A New C^2 Piecewise Bivariate Rational Interpolation Scheme with Bi-quadratic Denominator

Xiangbin Qin and Qingsong Xu

***Abstract*—We present a new C^2 piecewise bivariate rational interpolation scheme with bi-quadratic denominator. The given interpolation scheme based only on the values of the interpolated function which includes two steps: in the first step, we construct a kind of x-direction C^2 interpolation curves based on a new Hermite-type rational interpolation basis; in the second step, by using another new class of Hermite-type rational interpolation basis to interpolate the generated x-direction interpolation curves, a kind of piecewise bivariate rational interpolation surfaces with bi-quadratic denominator and two parameters is established in a rectangular domain. The conditions for the interpolation surface to be C^2 continuous in the whole rectangular domain are given in detail. And the interpolation surface is proved to be bounded and its error formula is provided. Several numerical examples are given and the numerical results show that the given interpolation scheme is effective and practical.**

***Index Terms*—Hermite-type rational interpolation basis, Interpolation surface, C^2 continuity, Bounded property, Error estimate**

I. Introduction

In Computer Aided Geometric Design (CAGD), Computer Graphics (CG) and scientific data visualization, constructing smooth interpolation surfaces to given data in rectangular grid is an essential issue. Generally speaking, for most applications, C^1 smoothness is sufficient, and there are many schemes to tackle this problem, see for example the classical Coos surface schemes [1], the bivariate rational interpolation schemes [2], [3], [4], [5], the bi-cubic blending rational interpolation schemes [6], [7], [8], the bivariate rational Hermite interpolation schemes [9], and the rational trigonometric interpolation schemes [10]. In some practical applications, curvature continuity is needed sometimes and this leads to the need for C^2 smoothness.

By using the classical Coons surface scheme, it is a more difficult task to construct C^2 interpolation surfaces for 3D data defined over rectangular grid. For example, for generating a C^2 bi-quintic Coons surface, there need to provide the second and higher mixed partial derivatives at the data points in advance. In practical applications, however, the second and higher mixed partial derivatives are hard to estimate and control, and there may also exist compatibility problem in generating the classical C^2 bi-quintic Coons surface, see [11]. Recently, a class of rational interpolation spline with bi-cubic denominator and two parameters was constructed in [12]. For generating interpolation surfaces, the given interpolant only use the values of the interpolated function and can be C^2 continuous for equally spaced knots. And the shape of the generated C^2 interpolation surfaces can be modified conveniently by using the parameters for the unchanged interpolating data.

The purpose of this paper is to present a class of piecewise bivariate rational interpolation surface scheme with bi-quadratic denominator and two parameters over rectangular domain. The given interpolation surface can be C^2 continuous in the whole rectangular domain without using the second or higher mixed partial derivatives at the knots. The values of the generated interpolation surface are bounded and stable no matter what the parameters might be. It improves on the existing schemes in some ways: (1) The classical C^2 bi-quintic Coons surface have to estimate the second or higher mixed partial derivatives at the knots in advance, while the given C^2 interpolation surface is based on the interpolated function only; (2) Compared with the rational interpolation spline with bi-cubic denominator developed in [12], the given interpolation scheme with bi-quadratic denominator has less computational cost. The rest of this paper is organized as follows. In section II, the construction of the new piecewise bivariate rational interpolation scheme is described. section III discusses the properties of the interpolation surface in detail, including C^2 continuity property, bounded property, and error formula. In section IV, several numerical examples are given to prove the effectiveness and practicability of the new developed schemes. Conclusion is given in the section V.

II. New piecewise bivariate rational interpolation scheme

In this section, we firstly construct two classes of Hermite-type interpolation basis functions. Then we use one of the two classes of Hermite-type interpolation basis functions to construct a kind of C^2 x-direction interpolation curve with a parameter. Based on this, by using another new kind of Hermite-type interpolation basis functions to interpolate the x-direction interpolation curve, we construct a class of C^2 piecewise bivariate rational interpolation surface scheme with bi-quadratic denominator and two local parameters in a rectangular domain.

A. Two new classes of Hermite-type rational interpolation basis functions

Firstly, for $t, s \in [0, 1]$, we construct two new classes of Hermite-type rational interpolation basis functions $H_k(t; \alpha)$

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Xiangbin Qin, Qingsong Xu are with the Department of School of Mathematics and Statistics, Central South University, Changsha 410083, PR China (e-mail: xiangbinqin@163.com; qsxu@csu.edu.cn).

and $G_k(s;\beta)$, $k=0,1,2,3$ respectively as follows

$$\begin{array}{l} H_0(t;\alpha)=\frac{(1-t)^2+\alpha(1-t)^3t(1+2t)}{(1-t)^2+\alpha(1-t)t+t^2},\\ H_1(t;\alpha)=\frac{t^2+\alpha(1-t)t^3(3-2t)}{(1-t)^2+\alpha(1-t)t+t^2},\\ H_2(t;\alpha)=\frac{(1-t)^3t+\alpha(1-t)^3t^2}{(1-t)^2+\alpha(1-t)t+t^2},\\ H_3(t;\alpha)=\frac{-\left[(1-t)t^3+\alpha(1-t)^2t^3\right]}{(1-t)^2+\alpha(1-t)t+t^2}, \end{array}$$

with $\alpha\geq 0$ and

$$\begin{array}{l} G_0(s;\beta)=\frac{(1-s)^3\left(1+s+2s^2\right)+\beta(1-s)^3s(1+2s)}{(1-s)^2+\beta(1-s)s+s^2},\\ G_1(s;\beta)=\frac{s^3\left(4-5s+2s^2\right)+\beta(1-s)s^3(3-2s)}{(1-s)^2+\beta(1-s)s+s^2},\\ G_2(s;\beta)=\frac{(1-s)^3s(1+s)+\beta(1-s)^3s^2}{(1-s)^2+\beta(1-s)s+s^2},\\ G_3(s;\beta)=\frac{-\left[(1-s)s^3(2-s)+\beta(1-s)^2s^3\right]}{(1-s)^2+\beta(1-s)s+s^2}, \end{array}$$

with $\beta\geq 0$.

