Periodic Solutions in Shifts Delta(+/-) for a Nabla Dynamic System of Nicholson’s Blowflies on Time Scales

Lili Wang, Pingli Xie, and Meng Hu

Abstract—In this paper, based on some properties of nabla exponential function $e_p(t, t_0)$ and shift operators $\delta_\pm$ on time scales, by using Krasnosel’skiı’s fixed point theorem in a cone and some mathematical methods, sufficient conditions are established for the existence and nonexistence of positive periodic solutions in shifts $\delta_\pm$ for a nabla dynamic system of Nicholson’s blowflies on time scales of the following form:

$$x^\nabla(t) = -a(t)x(t) + \sum_{i=1}^{m} b_i(t)\nabla(x(\delta_-(\tau_i, t)))e^{-c_i(t)}x(\delta_-(\tau_i, t)),$$

where $t \in \mathbb{T}, \mathbb{T} \subset \mathbb{R}$ is a periodic time scale in shifts $\delta_\pm$ with period $P \in [t_0, \infty)$ and $t_0 \in \mathbb{T}$ is nonnegative and fixed. Finally, two numerical examples are presented to illustrate the feasibility and effectiveness of the results.

Index Terms—positive periodic solution; Nicholson’s blowflies model; nabla dynamic equation; shift operator; time scale.

I. INTRODUCTION

T he theory of time scales was introduced by S. Hilger [1] in order to unify, extend, and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus. A time scale is a nonempty arbitrary closed subset of reals. The time scales approach not only unifies differential and difference equations, but also solves some other problems such as a mix of stop-start and continuous behaviors [2,3] powerfully. Nowadays the theory on time scales has been widely applied to ecological dynamic systems.

In 1980, Gurney et al. [4] proposed a mathematical model

$$x'(t) = -\delta x(t) + px(t - \tau)e^{-ax(t - \tau)}$$

to describe the dynamics of Nicholson’s blowflies, where $x(t)$ is the size of the population at time $t$, $p$ is the maximum per capita daily egg production, $1/\delta$ is the size at which the population reproduces at its maximum rate, $\delta$ is the per capita daily adult death rate, and $\tau$ is the generation time. Nicholson’s blowflies model and its analogous equations on time scales have attracted much attention in the past few years; see, for example, [5,6].

The existence problem of periodic solutions is of importance to biologists since most models deal with certain types of populations. In the paper of Kaufmann and Raffoul [7], the authors were the first to define the notion of periodic time scales, by satisfying the additivity “there exists a $\omega > 0$ such that $t \pm \omega \in \mathbb{T}, \forall t \in \mathbb{T}$.” Under this additivity all periodic time scales are unbounded above and below. However, there are many time scales that are of interest to biologists and scientists such as $q^\mathbb{Z}$ and $\cup_{k=1}^{\infty} [3^k \mathbb{Z}, 23^k \mathbb{Z}] \cup \{0\}$ which do not satisfy the additivity. To overcome such difficulties, Adıvar introduced a new periodicity concept on time scales which does not oblige the time scale to be closed under the operation $t \pm \omega$ for a fixed $\omega > 0$. He defined a new periodicity concept with the aid of shift operators $\delta_\pm$ which are first defined in [8] and then generalized in [9].

In recent years, periodic solutions in shifts $\delta_\pm$ for some nonlinear dynamic equations on time scales with delta derivative have been studied by many authors; see, for example, [10-13]. However, to the best of our knowledge, there are few papers published on the existence of periodic solutions in shifts $\delta_\pm$ for a dynamic equation on time scales with nabla derivative.

Motivated by the above, in the present paper, we first study some properties of the nabla exponential function $e_p(t, t_0)$ and shift operators $\delta_\pm$ on time scales, and then we consider the following nabla dynamic system of Nicholson’s blowflies on time scales:

$$x^\nabla(t) = -a(t)x(t) + \sum_{i=1}^{m} b_i(t)x(\delta_-(\tau_i, t))e^{-c_i(t)}x(\delta_-(\tau_i, t)),$$

where $t \in \mathbb{T}, \mathbb{T} \subset \mathbb{R}$ is a periodic time scale in shifts $\delta_\pm$ with period $P \in [t_0, \infty)$ and $t_0 \in \mathbb{T}$ is nonnegative and fixed; $a, b_i \in C_{\Delta}(\mathbb{T}, (0, \infty))$ for $i = 1, 2, \ldots, m$ are $\Delta$-periodic in shifts $\delta_\pm$ with period $\omega$ and $-a \in \mathbb{R}^\times$; $c_i \in C_{\Delta}(\mathbb{T}, (0, \infty))$ are periodic in shifts $\delta_\pm$ with period $\omega$ for $i = 1, 2, \ldots, m$; $\tau_i(i = 1, 2, \ldots, m)$ are fixed if $\mathbb{T} = \mathbb{R}$ and $\tau_i \in [P, \infty)$ if $\mathbb{T}$ is periodic in shifts $\delta_\pm$ with period $P$.

