Derivation and Application of Fourth Stage Inverse Polynomial Scheme to Initial Value Problems

O. E. Abolarin and S. W. Akingbade

Abstract— **In this paper, we propose the fourth stage of the inverse polynomial scheme for the numerical solution of initial value problems of ordinary differential equations. Binomial expansion and Taylor's series method are used towards the derivation of the scheme. We further study the analysis properties to show the efficiency of the method. Numerical experiments are carried out and the results are compared with the theoretical solution and some existing methods to show that it is adequate, effective and computationally time friendly.**

Index Terms— **inverse polynomial, initial value problem, ordinary differential equations, consistency, stability, convergence.**

I. INTRODUCTION

In science and engineering, modeling a system frequently amounts to solving an initial value problem. In this context, the differential equation is an evolution equation specifying how, given initial conditions, the system will evolve with time. Turning the rules that govern the evolution of a quantity into a differential equation is called modeling [2].

Numerical method is a substantial aspect in solving initial value problems in ordinary differential equations where the problems cannot be solved or difficult to obtain analytically. The numerical solutions of first order initial value problems have caught much attention recently; a new numerical scheme for the solution of initial value problems in ordinary differential equations was developed [10]. An integrator was also developed in [11] by representing the theoretical solution to initial value problems by an interpolating function which maybe linear or nonlinear.

There is a technique [20] for comparing numerical methods that have been designed to solve stiff systems of ordinary differential equations; the technique was applied to five methods of which three turn out to be quite good. However, each of the three has a weakness of its own, which can be identified with particular problem characteristics.

Wavelets was used in solving the first order ordinary differential equations which are either stiff or non-stiff [3]. It is worth mentioning that [1] works on the method for the numerical solution of the Painleve΄ equations (equations having singularities at points where the solution takes certain finite values). Researchers also worked on some other forms of equations like the integro-differential equations [15] and the integral equations where the Petrov-Galerkin method is employed for the numerical solution of stochastic volterra integral equations [4].

Other notable works are [6], [7], [8], [13], [16], and [19] to mention a few.

In this paper, we developed a numerical integrator capable of solving equations of the form

$$y' = f(x, y); y(x_0) = y_0 \tag{1}$$

The integrator is developed by representing the theoretical solution '$y(x)$' to (1) by an interpolating function.

In [14], the effectiveness of the first stage, second stage of the inverse polynomial method to solving ordinary differential equations with singularities was shown using a different integrator and [18] worked on the third stage of the method and analyzed its analysis properties; the local truncation error and the order were also determined towards its implementation.

II. PRELIMINARIES

In this section, we present some useful existing concept and works.

Definition II.1. The conventional one-step numerical integrator for initial value problem (1) is generally described according to [12] as

$$y_{n+1} = y_n + h\phi(x_n, y_n; h) \tag{2}$$

Examples of such method are Euler's method and Runge-Kutta's method.

Definition II.2. Truncation error is the error committed when the higher terms of the power series are ignored. Such errors are essentially algorithmic errors and we can predict the extent of the error that will occur in the method.

Definition II.3. An algorithm is said to be numerically stable if an error whatever the cause does not grow much larger during calculation. This happens if the problem is well posed, that is, the solution changes by only a small amount if the problem data are changed by small amounts.

Definition II.4. The simplest methods of order 'P' are always based on the Taylor series expansion of the solution '$y(x)$' of the IVP (1) as in [9]. If we assume '$y^{p+1}(x)$' to

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be continuous on the closed interval [a, b], then the Taylor's formula is given by

$$y(x_{n+1}) = y(x_n + h)$$
$$= y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) + \ldots + \frac{h^p}{p!}y^p(x) + \frac{h^{p+1}}{(p+1)!}y^{p+1}(\xi_n)$$

$$= y(x_n) + h[y'(x_n) + \frac{h}{2!}y''(x_n) + \ldots + \frac{h^{p-1}}{p!}y^p(x) + \frac{h^{p+1}}{(p+1)!}y^{p+1}(\xi_n)]$$

$$where \therefore\ x_n \le \xi_n \le x_{n+1}.$$
(3)

