In 2012, Huabsomboon et al. [12] presented a new Taylor-series expansion method to solve the one-dimensional Hammerstein equation. This paper extended the new Taylor-series expansion method to solving nonlinear integral equations. A resulting system of nonlinear equations is solved by the nonlinear solver such as the Newton's method which reduces the computational cost. In 2012, Novaprateep et al. [13] presented solution process for solving the multivariate integral equations of the second kind by the Taylor-series expansion method. In this paper, our aim is to combine the ideas of [12] and [13] to solving the the multivariate Hammerstein integral equation. The equation takes the following form:

$$f(\mathbf{t}) - \int_I k(\mathbf{t}, \mathbf{s}) \psi(\mathbf{s}, f(\mathbf{s})) d\mathbf{s} = g(\mathbf{t}), \ \mathbf{t} \in I \qquad (1)$$

where k is the kernel of integral equations, g is an external forcing term, ψ is a nonlinear term and $I \in R^n$. This paper is organized as follow. In Section II, a traditional Taylor-series expansion method is applied directly to one-dimensional Hammerstein equation. Moreover, the modified Taylor-series method is briefly discussed. In Section III, we present the multivariate Hammerstein equation which first transform equation (1) into another nonlinear equation. Then a Taylor-series expansion method with multi-variables and Newton's method are applied for solving the corresponding nonlinear system. In Section IV, the numerical examples are implemented with specific cases.

II. A Taylor-Series Expansion Method for One-Dimensional Hammerstein Equation

In this section, we review briefly the traditional Taylor-series expansion method and the modified Taylor-series ex-

pansion method for one-dimensional Hammerstein equation, see [12]. The one-dimensional Hammerstein equation is given by

$$f(t) - \int_0^1 k(t,s)\psi(s, f(s))ds = g(t), \qquad 0 \leq t \leq 1, \quad (2)$$

where k, ψ and g are known functions and f is the function to be determined. This type of equation arises as an integral reformulation of the two-point boundary value problem with nonlinear boundary term. Then differentiate (2) n times with respect to t , we obtain

$$f'(t) - \int_0^1 k_t'(t,s)\psi(s, f(s))ds = g'(t)$$

$$f''(t) - \int_0^1 k_t''(t,s)\psi(s, f(s))ds = g''(t) \qquad (3)$$

$$\vdots \quad \vdots \quad \vdots$$

$$f^{(n)}(t) - \int_0^1 k_t^{(n)}(t,s)\psi(s, f(s))ds = g^{(n)}(t)$$

where $k_t^{(i)} = \partial^{(i)} k(s,t)/\partial t^i$, $i = 1, \ldots, n$.
Recall the Taylor-series expansion with one variable is

$$f(s) \approx f(t) + f'(t)(s-t) + \ldots + \frac{1}{n!}f^{(n)}(t)(s-t)^n. \quad (4)$$

Substituting (4) for $f(s)$ into (2) and (3), for $0 < t < 1$, we obtain

$$f(t) - \int_0^1 k(s,t)\psi\left(s, f(t) + \ldots + \frac{f^{(n)}(t)}{n!}(s-t)^n\right) ds$$
$$\approx g(t)$$

$$f'(t) - \int_0^1 k_t'(s,t)\psi\left(s, f(t) + \ldots + \frac{f^{(n)}(t)}{n!}(s-t)^n\right) ds$$
$$\approx g'(t)$$

$$\vdots \quad \vdots \quad \vdots$$

$$f^{(n)}(t) - \int_0^1 k_t^{(n)}(s,t)\psi\left(s, f(t) + \ldots \frac{f^{(n)}(t)}{n!}(s-t)^n\right) ds$$
$$\approx g^{(n)}(t).$$

The above system is the nonlinear system for solving $f(t), f'(t), \ldots, f^{(n)}(t)$. The nonlinear solver, such as the secant method or Newton's method, is required. Note that all the unknowns $f(t), f'(t), \ldots, f^{(n)}(t)$ appear under nonlinear term ψ. This necessitates the calculations of the integrals at each iteration of the nonlinear solver. To avoid this costly calculation, we present next the modified Taylor-series expansion method. The method enables us to execute the nonlinear solver such as Newton's method more efficiently. More specifically, the integrals of the form $\int_0^1 k_t^{(i)}(s,t)\frac{(s-t)^j}{j!}dt$ are calculated only once throughout the solution process and all the unknowns $f(t), f'(t), \ldots, f^{(n)}(t)$ appear outside the integral, which results in the speedup in the execution time of the nonlinear solver.
First, define

$$z(t) = \psi(t, f(t)). \qquad (5)$$

Then equation (2) transforms to

$$f(t) - \int_0^1 k(s,t)z(s)ds = g(t), \quad \text{for } 0 \leq t \leq 1. \qquad (6)$$