For convenience, we introduce the notation

$$f^* = \sup_{t \in [t_0, \delta^\pm_+(t_0)]} f(t), \quad f_* = \inf_{t \in [t_0, \delta^\pm_-(t_0)]} f(t),$$

where $f$ is a positive and bounded periodic function.

Take the initial condition

$$x(s) = \phi(s), \phi \in C_{\Delta}([\delta_-(\tau^*, 0), 0], (0, \infty)), \phi \neq 0,$$

where $\tau^* = \max_{1 \leq i \leq m} \tau_i$.

It is easy to prove that the initial value problem (1) and (2) has a unique non-negative solution $x(t)$ on $[0, \infty)$.

The main purpose of this paper is to establish sufficient conditions for the existence and nonexistence of positive
periodic solutions in shifts $\delta_{\pm}$ of system (1) using Krasnoselskii’s fixed point theorem in a cone and some mathematical methods.

II. PRELIMINARIES

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ for all $t \in \mathbb{T}$, while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ for all $t \in \mathbb{T}$.

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf\mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum $m$, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum $m$, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$. The backwards graininess function $\nu : \mathbb{T} \to [0, +\infty)$ is defined by $\nu(t) = t - \rho(t)$.

A function $f : \mathbb{T} \to \mathbb{R}$ is $\nu$-continuous if $f$ is continuous and $\nu f \in C_{\text{rd}}(\mathbb{T})$. If $f$ is discontinuous at $t \in \mathbb{T}$, then $f$ is $\nu$-continuous if $f$ is left-scattered at $t$.

The set of all $\nu$-continuous functions is denoted by $\mathcal{R}_{\nu} = \mathcal{R}_{\nu}(\mathbb{T}, \mathbb{R})$.

Lemma 1. [14] If $P(f) \in \mathcal{R}_{\nu}$, and $a, b, c \in \mathbb{T}$, then

(i) $P(f)[a, b) = \{\nu f \in [b, c) : b \leq \nu f \leq c, a \leq \nu f \}$

(ii) $P(f)[a, b) = \{\nu f \in [b, c) : b \leq \nu f \leq c, a \leq \nu f \}$

(iii) $P(f)[a, b) = \{\nu f \in [b, c) : b \leq \nu f \leq c, a \leq \nu f \}$

(iv) $P(f)[a, b) = \{\nu f \in [b, c) : b \leq \nu f \leq c, a \leq \nu f \}$

(vi) $\int_{t_0}^{t_1} f(t) g(t) \nu(t) dt$ for all $t_0 \in \mathbb{T}$.

For more details about the calculus on time scales, see [14].

Let $\mathbb{T}$ be a non-empty subset of a time scale $\mathbb{T}$ and $t_0 \in \mathbb{T}$ be a fixed number, define operators $\delta_{\pm} : [t_0, +\infty) \times \mathbb{T} \to \mathbb{T}$: $\delta_{\pm}(s, t)$ is called to be rightward $\nu$-continuous at $t \in \mathbb{T}$, respectively. The variable $s \in [t_0, +\infty)$ in $\delta_{\pm}(s, t)$ is called the shift size. The value $\sigma_{\pm}(s, t)$ in $\mathbb{T}$ indicates $s$ units translation of the term $t \in \mathbb{T}$ to the right and left, respectively.

The sets

$$D_{\pm} = \{(s, t) \in [t_0, +\infty) \times \mathbb{T} : \delta_{\pm}(s, t) \in \mathbb{T}^*\}$$

are the domains of the shift operators $\delta_{\pm}$, respectively. Hereafter, $\mathbb{T}^*$ is the largest subset of the time scale $\mathbb{T}$ such that the shift operators $\delta_{\pm} : [t_0, +\infty) \times \mathbb{T} \to \mathbb{T}$ exist.

Definition 1. [15] (Periodicity in shifts $\delta_{\pm}$) Let $T$ be a time scale with the shift operators $\delta_{\pm}$ associated with the initial point $t_0 \in \mathbb{T}$. The time scale $\mathbb{T}$ is said to be periodic in shifts $\delta_{\pm}$ if there exists $p \in (t_0, +\infty)$ such that $(p, t) \in D_{\pm}$ for all $t \in \mathbb{T}^*$. Furthermore, if

$$P := \inf\{p \in (t_0, +\infty) \cap \mathbb{T}^* : (p, t_0) \in \delta_{\pm}, \forall t \in \mathbb{T^*}\} \neq t_0,$$

then $P$ is called the period of the time scale $\mathbb{T}$.