The continuity of '$y^{p+1}(x)$' implies that it is bounded on [a, b] and therefore,

$$\frac{y^{p+1}(\xi_n)h^{p+1}}{(p+1)!} = 0(h^{p+1}). \tag{4}$$

We introduce (4) in (3),

$$= y(x_n) + h[y'(x_n) + \frac{hy''(x_n)}{2!} + \ldots + \frac{h^{p-1}}{p!}y^p(x_n)] + O(h^{p+1})$$

Hence,

$$y(x_{n+1}) = y(x_n) + h[f(x_n, y(x_n)) + \frac{h}{2}f'(x_n, y(x_n)) + \ldots + \frac{h^{p-1}}{p!}f^{(p-1)}(x_n, y_n)] + O(h^{p+1}) \tag{5}$$

Equation (5) is called the Taylor Series Method of order P.

In this work, we denote the Taylor's series method of order 4 as TSM-4.

Definition II.5. Runge-Kutta Method (RK4) is a technique for approximating the solution of ordinary differential equations. It was developed by two mathematicians Carl Runge and Withen Kutta around 1900.

Runge-Kutta method is popular because of its efficient used in most computer programs for differential equations. The most widely used Runge-Kutta scheme is the fourth order scheme RK4 based on Simpson's rule [17].

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{6}$$

$$where$$
$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_n + \frac{1}{2}h, y_n + \frac{h}{2}k_1)$$
$$k_3 = f(x_n + \frac{1}{2}h, y_n + \frac{h}{2}k_2)$$
$$k_4 = f(x_n + h, y_n + hk_3)$$

III. FOURTH STAGE INVERSE POLYNOMIAL SCHEME

Let the numerical approximation 'y_{n+1}' evaluated at $x = x_{n+1}$ to exact solution '$y(x_{n+1})$' to the first order ordinary differential equation be represented as

$$y_{n+1} = y_n \left[\sum_{j=0}^{k} a_j x_n{}^j\right]^{-1} \tag{7}$$

The parameters $a_j s'$ are to be determined from the non-linear equations that will be generated by considering the following steps:

1. For the fourth order, k=4, and setting $a_0 = 1$

$$y_{n+1} = y_n\left[1 + (a_1x_n + a_2x_n{}^2 + a_3x_n{}^3 + a_4x_n{}^4)\right]^{-1} \tag{8}$$

2. Obtain the Binomial Expansion of (8)

$$y_{n+1} = y_n\begin{bmatrix} 1 + (-1)(a_1x_n + a_2x_n{}^2 + a_3x_n{}^3 + a_4x_n{}^4) + \\ \frac{(-1)(-2)}{2!}(a_1x_n + a_2x_n{}^2 + a_3x_n{}^3 + a_4x_n{}^4)^2 + \\ \frac{(-1)(-2)(-3)}{3!}(a_1x_n + a_2x_n{}^2 + a_3x_n{}^3 + a_4x_n{}^4)^3 + \\ \frac{(-1)(-2)(-3)(-4)}{4!}(a_1x_n + a_2x_n{}^2 + a_3x_n{}^3 + a_4x_n{}^4)^4 \end{bmatrix}$$

3. Express the left hand side of (8) in terms of its Taylor's series expansion

$$y_n + h = y_n + hy'_n + \frac{h^2y''_n}{2!} + \frac{h^3y'''_n}{3!} + \frac{h^4y^{(4)}{}_n}{4!}$$

4. Insert the obtained equations from steps 2 and 3 above into (8) and expand

$$y_n + hy'_n + \frac{h^2y''_n}{2!} + \frac{h^3y'''_n}{3!} + \frac{h^4y^{(4)}{}_n}{4!} = y_n - a_1x_ny_n + (a^2{}_1 - a_2)xn^2y_n +$$
$$(2a_1a_2 - a_1^3 - a_3)x_n^3y_n + (2a_1a_3 + a_2^2 - a_4 + a_1^4 - 3a_1^2a_2)x_n^4y_n$$

5. We make the expression above agrees term by term for each parameter.

$$hy'_n = -a_1x_ny_n$$
$$a_1 = \frac{-hy'_n}{x_ny_n} \tag{9}$$
$$\frac{h^2y''_n}{2!} = (a_1{}^2 - a_2\ )x_n{}^2y_n$$