For the two classes of Hermite-type rational interpolation basis functions $H_k(t;\alpha)$ and $G_k(s;\beta)$, $k=0,1,2,3$, by directly computing, we can obtain the following important end-point properties

$$\begin{array}{l} H_0(0;\alpha)=1,\ H_0'(0;\alpha)=0,\ H_0''(0;\alpha)=-2,\\ H_0(1;\alpha)=0,\ H_0'(1;\alpha)=0,\ H_0''(1;\alpha)=2,\\ H_1(0;\alpha)=0,\ H_1'(0;\alpha)=0,\ H_1''(0;\alpha)=2,\\ H_1(1;\alpha)=1, H_1'(1;\alpha)=0,\ H_1''(1;\alpha)=-2,\\ H_2(0;\alpha)=0,\ H_2'(0;\alpha)=1,\ H_2''(0;\alpha)=-2,\\ H_2(1;\alpha)=0,\ H_2'(1;\alpha)=0,\ H_2''(1;\alpha)=0,\\ H_3(0;\alpha)=0,\ H_3'(0;\alpha)=0,\ H_3''(0;\alpha)=0,\\ H_3(1;\alpha)=0,\ H_3'(1;\alpha)=1,\ H_3''(1;\alpha)=2, \end{array}$$

and

$$\begin{array}{l} G_0(0;\beta)=1,\ G_0'(0;\beta)=0,\ G_0''(0;\beta)=0,\\ G_0(1;\beta)=0,\ G_0'(1;\beta)=0,\ G_0''(1;\beta)=0,\\ G_1(0;\beta)=0,\ G_1'(0;\beta)=0,\ G_1''(0;\beta)=0,\\ G_1(1;\beta)=1, G_1'(1;\beta)=0,\ G_1''(1;\beta)=0,\\ G_2(0;\beta)=0,\ G_2'(0;\beta)=1,\ G_2''(0;\beta)=0,\\ G_2(1;\beta)=0,\ G_2'(1;\beta)=0,\ G_2''(1;\beta)=0,\\ G_3(0;\beta)=0,\ G_3'(0;\beta)=0,\ G_3''(0;\beta)=0,\\ G_3(1;\beta)=0,\ G_3'(1;\beta)=1,\ G_3''(1;\beta)=0. \end{array}$$

For any $t,s\in[0,1]$, it is easy to check that $H_0(t;\alpha)+H_1(t;\alpha)=1$ and $G_0(s;\beta)+G_1(s;\beta)=1$.

It is interesting to note that for a large value of α and β , the given Hermite-type rational interpolation basis functions $H_k(t;\alpha)$ and $G_k(s;\beta)$, $k=0,1,2,3$ will give approximation to the standard cubic Hermite interpolation basis functions.

B. Bivariate rational interpolation scheme

Let $\{(x_i,y_i,F_{ij}), i=1,2,\ldots,n; j=1,2,\ldots,m\}$ be a given set of data points defined over the rectangular domain $R=[x_1,x_n]\times[y_1,y_m]$, where $\pi_x: x_1<x_2<\ldots<x_n$ is the partition of $[x_1,x_n]$ and $\pi_y: y_1<y_2<\ldots<y_m$ is the partition of $[y_1,y_m]$. $D^x_{i,j}$ and $D^y_{i,j}$ are known as the first partial derivatives at the grid point (x_i,y_j). Denote $h^x_i=x_{i+1}-x_i$, $h^y_j=y_{j+1}-y_j$, $R_{i,j}=[x_i,x_{i+1}]\times[y_j,y_{j+1}]$, and for any $(x,y)\in R_{i,j}$, let $t=(x-x_i)/h^x_i$, $s=(y-y_j)/h^y_j$, and

$$\Delta^x_{i,j}=\frac{F_{i+1,j}-F_{i,j}}{h^x_i},\quad \Delta^y_{i,j}=\frac{F_{i,j+1}-F_{i,j}}{h^y_j}.$$

For each $y=y_j$, $j=1,2,\ldots,m$, and $x\in[x_i,x_{i+1}]$, by using the new Hermite-type rational interpolation basis functions $H_k(t;\alpha)$, $k=0,1,2,3$ given in the previous subsection, we construct a kind of x-direction interpolation curve with a free parameters $\alpha^x_{i,j}$ as follows

$$\begin{array}{l} P^*_{i,j}(x)=H_0(t;\alpha^x_{i,j})F_{i,j}+H_1(t;\alpha^x_{i,j})F_{i+1,j}\\ \quad +H_2(t;\alpha^x_{i,j})h^x_iD^x_{i,j}+H_3(t;\alpha^x_{i,j})h^x_iD^x_{i+1,j}, \end{array}\tag{1}$$

where $\alpha^x_{i,j}\geq 0$.