Definition 2. [15] (Periodic function in shifts $\delta_{\pm}$) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with the period $P$. We say that a real-valued function $f$ defined on $\mathbb{T}^*$ is periodic in shifts $\delta_{\pm}$ if there exists $\omega \in [P, +\infty)$ such that $(\omega, t) \in \mathbb{T}^*$ and $f(\delta_{\pm}(t)) = f(t)$ for all $t \in \mathbb{T}^*$, where $\delta_{\pm}(\omega, t)$. The smallest number $\omega \in [P, +\infty)$ is called the period of $f$.

Definition 3. (V-Periodic function in shifts $\delta_{\pm}$) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with the period $P$. We say that a real-valued function $f$ defined on $\mathbb{T}^*$ is $\nu$-periodic in shifts $\delta_{\pm}$ if there exists $\omega \in [P, +\infty)$ such that $(\omega, t) \in \mathbb{T}^*$, the shifts $\delta_{\pm}$ are $\nu$-differentiable with $\nu$-continuous derivatives and $f(\delta_{\pm}(t)) = f(t)$ for all $t \in \mathbb{T}^*$, where $\delta_{\pm}(\omega, t)$. The smallest number $\omega \in [P, +\infty)$ is called the period of $f$.

Similar to the proofs of Lemma 2, Corollary 1 and Theorem 2 in [15], we can get the following two lemmas.

Lemma 2. $\delta_{\pm}(\nu(t)) = \nu(\delta_{\pm}(t))$ and $\nu(\delta_{\pm}(t)) = \rho(\delta_{\pm}(t))$ for all $t \in \mathbb{T}^*$.

Lemma 3. Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with the period $P$, and let $f$ be a $\nu$-periodic function in shifts $\delta_{\pm}$ with the period $\omega \in [P, +\infty)$. Assume that $f \in C_{\text{rd}}(\mathbb{T})$, then

$$\int_{t_0}^{t_1} f(t) \nu(t) dt$$

for all $t_0 \in \mathbb{T}$.

Lemma 4. [16] Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with the period $P$. Assume that the shifts $\delta_{\pm}$ are $\nu$-differentiable on $t \in \mathbb{T}^*$, where $\omega \in [P, +\infty)$. Then the $\nu$-graininess function $\nu : \mathbb{T} \to [0, +\infty)$ satisfies

$$\nu(\delta_{\pm}(t)) = \delta_{\nu^{-1}}(t)\nu(t).$$

Lemma 5. [16] Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with the period $P$. Assume that the shifts $\delta_{\pm}$ are $\nu$-differentiable on $t \in \mathbb{T}^*$, where $\omega \in [P, +\infty)$. Then the $\nu$-graininess function $\nu : \mathbb{T} \to [0, +\infty)$ satisfies

$$\nu(\delta_{\pm}(t)) = \delta_{\nu^{-1}}(t)\nu(t).$$

Lemma 6. [14] Assume that $r$ is $\nu$-regressive and $f : \mathbb{T} \to \mathbb{R}$ is $\nu$-continuous. Let $t_0 \in \mathbb{T}$, $y_0 \in \mathbb{R}$, then the unique solution of the initial value problem

$$y' = f(t), \quad y(t_0) = y_0$$

is given by

$$y(t) = \left(\mathcal{E}_r(t, t_0)\right)y_0 + \int_{t_0}^{t} \mathcal{E}_r(t, \tau)f(\tau)\nu(\tau).$$

Set

$$X = \{x : x \in C_{\text{rd}}(\mathbb{T}, \mathbb{R}), x(\delta_{\pm}(t)) = x(t)\}$$
with the norm \( \|x\| = \sup_{t \in [t_0, \delta_+^+(t_0) \cap T]} |x(t)| \), then \( X \) is a Banach space.

**Lemma 7.** The function \( x(t) \in X \) is an \( \omega \)-periodic solution in shifts \( \delta_\pm \) of system (1) if and only if \( x(t) \) is an \( \omega \)-periodic solution in shifts \( \delta_\pm \) of

\[
x(t) = \int_t^{t+\omega} G(t, s) \sum_{i=1}^m b_i(s) x(\delta_-(\tau_i, s))
\]

\[\times e^{-c_i(s)x(\delta_-(\tau_i, s))} \nabla s,\]

where \( G(t, s) = \frac{\hat{e}_{-a}(t, \rho(s))}{\hat{e}_{-a}(0, \delta_+^+(t_0)) - 1} \).