Solve for a_2 by substituting (9)

$$a_2 = \frac{2h^2(y'_n)^2 - h^2y''_ny_n}{2x_n{}^2y_n{}^2} \tag{10}$$

$$\frac{h^3y'''_n}{3!} = (2a_1a_2 - a_3 - a_1{}^3)x_n{}^3y_n$$

Using (9) and (10),

$$a_3 = \frac{-h^3y'''_n}{6x_n{}^3y_n} - \frac{2hy'_n}{x_ny_n}\left(\frac{2h^2(y'_n)^2 - h^2y_ny''_n}{2x^2{}_ny_n{}^2}\right) + \left(\frac{h^3(y'_n)^3}{x^3{}_ny_n{}^3}\right)$$

$$a_3 = \frac{-h^3y_n{}^2y'''_n + 6h^3y_ny'_ny''_n\ - 6h^3(y'_n)^3}{6x_n{}^3y_n{}^3} \tag{11}$$

$$\frac{h^4y_n{}^{(4)}}{4!} = (2a_1a_3 + a_2{}^2 - a_4 + a_1{}^4 - 3a_1{}^2a_2)x_n{}^4y_n$$

$$a_4 = 2a_1a_3 + a_2{}^2 + a_1{}^4 - 3a_1{}^2a_2 - \frac{h^4y_n{}^{(4)}}{4!x_n{}^4y_n}$$

We express each term of a_4 above in relation to (9), (10), and (11)

$$2a_1a_3 = \frac{h^4{y_n}^2y_n'y_n'''}{3{x_n}^4{y_n}^4} + \frac{4h^4(y_n')^4}{{x_n}^4{y_n}^4} - \frac{2h^4(y_n')^2y_n''y_n}{{x_n}^4{y_n}^4} - \frac{2h^4(y_n')^4}{{x_n}^4{y_n}^4}$$

$$a_2{}^2 = \frac{h^4(y_n')^4}{{x_n}^4{y_n}^4} - \frac{h^4y_n(y_n')^2y_n''}{{x_n}^4{y_n}^4} + \frac{h^4(y_n'')^2{y_n}^2}{4{x_n}^4{y_n}^4}$$

$$a_1{}^4 - 3a_1{}^2a_2 = \frac{3h^4(y_n')^2y_n''y_n}{2{x_n}^4{y_n}^4} - \frac{2h^4(y_n')^4}{{x_n}^4{y_n}^4}$$

Finally,

$$a_4 = \frac{\begin{array}{c}24h^4(y_n')^4 - 36h^4y_n(y_n')^2y_n'' + \\ 8h^4{y_n}^2y_n'y_n'''' - h^4{y_n}^3{y_n}^{(4)} + 6h^4(y_n)^2(y_n'')^2\end{array}}{24x^4{}_n y^4{}_n} \quad (12)$$

By using (9), (10), (11), and (12) in (8), we have the fourth stage Inverse Polynomial:

$$24{y_n}^5\begin{bmatrix} 24y^4{}_n - 24h{y_n}^3y_n' + 24h^2y^2{}_n(y_n')^2 - 12h^2{y_n}^3y_n'' \\ 24h^3{y_n}^2y_n'y_n'' \quad - 24h^3y_n(y_n')^3 + \\ 24h^4(y_n')^4 - 4h^3{y_n}^3y_n''' + 8h^4{y_n}^2y_n'y_n'''' - \\ 36h^4y_n(y_n')^2y_n'' + 6h^4(y_n)^2(y_n'')^2 - h^4{y_n}^3y^{(4)}{}_n \end{bmatrix}^{-1} \quad (13)$$

Equation (13) is the Fourth Stage Inverse Polynomial Scheme (New Method).

IV. ANALYSIS OF THE BASIC PROPERTIES OF THE FOURTH STAGE SCHEME

A. Consistency

A numerical scheme with an increment function '$\phi(x_n, y_n; h)$' is said to be consistent with the initial value problem (1) if

$\phi(x_n, y_n; h) = f(x, y)$ when h =0.