Using (2) and (5),

$$z(t) = \psi\left(t, g(t) + \int_0^1 k(s,t)z(s)ds\right), 0 \leq t \leq 1. \quad (7)$$

Substituting the Taylor series of $z(s)$, we obtain

$$z(t) \approx \psi\left(t, g(t) + z(t)\int_0^1 k(s,t)ds - \ldots\right.$$
$$\left. - \frac{z^{(n)}(t)}{n!}\int_0^1 k(s,t)(s-t)^n ds\right).$$

After differentiating (7) i times with respect to t, for $i = 1, 2, \ldots, n$, and substituting the Taylor series of $z(s)$, we obtain

$$z^{(i)}(t) \approx \frac{\partial^{(i)}}{\partial t^{(i)}}\psi\left(t, g(t) - z(t)\int_0^1 k(s,t)ds - \ldots\right.$$
$$\left. - \frac{z^{(n)}(t)}{n!}\int_0^1 k(s,t)(s-t)^n ds\right).$$

The above system is the nonlinear system for solving $z(t), z'(t), \ldots, z^{(n)}(t)$ and the nonlinear solver is required. When $z(t), z'(t), \ldots, z^{(n)}(t)$ are found, use (6) to solve the approximate solution $f(t)$ of (2).

III. A MODIFIED TAYLOR-SERIES EXPANSION METHOD FOR MULTIVARIATE HAMMERSTEIN EQUATION

Let $\mathbf{s}, \mathbf{t} \in R^n, I \subset R^n$ be compact. The multivariate Hammerstein equation is

$$f(\mathbf{t}) - \int_I k(\mathbf{t},\mathbf{s})\psi(\mathbf{s}, f(\mathbf{s}))d\mathbf{s} = g(\mathbf{t}), \mathbf{t} \in I. \qquad (8)$$

To circumvent this difficulty, we first transform (8) as follows by defining

$$z(\mathbf{t}) = \psi(\mathbf{t}, f(\mathbf{t})). \qquad (9)$$

So that (8) becomes

$$f(\mathbf{t}) - \int_I k(\mathbf{t},\mathbf{s})z(\mathbf{s})d\mathbf{s} = g(\mathbf{t}). \qquad (10)$$

Putting f in (10) into (9),

$$z(\mathbf{t}) = \psi\left(\mathbf{t}, g(\mathbf{t}) + \int_I k(\mathbf{t},\mathbf{s})z(\mathbf{s})d\mathbf{s}\right). \qquad (11)$$

Recall the Taylor series with multivariable of $z(\mathbf{s})$ expanded around $\mathbf{t}$ is

$$z(\mathbf{s}) = z(\mathbf{t}) + \sum_{k=1}^r \frac{1}{k!}D^k z(\mathbf{t}) \cdot (\underbrace{\mathbf{h}, \mathbf{h}, \ldots, \mathbf{h}}_{k \text{ times}}) + R_r(\mathbf{t}, \mathbf{c})$$

or

$$z(\mathbf{s}) = z(\mathbf{t}) +$$
$$\sum_{k=1}^r \frac{1}{k!}\left(\frac{\partial}{\partial t_1} + \ldots + \frac{\partial}{\partial t_n}\right)^k z(\mathbf{t}) \cdot (\underbrace{\mathbf{h}, \mathbf{h}, \ldots, \mathbf{h}}_{k \text{ times}}) + R_r(\mathbf{t}, \mathbf{c})$$
$$(12)$$

where $\mathbf{h} = \mathbf{s} - \mathbf{t}$, $\mathbf{c}$ is a point on the line segment $\mathbf{s} - \mathbf{t}$ and $R_r(\mathbf{t}, \mathbf{h})/\|\mathbf{h}\| \to 0, \mathbf{h} \in R^n, \|\mathbf{h}\| \to 0$.
Here, $Du(\mathbf{t})$ is the gradient of z at $\mathbf{t}$ and

$$D^2 z(\mathbf{t}) = \begin{bmatrix} \frac{\partial^2 z}{\partial t_1^2}(\mathbf{t}) & \frac{\partial^2 z}{\partial t_1 \partial t_2}(\mathbf{t}) & \cdots & \frac{\partial^2 z}{\partial t_1 \partial t_n}(\mathbf{t}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 z}{\partial t_n \partial t_1}(\mathbf{t}) & \frac{\partial^2 z}{\partial t_n \partial t_2}(\mathbf{t}) & \cdots & \frac{\partial^2 z}{\partial t_n^2(\mathbf{t})} \end{bmatrix}.$$