From the end-point properties of the Hermite-type rational interpolation basis functions $H_k(t;\alpha)$, $k=0,1,2,3$, we have

$$\begin{array}{l} P^*_{i,j}(x_i^+)=F_{i,j},\ {P^*_{i,j}}'(x_i^+)=D^x_{i,j},\\ {P^*_{i,j}}''(x_i^+)=\frac{2\left(\Delta^x_{i,j}-D^x_{i,j}\right)}{h^x_i},\ P^*_{i,j}(x_{i+1}^-)=F_{i+1,j},\\ {P^*_{i,j}}'(x_{i+1}^-)=D^x_{i+1,j},\ {P^*_{i,j}}''(x_{i+1}^-)=\frac{2\left(D^x_{i+1,j}-\Delta^x_{i,j}\right)}{h^x_i}. \end{array}$$

Thus, we can see that if the first partial derivative values $D^x_{i,j}$, $i=2,3,\ldots,n-1$ are chosen as follows

$$D^x_{i,j}=\frac{h^x_{i-1}\Delta^x_{i,j}\qquad+h^x_i\Delta^x_{i-1,j}}{h^x_{i-1}+h^x_i},\tag{2}$$

then for $i=2,3,\ldots,n-1$, we have

$$\begin{array}{l} P^*_{i,j}(x_i^+)=P^*_{i,j}(x_i^-)=F_{i,j},\\ {P^*_{i,j}}'(x_i^+)={P^*_{i,j}}'(x_i^-)=D^x_{i,j},\\ {P^*_{i,j}}''(x_i^+)={P^*_{i,j}}''(x_i^-)=\frac{2\left(\Delta^x_{i,j}-\Delta^x_{i-1,j}\right)}{h^x_{i-1}+h^x_i}, \end{array}$$

which implies that the resulting interpolation function $P^*_{i,j}(x)$ defined by (1) is C^2 continuous in $[x_1,x_n]$. At the end knots x_1 and x_n, the derivative values are computed by the following formulas

$$\begin{array}{l} D^x_{1,j}=\Delta^x_{1,j}-\frac{h^x_1}{h^x_1+h^x_2}\left(\Delta^x_{2,j}-\Delta^x_{1,j}\right),\\ D^x_{n,j}=\Delta^x_{n-1,j}+\frac{h^x_{n-1}}{h^x_{n-2}+h^x_{n-1}}\left(\Delta^x_{n-1,j}-\Delta^x_{n-2,j}\right). \end{array}\tag{3}$$

For any $(x,y)\in R_{i,j}$, $i=1,2,\ldots,n-1$, $j=1,2,\ldots,m-1$, we further use the x-direction interpolant $P^*_{i,j}(x)$ given in (1) to construct a new kind of piecewise bivariate rational interpolation surfaces $P_{i,j}(x,y)$ as follows

$$\begin{array}{l} P_{i,j}(x,y)=G_0(s;\beta^y_{i,j})P^*_{i,j}(x)+G_1(s;\beta^y_{i,j})P^*_{i,j+1}(x)\\ \quad +G_2(s;\beta^y_{i,j})h^y_j\phi_{i,j}(x)+G_3(s;\beta^y_{i,j})h^y_j\phi_{i,j+1}(x), \end{array}\tag{4}$$

where the four Hermite-type interpolation basis functions $G_k(s;\beta^y_{i,j})$ with $\beta^y_{i,j}\geq 0$, $k=0,1,2,3$ are given in the previous subsection, and the functions $\phi_{i,l}(x), l=j,j+1$ are given by

$$\begin{array}{l} \phi_{i,l}(x)=(1-t)^3\left(1+4t+9t^2\right)D^y_{i,l}\\ \quad +t^3\left(6-8t+3t^2\right)D^y_{i+1,l}, \end{array}$$

where $l=j,j+1$.

From (1) and (4), after some manipulations, we can also rewrite the interpolation surfaces $P_{i,j}(x,y)$ as the following form

$$\begin{array}{l} P_{i,j}(x,y)=\sum_{k=i}^{i+1}\sum_{l=j}^{j+1}\Big[a_{k,l}(t,s)F_{k,l}+b_{k,l}(t,s)h^x_iD^x_{k,l}\\ \qquad +c_{k,l}(t,s)h^y_jD^y_{k,l}\Big] \end{array}\tag{5}$$

where

$$\begin{aligned}
&a_{i,j}(t,s)=H_0\left(t;\alpha_{i,j}^x\right)G_0\left(s;\beta_{i,j}^{y}\right),\\
&a_{i,j+1}(t,s)=H_0\left(t;\alpha_{i,j+1}^x\right)G_1\left(s;\beta_{i,j}^{y}\right),\\
&a_{i+1,j}(t,s)=H_1\left(t;\alpha_{i,j}^x\right)G_0\left(s;\beta_{i,j}^{y}\right),\\
&a_{i+1,j+1}(t,s)=H_1\left(t;\alpha_{i,j+1}^x\right)G_1\left(s;\beta_{i,j}^{y}\right),\\
&b_{i,j}(t,s)=H_2\left(t;\alpha_{i,j}^x\right)G_0\left(s;\beta_{i,j}^{y}\right),\\
&b_{i,j+1}(t,s)=H_2\left(t;\alpha_{i,j+1}^x\right)G_1\left(s;\beta_{i,j}^{y}\right),\\
&b_{i+1,j}(t,s)=H_3\left(t;\alpha_{i,j}^x\right)G_0\left(s;\beta_{i,j}^{y}\right),\\
&b_{i+1,j+1}(t,s)=H_3\left(t;\alpha_{i,j+1}^x\right)G_1\left(s;\beta_{i,j}^{y}\right),\\
&c_{i,j}(t,s)=(1-t)^3\left(1+4t+9t^2\right)G_2\left(s;\beta_{i,j}^{y}\right),\\
&c_{i,j+1}(t,s)=(1-t)^3\left(1+4t+9t^2\right)G_3\left(s;\beta_{i,j}^{y}\right),\\
&c_{i+1,j}(t,s)=t^3\left(6-8t+3t^2\right)G_2\left(s;\beta_{i,j}^{y}\right),\\
&c_{i+1,j+1}(t,s)=t^3\left(6-8t+3t^2\right)G_3\left(s;\beta_{i,j}^{y}\right).
\end{aligned}$$

We call the terms $a_{k,l}, b_{k,l}$ and $c_{k,l}$, $k=i,i+1$, $l=j,j+1$, as the interpolation basis functions of the interpolation surface defined by (5).