From (2)

$$y_{n+1} - y_n = h\phi(x_n, y_n; h) \quad (14)$$

$$\frac{y_{n+1} - y_n}{h} = \phi(x_{n,}, y_n; h)$$

In light of this, with respect to the scheme,

$$y_{n+1} - y_n =$$

$$\left(24{y_n}^5\begin{bmatrix} 24y^4{}_n - 24h{y_n}^3y_n' + 24h^2y^2{}_n(y_n')^2 - 12h^2{y_n}^3y''{}_n \\ 24h^3{y_n}^2y_n'y''{}_n \quad - 24h^3y_n(y_n')^3 + \\ 24h^4(y_n')^4 - 4h^3{y_n}^3y_n''' + 8h^4{y_n}^2y_n'y_n'''' - \\ 36h^4y_n(y_n')^2y''{}_n + 6h^4(y_n)^2(y''{}_n)^2 - h^4{y_n}^3y^{(4)}{}_n \end{bmatrix}^{-1}\right) - (y_n)$$

$$\frac{\begin{array}{l}24{y_n}^5 - 24{y_n}^5 + 24h{y_n}^4y_n' - 12h^2{y_n}^3\left[2(y_n')^2 - y_ny_n''\right] - 4h^3{y_n}^2[6y_ny_n'y_n'' - \\ 6(y_n')^3 - {y_n}^2y_n'''] - h^4[24y_n(y_n')^4 - 36{y_n}^2(y_n')^2y_n'' \; + 8{y_n}^3y_n'y_n''' \; + 6{y_n}^3(y_n'')^2 - \\ y_n{}^4y_n^{(4)}\end{array}}{\begin{array}{l}24{y_n}^4 - 24h{y_n}^3y_n' + 12h^2{y_n}^2\left[2(y_n')^2 - y_ny_n''\right] + 4h^3y_n\left[6y_ny_n'y_n'' - 6(y_n')^3 - {y_n}^2y_n'''\right] \\ + h^4\left[24(y_n')^4 - 36(y_n')^2y_n \; y_n'' + 8{y_n}^2y_n'y_n''' + 6{y_n}^2(y_n'')^2 - {y_n}^3y_n^{(4)}\right]\end{array}}$$

$$\frac{y_{n+1} - y_n}{h} =$$

$$\frac{h\begin{bmatrix}24{y_n}^4y_n' - 12h\;{y_n}^3\left[2(y_n')^2 - y_ny_n''\right] - 4h\;{y_n}^2[6y_ny_n'y_n'' - \\ 6(y_n')^3 - {y_n}^2y_n'''] - h^3[24y_n(y_n')^4 - 36{y_n}^2(y_n')^2y_n'' \; + 8{y_n}^3y_n'y_n''' \; + 6{y_n}^3(y_n'')^2 - \\ y_n{}^4y_n^{(4)}\end{bmatrix}}{\begin{array}{l}24{y_n}^4 - 24h{y_n}^3y_n' + 12h^2{y_n}^2\left[2(y_n')^2 - y_ny_n''\right] + 4h^3y_n\left[6y_ny_n'y_n'' - 6(y_n')^3 - {y_n}^2y_n'''\right] \\ + h^4\left[24(y_n')^4 - 36(y_n')^2y_n \; y_n'' + 8{y_n}^2y_n'y_n''' + 6{y_n}^2(y_n'')^2 - {y_n}^3y_n^{(4)}\right]\end{array}}$$

$$\frac{y_{n+1} - y_n}{h} =$$

$$\frac{\begin{array}{l}24hy_n'{y_n}^4 - 12h^2{y_n}^3\left[2(y_n')^2 - y_ny_n''\right] - 4h^3{y_n}^2\left[6y_ny_n'y_n'' - 6(y_n')^3 - {y_n}^2y_n'''\right] - \\ h^4\left[24y_n(y_n')^4 - 36(y_n')^2{y_n}^2y_n'' + 8{y_n}^3y_n'y_n''' + 6(y_n'')^2{y_n}^3 - {y_n}^4y^{iv}{}_n\right]\end{array}}{\begin{array}{l}24{y_n}^4 - 24hy_n'{y_n}^3 + 12h^2{y_n}^2\left[2(y_n')^2 - y_ny_n''\right] + 4h^3y_n\left[6y_ny_n'y_n'' - 6(y_n')^3 - {y_n}^2y_n'''\right] \\ + h^4\left[24(y_n')^4 - 36(y_n')^2y_n \; y'' + 8{y_n}^2y_n'y_n''' + 6(y_n'')^2{y_n}^2 - {y_n}^3y_n^{(4)}\right]\end{array}}$$