$D^i z(t)$, $i \geq 3$ are defined similarly. Substituting (12) into (11), we get

$$z(\mathbf{t}) \approx \psi\left(\mathbf{t}, g(\mathbf{t}) + \int_I k(\mathbf{t}, \mathbf{s})\left(z(\mathbf{t})\right.\right.$$

$$\left.\left. + \sum_{k=1}^{r} \frac{1}{k!}\left(\frac{\partial}{\partial t_1} + ... + \frac{\partial}{\partial t_n}\right)^k z(\mathbf{t}) \cdot (\underbrace{\mathbf{h}, \mathbf{h}, ..., \mathbf{h}}_{k \text{ times}})\right) ds\right)$$

or

$$z(\mathbf{t}) \approx \psi\left(\mathbf{t}, g(\mathbf{t}) + z(\mathbf{t})\int_I k(\mathbf{t}, \mathbf{s})d\mathbf{s}\right.$$

$$\left. + \sum_{k=1}^{r} \frac{1}{k!}\left(\frac{\partial}{\partial t_1} + ... + \frac{\partial}{\partial t_n}\right)^k z(\mathbf{t}) \cdot \int_I k(\mathbf{t}, \mathbf{s})(\underbrace{\mathbf{h}, \mathbf{h}, ..., \mathbf{h}}_{k \text{ times}})d\mathbf{s}\right).$$

After differentiating (11) i times with respect to t, for $i = 1, 2, \ldots, n$, we obtain

$$z^{(i)}(\mathbf{t}) \approx D^{(i)}\psi\left(\mathbf{t}, g(\mathbf{t}) + z(\mathbf{t})\int_I k(\mathbf{t}, \mathbf{s})d\mathbf{s}\right.$$

$$\left. + \sum_{k=1}^{r} \frac{1}{k!}\left(\frac{\partial}{\partial t_1} + ... + \frac{\partial}{\partial t_n}\right)^k z(\mathbf{t}) \cdot \int_I k(\mathbf{t}, \mathbf{s})(\underbrace{\mathbf{h}, \mathbf{h}, ..., \mathbf{h}}_{k \text{ times}})d\mathbf{s}\right).$$

The unknowns $z(\mathbf{t}), z'(\mathbf{t}), \ldots, z^{(n)}(\mathbf{t})$ are now placed outside the integrals so that the integrals of the form $\frac{1}{k!}\int_I k(\mathbf{t}, \mathbf{s})(\underbrace{\mathbf{h}, \mathbf{h}, ..., \mathbf{h}}_{k \text{ times}})d\mathbf{s}$ are computed once throughout iterations of a nonlinear solver such as the Newton's method. When $z(\mathbf{t})$ is found, $f(\mathbf{t})$ is obtained from substituting and solving (10).

A convergence analysis is the same as in the one-dimensional case which was presented in [12].

IV. NUMERICAL EXAMPLES

In this section, we present the four numerical examples using the modified Taylor-series expansion method. The computer programs are run on a personal computer with 2.8 GHz CPU and 8 GB memory.

Example 1: Consider the equation
$$f(t_1, t_2) - \int_0^1 \int_0^1 (t_1 + t_2 - s_1 - s_2)f^2(s_1, s_2)ds_1 ds_2$$
$$= g(t_1, t_2), \quad \{t_1, t_2\} \in [0, 1] \times [0, 1]$$
where $g(t_1, t_2)$ is chosen so that the isolated solution $f(t_1, t_2) = t_1 + 2t_2$. The numerical results using the Taylor-series expansion method with $r = 2$ can see in TABLE I.

In Example 1, the linear kernel is used, $t_1 + t_2 - s_1 - s_2$, and the solution is taken to be a linear polynomial. Thus, the current method with $r = 2$ delivers an approximation which is exact to the machine accuracy.