III. Properties of the interpolation surfaces

In this section, we shall discuss the properties of the interpolation surfaces in detail, including the C^2 continuous property, the bounded property, and the error formula.

A. C^2 continuity property

For any $(x,y)\in R_{i,j}$, from the interpolation surface $P_{i,j}(x,y)$ given in (4), direct computation gives that

$$\begin{aligned}
&P_{i,j}\left(x,y_j^+\right)=P_{i,j}^*(x),\quad P_{i,j}\left(x,y_{j+1}^-\right)=P_{i,j+1}^*(x),\\
&P_{i,j}(x_i^+,y)=G_0(s;\beta_{i,j}^y)F_{i,j}+G_1(s;\beta_{i,j}^y)F_{i,j+1}\\
&\quad+G_2(s;\beta_{i,j}^y)h_j^yD_{i,j}^y+G_3(s;\beta_{i,j}^y)h_j^yD_{i,j+1}^y,\\
&P_{i,j}(x_{i+1}^-,y)=G_0(s;\beta_{i,j}^y)F_{i+1,j}+G_1(s;\beta_{i,j}^y)F_{i+1,j+1}\\
&\quad+G_2(s;\beta_{i,j}^y)h_j^yD_{i+1,j}^y+G_3(s;\beta_{i,j}^y)h_j^yD_{i+1,j+1}^y,
\end{aligned}$$

Thus we have $P_{i,j}\left(x,y_j^+\right)=P_{i,j}\left(x,y_j^-\right), P_{i,j}(x_i^+,y)=P_{i,j}(x_i^-,y)$.

Furthermore,

$$\begin{aligned}
&\frac{\partial P_{i,j}\left(x,y_j^+\right)}{\partial x}=\frac{dP_{i,j}^*(x)}{dx},\quad \frac{\partial P_{i,j}\left(x,y_{j+1}^-\right)}{\partial x}=\frac{dP_{i,j+1}^*(x)}{dx},\\
&\frac{\partial P_{i,j}\left(x_i^+,y\right)}{\partial x}=G_0(s;\beta_{i,j}^y)D_{i,j}^x+G_1(s;\beta_{i,j}^y)D_{i,j+1}^x\\
&\quad+G_2(s;\beta_{i,j}^y)h_j^y\frac{D_{i,j}^y}{h_i^x}+G_3(s;\beta_{i,j}^y)h_j^y\frac{D_{i,j+1}^y}{h_i^x},\\
&\frac{\partial P_{i,j}\left(x_{i+1}^-,y\right)}{\partial x}=G_0(s;\beta_{i,j}^y)D_{i+1,j}^x+G_1(s;\beta_{i,j}^y)D_{i+1,j+1}^x\\
&\quad+G_2(s;\beta_{i,j}^y)h_j^y\frac{D_{i+1,j}^y}{h_i^x}+G_3(s;\beta_{i,j}^y)h_j^y\frac{D_{i+1,j+1}^y}{h_i^x},
\end{aligned}$$

thus we have $\frac{\partial P_{i,j}\left(x,y_j^+\right)}{\partial x}=\frac{\partial P_{i,j}\left(x,y_j^-\right)}{\partial x}$ and $\frac{\partial P_{i,j}\left(x_i^+,y\right)}{\partial x}=\frac{\partial P_{i,j}\left(x_i^-,y\right)}{\partial x}$ if $h_{i-1}^x=h_i^x$ and $\beta_{i-1,j}^y=\beta_{i,j}^y$.

Similarly, we have

$$\begin{aligned}
&\frac{\partial P_{i,j}\left(x,y_j^+\right)}{\partial y}=\phi_{i,j}(x),\quad \frac{\partial P_{i,j}\left(x,y_{j+1}^-\right)}{\partial y}=\phi_{i,j+1}(x),\\
&\frac{\partial P_{i,j}\left(x_i^+,y\right)}{\partial y}=\frac{dG_0(s;\beta_{i,j}^y)}{ds}\frac{F_{i,j}}{h_j^y}+\frac{dG_1(s;\beta_{i,j}^y)}{ds}\frac{F_{i,j+1}}{h_j^y}\\
&\quad+\frac{dG_2(s;\beta_{i,j}^y)}{ds}D_{i,j}^y+\frac{dG_3(s;\beta_{i,j}^y)}{ds}D_{i,j+1}^y,\\
&\frac{\partial P_{i,j}\left(x_{i+1}^-,y\right)}{\partial y}=\frac{dG_0(s;\beta_{i,j}^y)}{ds}\frac{F_{i+1,j}}{h_j^y}+\frac{dG_1(s;\beta_{i,j}^y)}{ds}\frac{F_{i+1,j+1}}{h_j^y}\\
&\quad+\frac{dG_2(s;\beta_{i,j}^y)}{ds}D_{i+1,j}^y+\frac{dG_3(s;\beta_{i,j}^y)}{ds}D_{i+1,j+1}^y,
\end{aligned}$$

it follows that $\frac{\partial P_{i,j}\left(x,y_j^+\right)}{\partial y}=\frac{\partial P_{i,j}\left(x,y_j^-\right)}{\partial y}$ and $\frac{\partial P_{i,j}\left(x_i^+,y\right)}{\partial y}=\frac{\partial P_{i,j}\left(x_i^-,y\right)}{\partial y}$ if $\beta_{i-1,j}^y=\beta_{i,j}^y$.