**Proof:** If \( x(t) \) is an \( \omega \)-periodic solution in shifts \( \delta_\pm \) of system (1). By using Lemmas 1 and 6, for any \( \delta \)

\[
(x(t) = \int_t^{t+\omega} G(t, s) \sum_{i=1}^m b_i(s) x(\delta_-(\tau_i, s))
\]

\[\times e^{-c_i(s)x(\delta_-(\tau_i, s))} \nabla s,\]

Let \( s = \delta_+^+(t) \), then

\[
x(\delta_+^+(t)) = \hat{e}_{-a}(t, \rho(s)) \int_{t_0}^{t+\omega} e^{-a(t, \rho(s))} \nabla \theta.
\]

Noticing that \( \hat{e}_{-a}(t, \delta_+^+(t)) = \hat{e}_{-a}(0, \delta_+^+(t_0)) \), \( x(\delta_+^+(t)) = x(t) \), by Lemma 1, then \( x(t) \) satisfies (3).

Let \( x(t) \) be an \( \omega \)-periodic solution in shifts \( \delta_\pm \) of (3). By

(3) and Lemmas 1, 2 and 5, we have

\[
x(\delta_+^+(t)) = -a(t)x(t)
\]

\[+ G(\rho(t, \delta_+^+(t))) \sum_{i=1}^m b_i(t)x(\delta_-(\tau_i, t))
\]

\[\times e^{-c_i(t)x(\delta_-(\tau_i, t))} \theta.
\]

So, \( x(t) \) is an \( \omega \)-periodic solution in shifts \( \delta_\pm \) of system (1).

This completes the proof.

It is easy to verify that the Green’s function \( G(t, s) \) satisfies the property

\[
0 < \frac{1}{\xi - 1} \leq G(t, s) \leq \xi - 1, \quad \forall s \in [t, \delta_+^+(t_0) \cap T],
\]

where \( \xi = \hat{e}_{-a}(0, \delta_+^+(t_0)) \). By Lemma 5, we have

\[
G(\delta_+^+(t), \delta_+^+(s)) = G(t, s), \quad \forall t \in T^+, s \in [t, \delta_+^+(t_0) \cap T].
\]

Define \( K \), a cone in \( X \), by

\[
K = \{ x \in X : x(t) \geq \frac{1}{\xi} ||x||, \forall t \in [t_0, \delta_+^+(t_0) \cap T] \}
\]

and an operator \( H : K \to X \) by

\[
(Hx)(t) = \int_t^{t+\omega} G(t, s) \sum_{i=1}^m b_i(s) x(\delta_-(\tau_i, s))
\]

\[\times e^{-c_i(s)x(\delta_-(\tau_i, s))} \nabla s.
\]

In the following, we shall give some lemmas concerning \( K \) and \( H \) defined by (6) and (7), respectively.

**Lemma 8.** \( H : K \to K \) is well defined.

**Proof:** For any \( x \in K, t \in [t_0, \delta_+^+(t_0) \cap T] \). In view of (7), by Lemma 3 and (5), we have

\[
Hx(\delta_+^+(t)) = \int_{\delta_+^+(t)}^{\delta_+^+(t+\omega)} G(\rho(t, \delta_+^+(t))) \sum_{i=1}^m b_i(s)x(\delta_-(\tau_i, s))
\]

\[\times e^{-c_i(s)x(\delta_-(\tau_i, s))} \nabla s.
\]

\[
\geq \frac{1}{\xi} ||Hx||,
\]

that is, \( Hx \in X \).

Furthermore, for any \( x \in K, t \in [t_0, \delta_+^+(t_0) \cap T] \), we have

\[
(Hx)(\delta_+^+(t)) \geq \frac{1}{\xi - 1} \int_{t_0}^{t+\omega} G(\rho(t, \delta_+^+(t))) \sum_{i=1}^m b_i(s)x(\delta_-(\tau_i, s))
\]

\[\times e^{-c_i(s)x(\delta_-(\tau_i, s))} \nabla s.
\]

\[
\geq \frac{1}{\xi} - \frac{B}{c_1} := M_1,
\]

where

\[
c_1 = \min_{1 \leq i \leq m} c_i, \quad B := \int_{t_0}^{\delta_+^+(t_0)} \sum_{i=1}^m b_i(s) \nabla s.
\]

Furthermore, for \( t \in T \), we have

\[
(Hx)(\delta_+^+(t)) = -a(t)(Hx)(t)
\]

\[+ \sum_{i=1}^m b_i(t)x(\delta_-(\tau_i, t))e^{-c_i(t)x(\delta_-(\tau_i, t))},
\]