Example 2: Consider the equation
$$f(t_1, t_2) - \int_0^1 \int_0^1 \exp(-t_1 - t_2 - s_1 - s_2)f^2(s_1, s_2)ds_1 ds_2$$
$$= g(t_1, t_2), \quad \{t_1, t_2\} \in [0, 1] \times [0, 1]$$
where $g(t_1, t_2)$ is chosen so that the isolated solution $f(t_1, t_2) = t_1^2 + t_2^2$. The numerical results using the Taylor-series expansion method with $r = 5$ can see in TABLE II.

In Example 2, the exponential kernel is used, $\exp(-t_1 - t_2 - s_1 - s_2)$, and the solution is taken to be a polynomial of degree 2. Thus, the current method with $r = 5$ delivers an approximation which is exact to the machine accuracy.

TABLE I
COMPUTATIONAL RESULTS WITH $r = 2$ FOR EXAMPLE 1.

t_1	t_2	Exact	Approx.	Absolute Error
0.00	0.00	0.0000	0.0000	4.5918e-41
0.00	0.25	0.2500	0.2500	0.0000e+00
0.00	0.50	1.0000	1.0000	0.0000e+00
0.00	0.75	2.2500	2.2500	9.7700e-15
0.00	1.00	4.0000	4.0000	5.3291e-15
0.25	0.00	0.0625	0.0625	0.0000e+00
0.25	0.25	0.5625	0.5625	0.0000e+00
0.25	0.50	1.5625	1.5625	4.4409e-15
0.25	0.75	3.0625	3.0625	4.4409e-15
0.25	1.00	5.0625	5.0625	1.7764e-15
0.50	0.00	0.2500	0.2500	0.0000e+00
0.50	0.25	1.0000	1.0000	7.7716e-16
0.50	0.50	2.2500	2.2500	3.5527e-15
0.50	0.75	4.0000	4.0000	1.7764e-15
0.50	1.00	6.2500	6.2500	3.5527e-15
0.75	0.00	0.5625	0.5625	6.1062e-15
0.75	0.25	1.5625	1.5625	2.6645e-15
0.75	0.50	3.0625	3.0625	1.3323e-15
0.75	0.75	5.0625	5.0625	3.5527e-15
0.75	1.00	7.5625	7.5625	3.5527e-15
1.00	0.00	1.0000	1.0000	1.9984e-15
1.00	0.25	2.2500	2.2500	1.3323e-15
1.00	0.50	4.0000	4.0000	3.1086e-15
1.00	0.75	6.2500	6.2500	2.6645e-15
1.00	1.00	9.0000	9.0000	3.5527e-15

TABLE II
COMPUTATIONAL RESULTS WITH $r = 5$ FOR EXAMPLE 2.

t_1	t_2	Exact	Approx.	Absolute Error
0.00	0.00	0.0000	0.0000	2.5017e-28
0.00	0.25	0.0039	0.0039	1.1874e-15
0.00	0.50	0.0625	0.0625	2.9837e-15
0.00	0.75	0.3164	0.3164	1.3767e-14
0.00	1.00	1.0000	1.0000	1.7986e-14
0.25	0.00	0.0039	0.0039	1.1866e-15
0.25	0.25	0.0156	0.0156	1.4537e-15
0.25	0.50	0.0977	0.0977	2.6090e-15
0.25	0.75	0.3906	0.3906	9.8810e-15
0.25	1.00	1.1289	1.1289	3.5527e-15
0.50	0.00	0.0625	0.0625	3.4001e-15
0.50	0.25	0.0977	0.0977	2.5951e-15
0.50	0.50	0.2500	0.2500	2.1649e-15
0.50	0.75	0.6602	0.6602	9.2149e-15
0.50	1.00	1.5625	1.5625	3.5527e-15
0.75	0.00	0.3164	0.3164	1.3711e-14
0.75	0.25	0.3906	0.3906	9.7700e-15
0.75	0.50	0.6602	0.6602	9.2149e-15
0.75	0.75	1.2656	1.2656	1.1546e-14
0.75	1.00	2.4414	2.4414	8.8818e-16
1.00	0.00	1.0000	1.0000	1.7764e-14
1.00	0.25	1.1289	1.1289	2.4425e-15
1.00	0.50	1.5625	1.5625	1.0214e-14
1.00	0.75	2.4414	2.4414	8.8818e-15
1.00	1.00	4.0000	4.0000	1.1546e-14