From the above analysis, we can see that the interpolation surface $P_{i,j}(x,y)$ is C^1 continuous in the whole rectangular domain R if h_i^x =constant and $\beta_{i,j}^y$ =constant for each $j\in\{1,2,\dots,m-1\}$ and all $i=1,2,\dots,n-1$, no matter what the parameters $\alpha_{i,j}^x$ might be. In the following, we shall further discuss the C^2 continuous property of the interpolation surface.

For any $(x,y)\in R_{i,j}$, straightforward computation gives the mixed partial derivatives $\frac{\partial^2 P_{i,j}(x,y)}{\partial x\partial y}$ and $\frac{\partial^2 P_{i,j}(x,y)}{\partial y\partial x}$ as follows

$$\begin{aligned}
\frac{\partial^2 P_{i,j}(x,y)}{\partial x\partial y}&=\frac{1}{h_j^y}\frac{dG_0\left(s;\beta_{i,j}^y\right)}{ds}\frac{dP_{i,j}^*(x)}{dx}+\frac{1}{h_j^y}\frac{dG_1\left(s;\beta_{i,j}^y\right)}{ds}\frac{dP_{i,j+1}^*(x)}{dx}\\
&\quad+\frac{dG_2\left(s;\beta_{i,j}^y\right)}{ds}\frac{d\phi_{i,j}(x)}{dx}+\frac{dG_3\left(s;\beta_{i,j}^y\right)}{ds}\frac{d\phi_{i,j+1}(x)}{dx}\\
&=\frac{\partial^2 P_{i,j}(x,y)}{\partial y\partial x}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\frac{\partial^2 P_{i,j}\left(x,y_j^+\right)}{\partial x\partial y}=\frac{d\phi_{i,j}(x)}{dx}=\frac{\partial^2 P_{i,j}\left(x,y_j^+\right)}{\partial y\partial x},\\
&\frac{\partial^2 P_{i,j}\left(x,y_{j+1}^-\right)}{\partial x\partial y}=\frac{d\phi_{i,j+1}(x)}{dx}=\frac{\partial^2 P_{i,j}\left(x,y_{j+1}^-\right)}{\partial y\partial x},\\
&\frac{\partial^2 P_{i,j}\left(x_i^+,y\right)}{\partial x\partial y}=\frac{dG_0(s;\beta_{i,j}^y)}{ds}\frac{D_{i,j}^x}{h_j^y}+\frac{dG_1(s;\beta_{i,j}^y)}{ds}\frac{D_{i,j+1}^x}{h_j^y}\\
&\quad+\frac{dG_2(s;\beta_{i,j}^y)}{ds}\frac{D_{i,j}^y}{h_i^x}+\frac{dG_3(s;\beta_{i,j}^y)}{ds}\frac{D_{i,j+1}^y}{h_i^x}\\
&=\frac{\partial^2 P_{i,j}\left(x_i^+,y\right)}{\partial y\partial x},\\
&\frac{\partial^2 P_{i,j}\left(x_{i+1}^-,y\right)}{\partial x\partial y}=\frac{dG_0(s;\beta_{i,j}^y)}{ds}\frac{D_{i+1,j}^x}{h_j^y}+\frac{dG_1(s;\beta_{i,j}^y)}{ds}\frac{D_{i+1,j+1}^x}{h_j^y}\\
&\quad+\frac{dG_2(s;\beta_{i,j}^y)}{ds}\frac{D_{i+1,j}^y}{h_i^x}+\frac{dG_3(s;\beta_{i,j}^y)}{ds}\frac{D_{i+1,j+1}^y}{h_i^x}\\
&=\frac{\partial^2 P_{i,j}\left(x_{i+1}^-,y\right)}{\partial y\partial x}.
\end{aligned}$$

These imply that $\frac{\partial^2 P_{i,j}\left(x,y_i^+\right)}{\partial x\partial y}=\frac{\partial^2 P_{i,j}\left(x,y_i^-\right)}{\partial x\partial y}=\frac{\partial^2 P_{i,j}\left(x,y_i^+\right)}{\partial y\partial x}=\frac{\partial^2 P_{i,j}\left(x,y_i^-\right)}{\partial y\partial x}$ and $\frac{\partial^2 P_{i,j}\left(x_i^+,y\right)}{\partial x\partial y}=\frac{\partial^2 P_{i,j}\left(x_i^-,y\right)}{\partial x\partial y}=\frac{\partial^2 P_{i,j}\left(x_i^+,y\right)}{\partial y\partial x}=\frac{\partial^2 P_{i,j}\left(x_i^-,y\right)}{\partial y\partial x}$ if $h_{i-1}^x=h_i^x$ and $\beta_{i-1,j}^y=\beta_{i,j}^y$.