(Advance online publication: 17 November 2017)
Lemma 10. Let \( H \) be a Banach space and \( K \subset \Omega \) be a cone in \( K \). Assume that \( \Omega_1, \Omega_2 \) are bounded open subsets of \( X \) with \( 0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2 \) and \( H : \Omega_2 \cap (\Omega_2 \setminus \Omega_1) \to K \) is a completely continuous operator such that, either

1. \( \|Hx\| \leq \|x\|, x \in K \cap \partial \Omega_1, \) and \( \|Hx\| \leq \|x\|, x \in K \cap \partial \Omega_2; \)

2. \( \|Hx\| \geq \|x\|, x \in K \cap \partial \Omega_1, \) and \( \|Hx\| \leq \|x\|, x \in K \cap \partial \Omega_2. \)

Then \( H \) has at least one fixed point in \( K \cap (\Omega_2 \setminus \Omega_1). \)

Lemma 11. Let

\[
\sum_{i=1}^{m} b_i(t) > a(t), t \in [t_0, \delta^*_+(t_0)].
\]

Then there exist positive constants \( M_1 \) and \( M_2 \) such that for \( x \in K, \)

\[
M_2 \leq \|Hx\| \leq M_1.
\]

Proof: From (8), for any \( x \in K, t \in [t_0, \delta^*_+(t_0)], \)

\[
\|Hx\| \leq M_1.
\]

From (9), there exists a \( q > 1 \) such that

\[
\sum_{i=1}^{m} b_i(t) > qa(t), t \in [t_0, \delta^*_+(t_0)].
\]

For any \( x \in K, t \in [t_0, \delta^*_+(t_0)], \)

\[
(Hx)(t) = \int_{t_0}^{\delta^*_+(t)} G(t, s) \sum_{i=1}^{m} b_i(s)x(\delta_-(\tau_i, s)) e^{-c_+s}(x(\delta_-(\tau_i, s))) \, ds
\]

\[
> q \int_{t_0}^{\delta^*_+(t_0)} a(s) e^{-a(t_0, \rho(s))} \, ds
\]

\[
\geq q \min \left\{ e^{-c_+s} \right\} \sum_{i=1}^{m} b_i(t_0) x(\delta_-(\tau_i, t_0))
\]

\[
\geq q \min \{ x \} e^{-c_+t_0}, x^* e^{-c_+ t^*},
\]

where \( c^* = \max_{1 \leq i \leq m} c_i. \)

Comparing (3) with (7), we also have for \( x \in K, t \in [t_0, \delta^*_+(t_0)], \)

\[
x(t) > q \min \{ x \} e^{-c_+t_0}, x^* e^{-c_+ t^*},
\]

which implies that

\[
x > q \min \{ x \} e^{-c_+t_0}, x^* e^{-c_+ t^*}.
\]

In the same way as (8), \( x(t) \leq M_1, \) which implies that

\[
x \leq M_1.
\]

If \( \min \{ x \} e^{-c_+t_0}, x^* e^{-c_+ t^*} \geq q \min \{ x \} e^{-c_+t_0}, x^* e^{-c_+ t^*}, \) then

\[
(Hx)(t) > qM_1 e^{-c_+t_0} := M_2 > 0.
\]

If \( \min \{ x \} e^{-c_+t_0}, x^* e^{-c_+ t^*} > q \min \{ x \} e^{-c_+t_0}, x^* e^{-c_+ t^*}, \) which implies that

\[
x > \frac{\ln q}{c^*}.
\]

From (13), we obtain

\[
(Hx)(t) > \frac{\ln q}{c^*} e^{-c_+t_0} \geq \ln \frac{q}{c^*} := M_2 > 0.
\]

Let \( M_2 = \min \{ M_2, 1 \}, \) then for \( x \in K, \)

\[
\|Hx\| \geq M_2.
\]

This completes the proof.

Theorem 1. Assume that

\[
\sum_{i=1}^{m} b_i(t) > a(t), t \in [t_0, \delta^*_+(t_0)].
\]

Then system (1) has at least one positive \( \omega \)-periodic solution in shifts \( \delta^*_+. \)

Proof: Let

\[
\Omega_1 = \{ x \in X : \|x\| \leq M_2 \},
\]

and

\[
\Omega_2 = \{ x \in X : \|x\| \leq M_1 \}.
\]

Clearly, \( \Omega_1 \) and \( \Omega_2 \) are open bounded subsets in \( X \), and \( \theta \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2 \). From Lemma 8, \( H : K \cap (\Omega_2 \setminus \Omega_1) \to K \) is completely continuous.