Example 3: Consider the equation
$$f(t_1, t_2) - \int_0^1 \int_0^1 \sin(t_1 + t_2 - s_1 - s_2)f^2(s_1, s_2)ds_1 ds_2$$
$$= g(t_1, t_2), \quad \{t_1, t_2\} \in [0, 1] \times [0, 1]$$
where $g(t_1, t_2)$ is chosen so that the isolated solution $f(t_1, t_2) = t_1 + t_2$. The numerical results using the Taylor-series expansion method with $r = 5$ can see in TABLE III. In Example 3, the trigonometric sine kernel is used, $\sin(t_1 + t_2 - s_1 - s_2)$, and the solution is taken to be a polynomial of degree 1. Thus, the current method with $r = 5$ delivers an approximation which is exact to the machine accuracy.

TABLE III
COMPUTATIONAL RESULTS WITH $r = 5$ FOR EXAMPLE 3.

t_1	t_2	Exact	Approx.	Absolute Error
0.00	0.00	0.0000	0.0000	1.5941e-32
0.00	0.25	0.0625	0.0625	2.7756e-17
0.00	0.50	0.2500	0.2500	0.0000e+00
0.00	0.75	0.5625	0.5625	0.0000e+00
0.00	1.00	1.0000	1.0000	0.0000e+00
0.25	0.00	0.0625	0.0625	1.1796e-16
0.25	0.25	0.2500	0.2500	0.0000e+00
0.25	0.50	0.5625	0.5625	2.2204e-15
0.25	0.75	1.0000	1.0000	2.2204e-16
0.25	1.00	1.5625	1.5625	2.2204e-16
0.50	0.00	0.2500	0.2500	1.1102e-16
0.50	0.25	0.5625	0.5625	0.0000e+00
0.50	0.50	1.0000	1.0000	2.8866e-15
0.50	0.75	1.5625	1.5625	0.0000e+00
0.50	1.00	2.2500	2.2500	0.0000e+00
0.75	0.00	0.5625	0.5625	1.1102e-16
0.75	0.25	1.0000	1.0000	0.0000e+00
0.75	0.50	1.5625	1.5625	4.8850e-15
0.75	0.75	2.2500	2.2500	0.0000e+00
0.75	1.00	3.0625	3.0625	0.0000e+00
1.00	0.00	1.0000	1.0000	3.3307e-16
1.00	0.25	1.5625	1.5625	4.4409e-16
1.00	0.50	2.2500	2.2500	9.3259e-15
1.00	0.75	3.0625	3.0625	0.0000e+00
1.00	1.00	4.0000	4.0000	0.0000e+00

Next, we show the comparison of numerical experimentations with several orders of Taylor-series expansion method.

Example 4: Consider the equation
$$f(t_1, t_2) - \int_0^1 \int_0^1 \exp(-t_1 - t_2 - s_1 - s_2) f^2(s_1, s_2) ds_1 ds_2$$
$$= g(t_1, t_2), \quad \{t_1, t_2\} \in [0,1] \times [0,1]$$
where $g(t_1, t_2)$ is chosen so that the isolated solution $f(t_1, t_2) = \sin(t_1 + t_2)$. The numerical results using the Taylor-series expansion method with $r = 4, r = 5$ and $r = 6$ can see in TABLE IV - TABLE VI.

TABLE IV
COMPUTATIONAL RESULTS WITH $r = 4$ FOR EXAMPLE 4.

t_1	t_2	Exact	$r = 4$	
			Approx.	Absolute Error
0.00	0.00	0.0000	0.0024	2.3975e-3
0.00	0.25	0.0612	0.0494	1.1821e-2
0.00	0.50	0.2298	0.2224	7.4808e-3
0.00	0.75	0.4646	0.4627	1.9313e-3
0.00	1.00	0.7081	0.7096	1.5660e-3
0.25	0.00	0.0612	0.0494	1.1821e-2
0.25	0.25	0.2298	0.2224	7.4808e-3
0.25	0.50	0.4646	0.4627	1.9313e-3
0.25	0.75	0.7081	0.7096	1.5660e-3
0.25	1.00	0.9006	0.9067	6.1777e-3
0.50	0.00	0.2298	0.2224	7.4808e-3
0.50	0.25	0.4646	0.4627	1.9313e-3
0.50	0.50	0.7081	0.7096	1.5660e-3
0.50	0.75	0.9006	0.9067	6.1777e-3
0.50	1.00	0.9950	1.0053	1.0353e-2
0.75	0.00	0.4646	0.4627	1.9313e-3
0.75	0.25	0.7081	0.7096	1.5660e-3
0.75	0.50	0.9006	0.9067	6.1777e-3
0.75	0.75	0.9950	1.0053	1.0353e-2
0.75	1.00	0.9682	0.9884	2.0193e-2
1.00	0.00	0.7081	0.7096	1.5660e-3
1.00	0.25	0.9006	0.9067	6.1777e-3
1.00	0.50	0.9950	1.0053	1.0353e-2
1.00	0.75	0.9682	0.9884	2.0193e-2
1.00	1.00	0.8268	0.8637	3.6860e-2