$$\frac{y_{n+1} - y_n}{h} =$$

$$\frac{\begin{array}{l}24y_n'{y_n}^4 - 12h{y_n}^3\left[2(y_n')^2 - y_ny_n''\right] - 4h^2{y_n}^2\left[6y_ny_n'y_n'' - 6(y_n')^3 - {y_n}^2y_n'''\right] - \\ h^3\left[24y_n(y_n')^4 - 36(y_n')^2{y_n}^2y_n'' + 8{y_n}^3y_n'y_n''' + 6(y_n'')^2{y_n}^3 - {y_n}^4y^{iv}{}_n\right]\end{array}}{\begin{array}{l}24{y_n}^4 - 24hy_n'{y_n}^3 + 12h^2{y_n}^2\left[2(y_n')^2 - y_ny_n''\right] + 4h^2y_n\left[6y_ny_n'y_n'' - 6(y_n')^3 - {y_n}^2y_n'''\right] \\ + h^4\left[24(y_n')^4 - 36(y_n')^2y_n \; y'' + 8{y_n}^2y_n'y_n''' + 6(y_n'')^2{y_n}^2 - {y_n}^3y_n^{(4)}\right]\end{array}}$$

$$\frac{y_{n+1} - y_n}{h} =$$

$$\frac{\begin{array}{l}24{y_n}^4y_n' - 12h{y_n}^3\left[2(y_n')^2 - y_ny_n''\right] - 4h^2{y_n}^2\left[6y_ny_n'y_n'' - 6(y_n')^3 - {y_n}^2y_n'''\right] - \\ h^3\left[24y_n(y_n')^4 - 36{y_n}^2(y_n')^2y_n'' + 8{y_n}^3y_n'y_n''' + 6{y_n}^3(y_n'')^2 - {y_n}^4y^{iv}{}_n\right]\end{array}}{\begin{array}{l}24{y_n}^4 - 24h{y_n}^3y_n' + 12h^2{y_n}^2\left[2(y_n')^2 - y_ny_n''\right] + 4h^2y_n\left[6y_ny_n'y_n'' - 6(y_n')^3 - {y_n}^2y_n'''\right] \\ + h^4\left[24(y_n')^4 - 36(y_n')^2y_n \; y'' + 8{y_n}^2y_n'y_n''' + 6(y_n'')^2{y_n}^2 - {y_n}^3y_n^{(4)}\right]\end{array}}$$

Taking the limit as h approaches zero,

$$\frac{y_{n+1} - y_n}{h} = y_n' \quad (15)$$

B. Stability

One-step scheme is said to be stable if for any initial error e_0, there exist a constant M and h_0>0 such that when the general one-step scheme is applied to initial value problems with step size $h \in (0, h_0)$, the ultimate error e_n satisfies the following inequalities

$e_n \leq Me_0$ and

$0 < M < 1$

Using the general form,

$$y_{n+h} = y_{n+h-1}\left(\sum_{j=0}^{k} a_j {x_{n+h}}^j\right)^{-1} \quad (16)$$

The theoretical solution y(x) is given as

$$y(x_{n+h}) = y(x_{n+h-1})\left(\sum_{j=0}^{k} a_j {x_{n+h}}^j\right)^{-1} + T_{n+h} \quad (17)$$

$$y(x_{n+h}) - y_{n+h} = y(x_{n+h-1})\left(\sum_{j=0}^{k} a_j {x_{n+h}}^j\right)^{-1} - y_{n+h-1}\left(\sum_{j=0}^{k} a_j {x_{n+h}}^j\right)^{-1} + T_{n+h}$$

$$e_{n+h} = e_{n+h-1}\left(\sum_{j=0}^{k} a_j {x_{n+h}}^j\right)^{-1} + T_{n+h} \quad (18)$$

We take the absolute value of both sides,

$$|e_{n+h}| = \left|e_{n+h-1}\left(\sum_{j=0}^{k} a_j {x_{n+h}}^j\right)^{-1}\right| + |T_{n+h}|$$