If \( x \in K \cap \partial \Omega_2, \) which implies that \( \|x\| = M_1 \), from Lemma 11, \( \|Hx\| \leq M_1. \) Hence \( \|Hx\| \leq \|x\| \) for \( x \in K \cap \partial \Omega_2. \)

If \( x \in K \cap \partial \Omega_1, \) which implies that \( \|x\| = M_2, \) from Lemma 11, \( \|Hx\| \leq M_2. \) Hence \( \|Hx\| \geq \|x\| \) for \( x \in K \cap \partial \Omega_1. \)

From the cone fixed point theorem (Lemma 10), the operator \( H \) has at least one fixed point lying in \( K \cap (\Omega_2 \setminus \Omega_1), \) i.e., system (1) has at least one positive \( \omega \)-periodic solution in shifts \( \delta^*_+. \) This completes the proof.
IV. NONEXISTENCE RESULT

In this section, we shall state and prove our main result about the nonexistence of positive periodic solution in shifts $\delta_{\pm}$ of system (1).

**Lemma 12.** Assume that

$$
\sum_{i=1}^{m} b_i(t) \leq \frac{1}{2} a(t), t \in [t_0, \delta_+^{(t_0)}].
$$

(19)

Then every positive solution of system (1) tends to zero as $t \to \infty$.

**Proof:** Let $x(t)$ be any positive solution of system (1). By using Lemma 5, integrating system (1) from $t_0$ to $t(> t_0)$, we have

$$
x(t) = \dot{e}_{-a}(t, t_0) x(t_0) + \int_{t_0}^{t} \dot{e}_{-a}(t, \rho(s)) \sum_{i=1}^{m} b_i(s) x(\delta_-(\tau_i, s)) \times e^{-c_i(s) \delta_-(\tau_i, \tau_i)} \nabla s.
$$

(20)

From (19),

$$
x(t) \leq \dot{e}_{-a}(t, t_0) x(t_0) + \frac{1}{2c_{a}} \int_{t_0}^{t} a(s) \dot{e}_{-a}(t, \rho(s)) \nabla s
$$

$$
= \dot{e}_{-a}(t, t_0) x(t_0) + \frac{1}{2c_{a}} \int_{t_0}^{t} \nabla [\dot{e}_{-a}(t, s)]
$$

$$
= \dot{e}_{-a}(t, t_0) x(t_0) + \frac{1}{2c_{a}} [1 - \dot{e}_{-a}(t, t_0)].
$$

Let $\beta = \limsup_{t \to \infty} x(t)$, then $0 \leq \beta < \infty$.

Next, we shall prove $\beta = 0$. We have some possible cases to consider.

**Case 1.** $x^{\nabla}(t) > 0$ eventually. Choose $t_0 > 0$ such that $x^{\nabla}(t) > 0$ for $t \geq t_0$. Let $\eta > 0$ be a sufficient large number with $\delta_-(\tau_i, t) > t_0, i = 1, 2, \ldots, m$ for $t \geq t_0 + \eta$. Then $0 < x(\delta_-(\tau_i, t)) < x(t)$ for $t \geq t_0 + \eta$ and $i = 1, 2, \ldots, m$. From (1), for $t \geq t_0 + \eta$,

$$
0 < -a(t)x(t) + \sum_{i=1}^{m} b_i(t) x(\delta_-(\tau_i, t)) e^{-c_i(t) \delta_-(\tau_i, t)}
$$

$$
< \left[ \sum_{i=1}^{m} b_i(t) - a(t) \right] x(t) < 0.
$$

This contradiction shows that Case 1 is impossible.

**Case 2.** $x^{\nabla}(t) < 0$ eventually. Choose $t_0 > 0$ such that $x^{\nabla}(t) < 0$ for $t \geq t_0$. Then $\beta < x(\delta_-(\tau_i, t)) < x(\delta_-(\tau_i, t_0))$ for $t \geq t_0 + \eta$ and $i = 1, 2, \ldots, m$. From (19) and (20), we have

$$
x(t) \leq \dot{e}_{-a}(t, t_0) x(t_0) + \frac{1}{2} \max_{1 \leq i \leq m} x(\delta_-(\tau_i, t_0)) e^{-c_i \beta}
$$

$$
\times [1 - \dot{e}_{-a}(t, t_0)].
$$

(21)

Let $t \to \infty$ in (21), we obtain

$$
\beta \leq \frac{1}{2} \max_{1 \leq i \leq m} x(\delta_-(\tau_i, t_0)) e^{-c_i \beta}.
$$

(22)

Again let $t_0 \to \infty$ in (22), we have that $\beta \leq \beta(\frac{1}{2} e^{-c_i \beta})$, which implies that $\beta = 0$.