TABLE V
COMPUTATIONAL RESULTS WITH $r = 5$ FOR EXAMPLE 4.

t_1	t_2	Exact	$r = 5$	
			Approx.	Absolute Error
0.00	0.00	0.0000	0.0022	2.1817e-3
0.00	0.25	0.0612	0.0570	4.2257e-3
0.00	0.50	0.2298	0.2291	7.0756e-4
0.00	0.75	0.4646	0.4648	1.2036e-4
0.00	1.00	0.7081	0.7082	1.6443e-4
0.25	0.00	0.0612	0.0570	4.2257e-3
0.25	0.25	0.2298	0.2291	7.0756e-4
0.25	0.50	0.4646	0.4648	1.2036e-4
0.25	0.75	0.7081	0.7082	1.6443e-4
0.25	1.00	0.9006	0.9021	1.4882e-3
0.50	0.00	0.2298	0.2291	7.0756e-4
0.50	0.25	0.4646	0.4648	1.2036e-4
0.50	0.50	0.7081	0.7082	1.6443e-4
0.50	0.75	0.9006	0.9021	1.4882e-3
0.50	1.00	0.9950	1.0022	7.2064e-3
0.75	0.00	0.4646	0.4648	1.0236e-4
0.75	0.25	0.7081	0.7082	1.6443e-4
0.75	0.50	0.9006	0.9021	1.4882e-3
0.75	0.75	0.9950	1.0022	7.2064e-3
0.75	1.00	0.9682	0.9719	3.7189e-3
1.00	0.00	0.7081	0.7082	1.6443e-4
1.00	0.25	0.9006	0.9021	1.4882e-3
1.00	0.50	0.9950	1.0022	7.2064e-3
1.00	0.75	0.9682	0.9719	3.7189e-3
1.00	1.00	0.8268	0.8015	2.5345e-2

TABLE VI
COMPUTATIONAL RESULTS WITH $r = 6$ FOR EXAMPLE 4.

t_1	t_2	Exact	$r = 6$	
			Approx.	Absolute Error
0.00	0.00	0.0000	0.0009	9.1384e-4
0.00	0.25	0.0612	0.0620	1.0893e-3
0.00	0.50	0.2298	0.2295	3.9584e-4
0.00	0.75	0.4646	0.4646	8.4802e-5
0.00	1.00	0.7081	0.7081	7.2459e-5
0.25	0.00	0.0612	0.0620	1.0893e-3
0.25	0.25	0.2298	0.2297	1.5892e-4
0.25	0.50	0.4646	0.4646	8.4802e-5
0.25	0.75	0.7081	0.7081	7.2459e-5
0.25	1.00	0.9006	0.9014	8.9358e-4
0.50	0.00	0.2298	0.2297	1.5892e-4
0.50	0.25	0.4646	0.4646	8.4802e-5
0.50	0.50	0.7081	0.7081	6.2459e-5
0.50	0.75	0.9006	0.9014	8.9358e-4
0.50	1.00	0.9950	0.9959	9.4391e-4
0.75	0.00	0.4646	0.4646	8.4802e-5
0.75	0.25	0.7081	0.7081	6.2459e-5
0.75	0.50	0.9006	0.9014	8.9358e-4
0.75	0.75	0.9950	0.9959	9.4391e-4
0.75	1.00	0.9682	0.9677	5.3205e-4
1.00	0.00	0.7081	0.7081	7.2459e-5
1.00	0.25	0.9006	0.9010	4.9384e-4
1.00	0.50	0.9950	0.9959	9.4391e-4
1.00	0.75	0.9682	0.9677	5.3205e-4
1.00	1.00	0.8268	0.8282	1.4873e-3
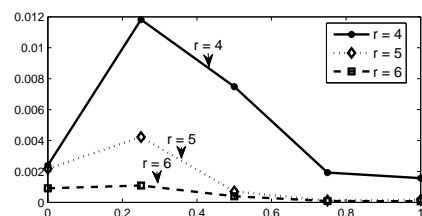
Fig. 1. The comparison of the absolute errors with fixed $t_1 = 0.00$.