For $\frac{\partial^2 P_{i,j}(x,y)}{\partial x^2}$, since the x-direction interpolation curve $P_{i,j}^*(x)$ is C^2 continuous if the first partial derivative values $D_{i,j}^x$, $i=2,3,\dots,n-1$ are given by (2) and

$$\frac{d^2\phi_{i,l}\left(x_i^+\right)}{dx^2}=\frac{d^2\phi_{i,l}\left(x_{i+1}^-\right)}{dx^2}=0,\quad l=j,j+1,$$

we have

$$\begin{aligned}
&\frac{\partial^2 P_{i,j}\left(x,y_j^+\right)}{\partial x^2}=\frac{d^2P_{i,j}^*(x)}{dx^2},\quad \frac{\partial^2 P_{i,j}\left(x,y_{j+1}^-\right)}{\partial x^2}=\frac{d^2P_{i,j+1}^*(x)}{dx^2},\\
&\frac{\partial^2 P_{i,j}\left(x_i^+,y\right)}{\partial x^2}=G_0\left(s;\beta_{i,j}^y\right)\frac{2\left(\Delta_{i,j}^x-\Delta_{i-1,j}^x\right)}{h_{i-1}^x+h_i^x}\\
&\quad+G_1\left(s;\beta_{i,j}^y\right)\frac{2\left(\Delta_{i,j+1}^x-\Delta_{i-1,j+1}^x\right)}{h_{i-1}^x+h_i^x},\\
&\frac{\partial^2 P_{i,j}\left(x_{i+1}^-,y\right)}{\partial x^2}=G_0\left(s;\beta_{i,j}^y\right)\frac{2\left(\Delta_{i+1,j}^x-\Delta_{i,j}^x\right)}{h_i^x+h_{i+1}^x}\\
&\quad+G_1\left(s;\beta_{i,j}^y\right)\frac{2\left(\Delta_{i+1,j+1}^x-\Delta_{i,j+1}^x\right)}{h_i^x+h_{i+1}^x},
\end{aligned}$$

these indicate that $\frac{\partial^2 P_{i,j}\left(x,y_j^+\right)}{\partial x^2}=\frac{\partial^2 P_{i,j}\left(x,y_j^-\right)}{\partial x^2}$ and $\frac{\partial^2 P_{i,j}\left(x_i^+,y\right)}{\partial x^2}=\frac{\partial^2 P_{i,j}\left(x_i^-,y\right)}{\partial x^2}$ if $\beta_{i-1,j}^y=\beta_{i,j}^y$.

Finally, for $\frac{\partial^2 P_{i,j}(x,y)}{\partial y^2}$, we have

$$\begin{aligned}
&\frac{\partial^2 P_{i,j}\left(x,y_j^+\right)}{\partial y^2}=0, \quad \frac{\partial^2 P_{i,j}\left(x,y_{j+1}^-\right)}{\partial y^2}=0,\\
&\frac{\partial^2 P_{i,j}\left(x_i^+,y\right)}{\partial y^2}=\frac{d^2G_0\left(s;\beta_{i,j}^y\right)}{ds^2}\frac{F_{i,j}}{\left(h_j^y\right)^2}+\frac{d^2G_1\left(s;\beta_{i,j}^y\right)}{ds^2}\frac{F_{i,j+1}}{\left(h_j^y\right)^2}\\
&\quad+\frac{d^2G_2\left(s;\beta_{i,j}^y\right)}{ds^2}\frac{D_{i,j}^y}{h_j^y}+\frac{d^2G_3\left(s;\beta_{i,j}^y\right)}{ds^2}\frac{D_{i,j+1}^y}{h_j^y},\\
&\frac{\partial^2 P_{i,j}\left(x_{i+1}^-,y\right)}{\partial y^2}=\frac{d^2G_0\left(s;\beta_{i,j}^y\right)}{ds^2}\frac{F_{i+1,j}}{\left(h_j^y\right)^2}+\frac{d^2G_1\left(s;\beta_{i,j}^y\right)}{ds^2}\frac{F_{i+1,j+1}}{\left(h_j^y\right)^2}\\
&\quad+\frac{d^2G_2\left(s;\beta_{i,j}^y\right)}{ds^2}\frac{D_{i+1,j}^y}{h_j^y}+\frac{d^2G_3\left(s;\beta_{i,j}^y\right)}{ds^2}\frac{D_{i+1,j+1}^y}{h_j^y},
\end{aligned}$$

it follows that $\frac{\partial^2 P_{i,j}\left(x,y_j^+\right)}{\partial y^2}=\frac{\partial^2 P_{i,j}\left(x,y_j^-\right)}{\partial y^2}$ and $\frac{\partial^2 P_{i,j}\left(x_i^+,y\right)}{\partial y^2}=\frac{\partial^2 P_{i,j}\left(x_i^-,y\right)}{\partial y^2}$ if $\beta_{i-1,j}^y=\beta_{i,j}^y$.

Summarizing the above discussion, we can conclude the following theorem.

Theorem 1: If the knots are equally spaced for variable x, that is h_i^x =constant, and the first partial derivative values $D_{i,j}^x$ are given by (2), then a sufficient condition for the interpolation surface $P_{i,j}(x,y)$ to be C^2 continuous in the whole rectangular domain R is that $\beta_{i,j}^y$ =constant for each $j\in\{1,2,\ldots,m-1\}$ and all $i=1,2,\ldots,n-1$, no matter what the parameters $\alpha_{i,j}^x$ might be.

For generating the interpolation surface $P_{i,j}(x,y)$, we also need to provide the first partial derivative values $D_{i,j}^y$, $i=1,2,\ldots,n$, $j=1,2,\ldots,m$ in advance. In this paper, they are computed by the following formula

$$\begin{aligned}
&D_{i,1}^y=\Delta_{i,1}^y-\frac{h_1^y}{h_1^y+h_2^y}\left(\Delta_{i,2}^y-\Delta_{i,1}^y\right),\\
&D_{i,j}^y=\frac{h_{j-1}^y\Delta_{i,j}^y+h_j^y\Delta_{i,j-1}^y}{h_{j-1}^y+h_j^y}, \quad j=2,3,\ldots,m-1,\\
&D_{i,m}^y=\Delta_{i,m-1}^y+\frac{h_{m-1}^y}{h_{m-2}^y+h_{m-1}^y}\left(\Delta_{i,m-1}^y-\Delta_{i,m-2}^y\right),
\end{aligned} \tag{6}$$

where $i=1,2,\ldots,n$.