We assume that $Q = \sum_{j=0}^{k} a_j x^j{}_{n+h}$ (19)

Then,

$$\left|\left(\sum_{j=0}^{k} a_j x_{n+h}{}^{j}\right)^{-1}\right| = \left(\sum_{j=0}^{k} a_j x_{n+h}{}^{j}\right)^{-1}$$

$$\left(\sum_{j=0}^{k} a_j x_{n+h}{}^{j}\right)^{-1} = Q^{-1} = M \tag{20}$$

$$|e_{n+h}| \leq M|e_{n+h-1}| + |T_{n+h}| \tag{21}$$

Let $E_{n+h} = \sup(e_{n+h})$ and $T = \sup(T_{n+h})$

Similarly, $E_{n+h-1} = \sup(e_{n+h-1})$ with $0 < n < \infty$

We then have in the form

$$E_{n+h} \leq ME_{n+h-1} + T \tag{22}$$

For h=1,

$$E_{n+1} \leq ME_n + T$$

For h=2,

$$\begin{aligned} E_{n+2} &\leq ME_{n+1} + T \\ &\leq M(ME_n + T) + T \\ &= M^2 E_n + MT + T \end{aligned}$$

$$E_{n+2} \leq M^2 E_n + MT + T$$

For h=3,

$$\begin{aligned} E_{n+3} &\leq ME_{n+2} + T \\ &\leq M(M^2 E_n + MT + T) + T \end{aligned}$$

$$E_{n+3} \leq M^3 E_n + M^2 T + MT + T$$

The general form is

$$E_{n+h} \leq M^k E_n + \sum_{r=0} M^r T \tag{23}$$

$$|e_{n+k}| < E_{n+h} < M^k E_n + \sum_{r=0} M^r T$$

Since M<1 and as $n \to \infty, E_{n+h} \to 0$

This shows that the method is stable and also convergent.

C. Convergence

The necessary and sufficient conditions for a numerical method to be convergent are stability and consistency. Since these conditions are satisfied, we can conclude that the proposed method is Convergent.

More of stability, consistency and convergence shall be discussed in the next section by considering some test problems.

V. NUMERICAL RESULTS AND INTERPRETATIONS.

A. Test problems

We require that the first, second, third and fourth derivatives with respect to 'x' of the interpolating function, respectively coincide with the differential equation as well as its first, second, third and fourth derivatives with respect to x at x_n. In other words, we require that,

$$y'(x_n) = f_n$$

$$y''(x_n) = f_n'$$

$$y'''(x_n) = f_n{}^2$$

$$y^{(4)}(x_n) = f_n{}^3$$

Now for $h = \dfrac{x_n - x_0}{n}$, (24)

We generate iterations to determine 'y_n' at each value of 'x_n'.

We shall compare our results with the Taylor's series method of order P=4 (TSM-4), Runge-kutta of order 4 (RK4) and the exact solution of each of the initial value problems in order to show the effectiveness of the new method. We consider the following examples:

Example 1

The logistics growth modelled by the differential equation

$$\frac{dp}{dt} = kp(1 - \frac{p}{m})$$

for some positive constants k and m

We now take the IVP $(x = t)$:

$$y' = y(1-y); y(0) = 0.5 \tag{a}$$

$$h = 0.1$$

The exact solution is given as

$$y(t) = \frac{0.5}{0.5 + 0.5e^{-t}}$$

Table I

Results of Example 1

t	New Method	Exact solution	Error
0.0	0.500000000	0.500000000	0.000000000
0.1	0.524979165	0.524979187	0.000000022
0.2	0.549833952	0.549833997	0.000000045
0.3	0.574442450	0.574442516	0.000000066
0.4	0.598687573	0.598687660	0.000000087
0.5	0.622459225	0.622459331	0.000000106
0.6	0.645656182	0.645656306	0.000000124
0.7	0.668187632	0.668187772	0.000000140
0.8	0.689974326	0.689974481	0.000000157
0.9	0.710949335	0.710949502	0.000000167
1.0	0.731058400	0.731058578	0.000000178