In Example 4, the exponential kernel is used, $\exp(-t_1 - t_2 - s_1 - s_2)$, and the solution is taken to be a nonlinear term, $\sin(t_1 + t_2)$. From TABLE VII, the comparison of numerical experimentations with $r = 4, r = 5$ and $r = 6$ of Taylor-series expansion method is shown. Thus, we found that the numerical errors decreased when order of Taylor-series expansion increased for each point. Moreover, we may see it from Fig. 1 to Fig. 3.

TABLE VII
COMPARISON BETWEEN $r = 4$, $r = 5$ AND $r = 6$ FOR EXAMPLE 4.

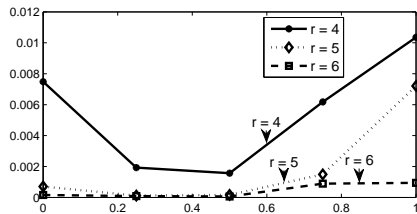| t_1	t_2	Absolute Error		
		$r = 4$	$r = 5$	$r = 6$
0.00	0.00	2.3975e-3	2.1817e-3	9.1384e-4
0.00	0.25	1.1821e-2	4.2257e-3	1.0893e-3
0.00	0.50	7.4808e-3	7.0756e-4	3.9584e-4
0.00	0.75	1.9313e-3	1.2036e-4	8.4802e-5
0.00	1.00	1.5660e-3	1.6443e-4	7.2459e-5
0.25	0.00	1.1821e-2	4.2257e-3	1.0893e-3
0.25	0.25	7.4808e-3	7.0756e-4	1.5892e-4
0.25	0.50	1.9313e-3	1.2036e-4	8.4802e-5
0.25	0.75	1.5660e-3	1.6443e-4	7.2459e-5
0.25	1.00	6.1777e-3	1.4882e-3	8.9358e-4
0.50	0.00	7.4808e-3	7.0756e-4	1.5892e-4
0.50	0.25	1.9313e-3	1.2036e-4	8.4802e-5
0.50	0.50	1.5660e-3	1.6443e-4	6.2459e-5
0.50	0.75	6.1777e-3	1.4882e-3	8.93584-4
0.50	1.00	1.0353e-2	7.2064e-3	9.4391e-4
0.75	0.00	1.9313e-3	1.0236e-4	8.4802e-5
0.75	0.25	1.5660e-3	1.6443e-4	6.2459e-5
0.75	0.50	6.1777e-3	1.4882e-3	8.9358e-4
0.75	0.75	1.0353e-2	7.2064e-3	9.4391e-4
0.75	1.00	2.0193e-2	3.7189e-3	5.3205e-4
1.00	0.00	1.5660e-3	1.6443e-4	7.2459e-5
1.00	0.25	6.1777e-3	1.4882e-3	4.9384e-4
1.00	0.50	1.0353e-2	7.2064e-3	9.4391e-4
1.00	0.75	2.0193e-2	3.7189e-3	5.3205e-4
1.00	1.00	3.6860e-2	2.5345e-2	1.4873e-3

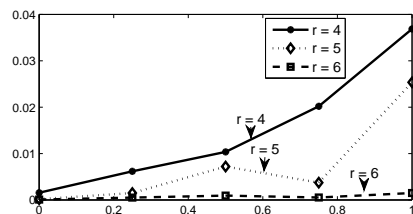Fig. 2. The comparison of the absolute errors with fixed $t_1 = 0.50$.

Fig. 3. The comparison of the absolute errors with fixed $t_1 = 1.00$.

V. CONCLUSION

In this paper, we have extended the idea of the modified Taylor-series expansion method from one dimensional Hammerstein equation to multivariate Hammerstein equation. Then we have applied it to find the numerical solution of multivariate Hammerstein equation. This new method finds an approximation of the solution pointwise and therefore lends itself to numerical computations which can be done in parallel. The method also computes the derivatives of the solution concurrently. The results of several numerical experiments have shown the efficiency and accuracy of this new method.

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