B. Bounded property

We denote

$$\begin{aligned}
&M=\max\left\{\left|F_{k,l}\right|,k=i,i+1,l=j,j+1\right\},\\
&Q_1=\max\left\{h_i^x\left|D_{k,l}^x\right|,k=i,i+1,l=j,j+1\right\},\\
&Q_2=\max\left\{h_j^y\left|D_{k,l}^y\right|,k=i,i+1,l=j,j+1\right\}.
\end{aligned}$$

By directly computing, we can obtain the following properties of the basis functions

$$\begin{aligned}
&a_{i,j}(t,s)+a_{i,j+1}(t,s)+a_{i+1,j}(t,s)+a_{i+1,j+1}(t,s)\\
&=1, b_{i,j}(t,s)+b_{i,j+1}(t,s)-b_{i+1,j}(t,s)-b_{i+1,j+1}(t,s)\\
&=(1-t)\,t,\\
&c_{i,j}(t,s)-c_{i,j+1}(t,s)+c_{i+1,j}(t,s)-c_{i+1,j+1}(t,s)\\
&=\left(1+t-10t^3+15t^4-6t^5\right)(1-s)\,s\left[1\right.\\
&\left.+\frac{(1-s)s}{(1-s)^2+\beta_{i,j}^y(1-s)s+s^2}\right].
\end{aligned}$$

Thus, for the given data, from the expression of the interpolation surface $P_{i,j}(x,y)$ given in [5], we have

$$\begin{aligned}
&\left|P_{i,j}(x,y)\right|\le M\sum_{k=i}^{i+1}\sum_{l=j}^{j+1}\left|a_{k,l}(t,s)\right|+Q_1\sum_{k=i}^{i+1}\sum_{l=j}^{j+1}\left|b_{k,l}(t,s)\right|\\
&\qquad+Q_2\sum_{k=i}^{i+1}\sum_{l=j}^{j+1}\left|c_{k,l}(t,s)\right|\\
&=M+Q_1(1-t)\,t+Q_2\left(1+t-10t^3\right.\\
&\left.+15t^4-6t^5\right)(1-s)\,s\left[1+\frac{(1-s)s}{(1-s)^2+\beta_{i,j}^y(1-s)s+s^2}\right]\\
&\le M+0.25Q_1+Q_2\left(1+t-10t^3\right.\\
&\left.+15t^4-6t^5\right)(1-s)\,s\left[1+\frac{(1-s)s}{(1-s)^2+s^2}\right]\\
&\le M+0.25Q_1+1.5Q_2\left(1+t-10t^3\right.\\
&\left.+15t^4-6t^5\right)(1-s)\,s\\
&\le M+0.25Q_1+0.375Q_2\left(1+t-10t^3+15t^4-6t^5\right).
\end{aligned}$$

Since

$$\max_{t\in[0,1]}\left(1+t-10t^3+15t^4-6t^5\right)=1.14675,$$

we can immediately conclude the following theorem.

Theorem 2: For any nonnegative free parameters $\alpha_{i,j}^x,\beta_{i,j}^y$, the values of the resulting interpolation surface $P_{i,j}(x,y)$ on $R_{i,j}$ are bounded by

$$\left|P_{i,j}(x,y)\right|\le M+0.25Q_1+0.43003125Q_2.$$

C. Error formula

For any $(x,y)\in R_{i,j}$, let $F_{i,j}=F(x_i,y_j)$, $D_{i,j}^x=\frac{\partial F(x_i,y_j)}{\partial x}$, $D_{i,j}^y=\frac{\partial F(x_i,y_j)}{\partial y}$, and denote

$$\begin{aligned}
\left\|\frac{\partial F(x,y)}{\partial x}\right\|&=\max_{(x,y)\in R_{i,j}}\left|\frac{\partial F(x,y)}{\partial x}\right|,\\
\left\|\frac{\partial F(x,y)}{\partial y}\right\|&=\max_{(x,y)\in R_{i,j}}\left|\frac{\partial F(x,y)}{\partial y}\right|.
\end{aligned}$$

For any $(x,y)\in R_{i,j}$, by using the Taylor formula of $F(x,y)$ at the points (x_k,y_l), $k=i,i+1$, $l=j,j+1$, we have

$$F(x,y)-F(x_k,y_l)=(x-x_k)\frac{\partial F(\theta_k,\eta_l)}{\partial x}+(y-y_l)\frac{\partial F(\theta_k,\eta_l)}{\partial y},$$

where θ_k and η_l are between x and x_k, y and y_l, respectively. It follows that

$$\max_{(x,y)\in R_{i,j}}\left|F(x,y)-F(x_k,y_l)\right|\le h_i^x\left\|\frac{\partial F(x,y)}{\partial x}\right\|+h_j^y\left\|\frac{\partial F(x,y)}{\partial y}\right\|.$$