Example 2

The non-linear initial value problem as in [5]

$$y'' + (y')^2 = 0; y(0) = 1, y'(0) = 1 \tag{b}$$

$$h = 0.1, \quad x_n = 1.0$$

This can be reduced to the desired order by the method of reduction of order. The exact solution is then given as

$$y(x) = \frac{1}{x+1}$$

Table II
Results of Example 2

x	TSM-4	RK4	New Method
0.0	1.000000000	1.000000000	1.000000000
0.1	0.909033333	0.909091186	0.909090909
0.2	0.833290126	0.833333729	0.833333333
0.3	0.769197044	0.769231206	0.769230769
0.4	0.714258558	0.714286154	0.714285714
0.5	0.666644250	0.66666709	0.666666667
0.6	0.624981120	0.624980325	0.625000000
0.7	0.588219131	0.588217885	0.588235294
0.8	0.555541530	0.555540041	0.555555556
0.9	0.526303480	0.525828299	0.526315790
1.0	0.499989092	0.499560026	0.500000000

The exact solution is given as follows

Table III
Exact Solution to (b)

x	Exact solution
0.0	1.000000000
0.1	0.909090909
0.2	0.833333333
0.3	0.769230769
0.4	0.714285714
0.5	0.666666667
0.6	0.625000000
0.7	0.588235294
0.8	0.555555556
0.9	0.526315790
1.0	0.500000000

Table IV
Error Analysis from the Results of Example 2

x	Error in TSM-4	Error in RK4	Error in New Method
0.0	0.000000000	0.000000000	0.000000000
0.1	0.000057576	0.000000277	0.000000000
0.2	0.000043207	0.000000396	0.000000000
0.3	0.000033725	0.000000437	0.000000000
0.4	0.000027156	0.000000440	0.000000000
0.5	0.000022417	0.000000423	0.000000000
0.6	0.000018880	0.000019675	0.000000000
0.7	0.000016160	0.000017409	0.000000000
0.8	0.000014026	0.000015515	0.000000000
0.9	0.000012310	0.000487491	0.000000000
1.0	0.000010908	0.000439974	0.000000000

Example 3
We consider the special initial value problem in ordinary differential equation of the form

$$y' = 1 + y^2; y(0) = 1, h = 0.01, \quad 0 \le x \le 0.04 \qquad \text{(c)}$$

The exact result is given as

$$y(x) = \tan(x + \frac{\pi}{4}); 0 \le x \le 1$$

Table V
Results of Example 3

x	TSM-4	RK4	New Method
0.00	1.0000000	1.0000000	1.0000000
0.01	1.0202027	1.0202027	1.0202027
0.02	1.0408219	1.0408219	1.0408219
0.03	1.0618748	1.0618748	1.0618748
0.04	1.0833797	1.0833797	1.0833797
0.05	1.1053556	1.1053556	1.1053556
0.06	1.1278228	1.1278228	1.1278228
0.07	1.1508026	1.1508026	1.1508026
0.08	1.1743174	1.1743174	1.1743174
0.09	1.1983911	1.1983911	1.1983911
0.10	1.2230489	1.2230489	1.2230489

The exact solution is given as follows

Table VI
Exact Solution to (c)

x	Exact Solution
0.00	1.0000000
0.01	1.0202027
0.02	1.0408219
0.03	1.0618748
0.04	1.0833797
0.05	1.1053556
0.06	1.1278228
0.07	1.1508026
0.08	1.1743174
0.09	1.1983911
0.10	1.2230489

B. *Interpretation of results*

We have implemented the fourth stage of the Inverse Polynomial Scheme which has an advantage over all previously proposed methods of the same order as it is seen in Table II when it was compared with Runge-Kutta method of order 4 (RK4) and the Taylor's series method P=4 (TSM-4). Table IV showed the analysis of error in each of the methods. In Table I, the new method is well behaved when compared with the exact solution. This makes it to be more accurate and reliable.

Example 3 is a special initial value problem which is unbounded or undefined at '$x = \frac{\pi}{4}$'

So, the new method compared favourably with the existing methods and the exact method when called upon to solve this form of initial value problem as shown above in Table V.

From all the examples, we can see that the issue of stability and consistency of the new scheme is well demonstrated and thereby showing a measure of convergence towards the exact solution.

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