**Case 3.** $x^{\nabla}(t)$ is oscillatory. By the definition of oscillatory, then

(i) there exists $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$
x^{\nabla}(t) = 0 \quad \text{and} \quad \lim_{n \to \infty} x(t_n) = \beta;
$$

or

(ii) there exists $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$
x^{\nabla}(t_n) x^{\nabla}(\rho(t_n)) < 0 \quad \text{for} \quad n = 1, 2, \ldots,
$$

and

$$
\lim_{n \to \infty} x(t_n) = \lim_{n \to \infty} x(\rho(t_n)) = \beta.
$$

In case (i), from (1),

$$
a(t_n) x(t_n)
$$

$$
= \sum_{i=1}^{m} b_i(t_n) x(\delta_-(\tau_i, t_n)) e^{c_i(t_n) x(\delta_-(\tau_i, t_n))}
$$

$$
\leq x(\delta_-(\tau_i, t_n)) e^{-c_i x(\delta_-(\tau_i, \tau_i))} \sum_{i=1}^{m} b_i(t_n),
$$

(23)

where $l = l(n) \in \{1, 2, \ldots, m\}$ such that

$$
x(\delta_-(\tau_i, t_n)) e^{-c_i x(\delta_-(\tau_i, \tau_i))}
$$

$$
= \max_{1 \leq i \leq m} x(\delta_-(\tau_i, t_n)) e^{-c_i x(\delta_-(\tau_i, \tau_i))}.
$$

From (19) and (23), we have

$$
2x(t_n) e^{c_i x(\delta_-(\tau_i, t_n))} < x(\delta_-(\tau_i, t_n)).
$$

(24)

Set $\alpha = \limsup_{n \to \infty} x(\delta_-(\tau_i, t_n))$, then $\alpha \leq \beta$. Finding the superior limit of both sides of (24), we obtain

$$
\beta(2e^{c_i \alpha}) \leq \alpha,
$$

then

$$
\beta(2e^{c_i \alpha}) \leq \alpha \leq \beta,
$$

which implies that $\beta = \alpha = 0$.

In case (ii), from (1),

$$
a(t_n) a(\rho(t_n)) x(t_n) x(\rho(t_n))
$$

$$
+ \sum_{i=1}^{m} b_i(t_n) x(\delta_-(\tau_i, t_n)) e^{-c_i(t_n) x(\delta_-(\tau_i, t_n))}
$$

$$
\times \sum_{i=1}^{m} b_i(t_n) x(\delta_-(\tau_i, \rho(t_n)))
$$

$$
\times e^{-c_i(t_n) x(\delta_-(\tau_i, \rho(t_n)))}
$$

$$
< a(t_n) x(t_n) \sum_{i=1}^{m} b_i(\rho(t_n)) x(\delta_-(\tau_i, \rho(t_n)))
$$

$$
\times e^{-c_i(\rho(t_n)) x(\delta_-(\tau_i, \rho(t_n)))}
$$

$$
+ a(\rho(t_n)) x(\rho(t_n)) \sum_{i=1}^{m} b_i(t_n) x(\delta_-(\tau_i, t_n))
$$

$$
\times e^{-c_i(t_n) x(\delta_-(\tau_i, t_n))}
$$

$$
\leq [a(t_n) x(t_n) \sum_{i=1}^{m} b_i(\rho(t_n))
$$

$$
+ a(\rho(t_n)) x(\rho(t_n)) \sum_{i=1}^{m} b_i(t_n)]
$$

$$
\times x(\delta_-(\tau_i, t_n)) e^{-c_i x(\delta_-(\tau_i, t_n))},
$$

(25)

where $l = l(n) \in \{1, 2, \ldots, m\}$, $t_n = \{t_n, \rho(t_n)\}$, such that

$$
x(\delta_-(\tau_i, t_n)) e^{-c_i(t_n) x(\delta_-(\tau_i, t_n))}
$$

$$
= \max_{1 \leq i \leq m} \{x(\delta_-(\tau_i, t_n)) e^{-c_i(t_n) x(\delta_-(\tau_i, t_n))},
$$

$$
x(\delta_-(\tau_i, \rho(t_n))) e^{-c_i(\rho(t_n)) x(\delta_-(\tau_i, \rho(t_n)))}\}.
$$
From (19) and (25), we have
\[
2x(t_n)x(\rho(t_n))e^{c\alpha x(\delta_-(\tau_1, t_n))} \\
\leq [x(t_n) + x(\rho(t_n))]^2 x(\delta_-(\tau_1, t_n)).
\] (26)
Set \( \alpha = \limsup x(\delta_-(\tau_1, t_n)) \), then \( \alpha \leq \beta \). Finding the superior limit of both sides of (26), we obtain
\[
\beta e^{c \alpha} \leq \alpha,
\]
therefore
\[
\beta e^{c \alpha} \leq \alpha \leq \beta,
\]
which implies that \( \beta = \alpha = 0 \). This completes the proof. \( \blacksquare \)

From Lemma 12, we can get the following Theorem.