Thus for any $(x,y)\in R_{i,j}$, we have

$$\begin{aligned}
&\left|F(x,y)-P_{i,j}(x,y)\right|=\left|\sum_{k=i}^{i+1}\sum_{l=j}^{j+1}\left[a_{k,l}(t,s)\left(F(x,y)-F(x_k,y_l)\right)\right.\right.\\
&\qquad\left.\left.+b_{k,l}(t,s)\,h_i^x\frac{\partial F(x_k,y_l)}{\partial x}+c_{k,l}(t,s)\,h_j^y\frac{\partial F(x_k,y_l)}{\partial y}\right]\right|\\
&\le\max_{(x,y)\in R_{i,j}}\left|F(x,y)-F(x_k,y_l)\right|\left|\sum_{k=i}^{i+1}\sum_{l=j}^{j+1}a_{k,l}(t,s)\right|\\
&\qquad+h_i^x\left\|\frac{\partial F(x,y)}{\partial x}\right\|\left|\sum_{k=i}^{i+1}\sum_{l=j}^{j+1}b_{k,l}(t,s)\right|\\
&\quad h_j^y\left\|\frac{\partial F(x,y)}{\partial y}\right\|\left|\sum_{k=i}^{i+1}\sum_{l=j}^{j+1}c_{k,l}(t,s)\right|\\
&\le 1.25h_i^x\left\|\frac{\partial F(x,y)}{\partial x}\right\|+1.43003125h_j^y\left\|\frac{\partial F(x,y)}{\partial y}\right\|.
\end{aligned}$$

Summarize the above analysis, we have the following theorem.

***Theorem** 3:* Let $F(x,y) \in C^1(R)$ be the interpolated function with $P_{i,j}(x,y)$ is compared. Then for any $(x,y) \in R_{i,j}$, the following error formula holds

$$|F(x,y) - P_{i,j}(x,y)| \leq 1.25h_i^x \left\|\frac{\partial F(x,y)}{\partial x}\right\| + 1.43003125h_j^y \left\|\frac{\partial F(x,y)}{\partial y}\right\|.$$

From Theorems 2 and 3, we can see that the generated interpolation surface is stable for the parameters.

IV. Numerical Examples

In this section, we shall give two numerical examples to show that the proposed C^2 interpolation surface scheme can give a good approximation to the interpolated function. And for the unchanged interpolating data, the shape of the interpolation surface can be modified by selecting parameters according to the control need. In the following figures, the interpolating data points have been marked with solid black dots.

***Example** 1:* Let the interpolated function be $F(x,y) = sin(x+y), (x,y) \in [0,1] \times [0,1]$, and $x_i = 0.2(i-1), y_j = 0.2(j-1), i,j = 1,2,\ldots,6$. The parameters are chosen as $\alpha_{i,j}^x = 5+5i+10j, \beta_{i,j}^y = 10+10j, i,j = 1,2,\ldots,6$. Fig. 1 shows the resulting interpolation surface $P(x,y)$ defined by (4) and the error surface $F(x,y) - P(x,y)$. From the results, we can see that the interpolation surface gives a good approximation to the interpolated function.

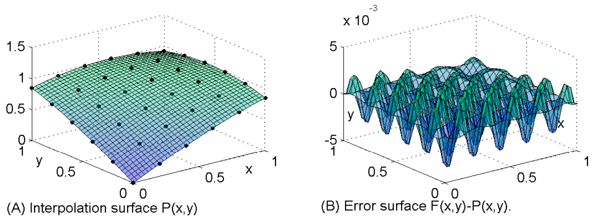

Fig. 1. Interpolation surface and the error surface.

***Example** 2:* Fig. 2 shows the C^2 interpolation surface with different parameters for the 3D data set given in Tab. I. It can be seen that the interpolation surface can be modified conveniently by selecting suitable parameters according to needs of practical design.

TABLE I
A 3D data set.

y/x	0	1	2	3	4	5	6
0	0	1	3	4	3	1	0
1	1	2	4	5	4	2	1
3	3	4	5	8	5	4	3
5	1	2	4	5	4	2	1
6	0	1	3	4	3	1	0

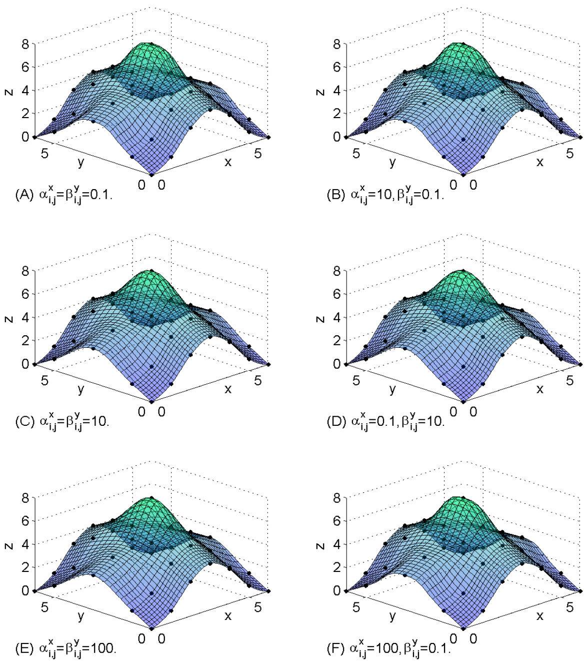

Fig. 2. C^2 interpolation surface with different parameters.

V. Conclusion

As stated above, the developed bivariate rational interpolation surface can be C^2 continuous based only on the function values. The shape of the interpolation surface can be modified conveniently by using the parameters under the unchanged interpolating data. And the interpolation surface is bounded and stable for the parameters. Compared with the rational interpolation spline with bi-cubic denominator developed in [12], the given interpolation scheme with bi-quadratic denominator has less computational cost. There are still some problems worthy of further study, such as the convexity control of the new constructed interpolation surfaces. These will be our future work.

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