**Theorem 2.** Assume that the condition (19) hold. Then system (1) has no positive \( \omega \)-periodic solution in shifts \( \delta_\pm \).

**V. NUMERICAL EXAMPLES**

Consider the following Nicholson’s blowflies model on time scales \( \mathbb{T} \)
\[
x^{\nabla}(t) = -a(t)x(t) + \sum_{i=1}^{2} b_i(t)x(\delta_-(\tau_i, t))e^{-c_i(t)x(\delta_-(\tau_i, t))}.
\] (27)

**Example 1.** Take
\[
a(t) = a_0 + \frac{|\sin 2t + \cos 3t|}{2},
b_1(t) = e^{-1}(10 + 0.005)\sin t,
b_2(t) = e^{-1}(10 + 0.005)\cos t,
c_i(t) = c_2(t) = 0.25 + 0.025|\sin 3t + \cos 2t|.
\]
Let \( \mathbb{T} = \mathbb{R} \), \( t_0 = 0 \), then \( \omega = \pi \) and \( \delta_\pm(t) = t + \pi \). It is easy to verify \( a(t), b_i(t), c_i(t) \) satisfy
\[
a(\delta_\pm(t))\delta_\pm^{\nabla}(t) = a(t),
b_i(\delta_\pm(t))\delta_\pm^{\nabla}(t) = b_i(t),
c_i(\delta_\pm(t)) = c_i(t), \quad \forall t \in \mathbb{T}^*, \quad i = 1, 2,
\]
and \(-a \in \mathbb{R}^+\).

**Case I.** If \( a_0 = 18 \), by a direct calculation, we can get
\[
\sum_{i=1}^{2} b_i(t) \geq 20 e^{-1} = a(t), \quad t \in \mathbb{R}.
\]
According to Theorem 1, when \( \mathbb{T} = \mathbb{R} \), system (27) exists at least one positive \( \pi \)-periodic solution in shifts \( \delta_\pm \).

**Case II.** If \( a_0 = 240 \), by a direct calculation, we can get
\[
\sum_{i=1}^{2} b_i(t) \leq 20.02 e^{-1} < \frac{1}{2} a(t), \quad t \in \mathbb{R}.
\]
According to Theorem 2, when \( \mathbb{T} = \mathbb{R} \), system (27) has no positive periodic solution in shifts \( \delta_\pm \).

**Example 2.** Take
\[
a(t) = \frac{1}{a_0 t},
b_1(t) = \frac{1}{2t},
b_2(t) = \frac{1}{3t},
c_1(t) = c_2(t) = 0.25.
\]
Let \( \mathbb{T} = 2^{N_0}, t_0 = 1 \), then \( \omega = 4 \) and \( \delta_\pm(t) = 4t \). It is easy to verify \( a(t), b_i(t), c_i(t) \) satisfy
\[
a(\delta_\pm(t))\delta_\pm^{\nabla}(t) = a(t),
b_i(\delta_\pm(t))\delta_\pm^{\nabla}(t) = b_i(t),
c_i(\delta_\pm(t)) = c_i(t), \quad \forall t \in \mathbb{T}^*, \quad i = 1, 2,
\]
and \(-a \in \mathbb{R}^+\).

**Case I.** If \( a_0 = 6 \), by a direct calculation, we can get
\[
\sum_{i=1}^{2} b_i(t) = \frac{5}{6t} > a(t), \quad t \in 2^{N_0}.
\]
According to Theorem 1, when \( \mathbb{T} = 2^{N_0} \), system (27) exists at least one positive \( 4 \)-periodic solution in shifts \( \delta_\pm \).

**Case II.** If \( a_0 = \frac{1}{2} \), by a direct calculation, we can get
\[
\sum_{i=1}^{2} b_i(t) = \frac{5}{6t} < \frac{1}{2} a(t), \quad t \in 2^{N_0}.
\]
According to Theorem 2, when \( \mathbb{T} = 2^{N_0} \), system (27) has no positive periodic solution in shifts \( \delta_\pm \).

**VI. CONCLUSION**

Two problems for a Nicholson’s blowflies model with time delays on time scales have been studied, namely, existence and nonexistence of positive periodic solutions in shifts \( \delta_\pm \) on time scales. It is important to notice that the methods used in this paper can be extended to other types of biological models; see, for example, [18-20]. Future work will include biological dynamic systems modeling and analysis on time scales.

**REFERENCES**


(Advance online publication: 17 November 2017)