A Domain Decomposition Method using Elliptical Arc Artificial Boundary for Exterior Problems

Yajun Chen, and Qikui Du

***Abstract*—In this paper, a Dirichlet-Neumann alternating method using elliptical arc artificial boundary is designed to solve exterior Poisson problem with a concave angle. It is shown that the method is equivalent to a preconditioned Richardson iteration. The convergence of this method and its discretization are studied. Finally, some numerical examples are given to show the effectiveness of this method.**

***Index Terms*—Dirichlet-Neumann alternating method, elliptical arc artificial boundary, exterior problem.**

I. Introduction

MANY scientific and engineering problems can be modeled by exterior boundary value problems of partial differential equations which are required to be solved in unbounded domains. In the last three decades, some methods for solving problems over unbounded domains have been developed. One of the commonly used techniques is the method of artificial boundary conditions [1]-[9]. The method may be summarized as follows: (i) Introduce an artificial boundary Γ_μ, which divides the original unbounded domain into two non-overlapping subdomains: a bounded computational domain Ω_i and infinite residual domain Ω_e. (ii) By analyzing the problem in Ω_e, obtain a relation on Γ_μ (exact or approximate) involving the unknown function u and its derivatives. (iii) Using the relation as a boundary condition on Γ_μ, to obtain a well-posed problem in Ω_i. (iv) Solve the problem in Ω_i be the standard finite element methods or some other numerical methods.The relation obtained in Step (ii) and used as a boundary condition in Step (iii) is called an artificial boundary condition.

Based on artificial boundary conditions, the overlapping and non-overlapping domain decomposition methods can be viewed as effective ways to solve problems in unbounded domains. These techniques have been used to solve many linear or nonlinear problems [10]-[17]. Recently, the authors used a new elliptical arc artificial boundary to solve Poisson problems and anisotropic problems [18]-[19], and construct an iteration method which is equivalent to a Schwarz alternating method to solve Poisson problems in concave angle domains [20]. In this paper, we design a Dirichlet-Neumann alternating method based on an elliptical arc artificial boundary to solve exterior Poisson problem with a concave angle.

Let Ω be an exterior concave angle domain with angle α, and $0 < \alpha \leq 2\pi$. The boundary of domain Ω is decomposed into three disjoint parts: Γ, Γ_0 and Γ_α(see Fig. 1), i.e. $\partial\Omega = \overline{\Gamma \cup \Gamma_0 \cup \Gamma_\alpha}$, $\Gamma_0 \cap \Gamma_\alpha = \emptyset$, $\Gamma \cap \Gamma_0 = \emptyset$, $\Gamma \cap \Gamma_\alpha = \emptyset$. The boundary Γ is a simple smooth curve part, Γ_0 and Γ_α are two half lines.

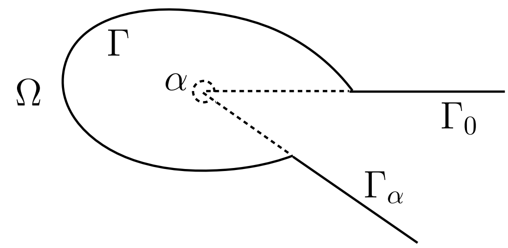

Fig. 1. The Illustration of Domain Ω

We consider the Poisson problem in two cases:

$$\begin{cases} -\Delta u = f, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \Gamma_0 \cup \Gamma_\alpha, \\ \dfrac{\partial u}{\partial n} = g, \quad \text{on } \Gamma, \\ u \text{ is vanish at infinity}, \end{cases} \tag{1}$$

and

$$\begin{cases} -\Delta u = f, \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0, \quad \text{on } \Gamma_0 \cup \Gamma_\alpha, \\ u = h, \quad \text{on } \Gamma, \\ u \text{ is bounded at infinity}, \end{cases} \tag{2}$$

where u is the unknown function, $f \in L^2(\Omega)$ and $g, h \in L^2(\Gamma)$ are given functions, supp(f) is compact.

The rest of the paper is organized as follows. In Section 2, we introduce an elliptical arc artificial boundary which divide the original domain Ω into two non-overlapping subdomains, then we construct a Dirichlet-Neumann alternating method. In Section 3, we give the weak form and discretization. In Section 4, we analyze the convergence of the method. Finally, in Section 5 we present some numerical results to show its accuracy and the effectiveness of our method.

II. Dirichlet-Neumann alternating method

Draw a elliptical arc $\Gamma_1 = \{(\mu, \varphi) | \mu = \mu_1, 0 < \varphi < \alpha\}$, which enclose Γ such that dist$(\Gamma, \Gamma_1) > 0$. Then Ω is divided into two non-overlapping subdomains Ω_1 and Ω_2(see Fig. 2). Let Ω_1 be the bounded domain among Γ, Γ_0, Γ_α and Γ_1, and Ω_2 be the unbounded domain outside Γ_1, Γ_0 and Γ_α. Then the original problems (1) is decomposed into two subproblems in domains Ω_1 and Ω_2 with $\Omega_1 \cap \Omega_2 = \emptyset$, $\partial\Omega_1 = \Gamma \cup \Gamma_1 \cup \Gamma_{01} \cup \Gamma_{\alpha 1}$, $\partial\Omega_2 = \Gamma_1 \cup \Gamma_{02} \cup \Gamma_{\alpha 2}$, where $\Gamma_{0i} = \overline{\Omega}_i \cap \Gamma_0$, $\Gamma_{\alpha i} = \overline{\Omega}_i \cap \Gamma_\alpha$, $i = 1, 2$.

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In the first case, we proposed the Dirichlet-Neumann alternating method as follows.

Step 0. Pick an initial value $\lambda^0 \in H^{\frac{1}{2}}(\Gamma_1)$, and put $k = 0$.

Step 1. Solve a Dirichlet problem in Ω_2

$$\begin{cases} -\Delta u_2^k = f, \quad \text{in } \Omega_2, \\ u_2^k = 0, \quad \text{on } \Gamma_{02} \cup \Gamma_{\alpha 2}, \\ u_2^k = \lambda^k, \quad \text{on } \Gamma_1, \\ u_2^k \text{ is vanish at infinity.} \end{cases} \tag{3}$$

Step 2. Solve a mixed problem in Ω_1

$$\begin{cases} -\Delta u_1^k = f, \quad \text{in } \Omega_1, \\ u_1^k = 0, \quad \text{on } \Gamma_{01} \cup \Gamma_{\alpha 1}, \\ \dfrac{\partial u_1^k}{\partial n} = g, \quad \text{on } \Gamma, \\ \dfrac{\partial u_1^k}{\partial n} = -\dfrac{\partial u_2^k}{\partial n}, \quad \text{on } \Gamma_1. \end{cases} \tag{4}$$

Step 3. Update the boundary value on Γ_1 by

$$\lambda^{k+1} = \theta_k u_1^k + (1-\theta_k)\lambda^k. \tag{5}$$

Step 4. Set $k = k+1$, then goto Step 1.

where u_1^k and u_2^k are the kth approximate solutions in Ω_1 and Ω_2, respectively. θ_k denotes the kth relaxation factor and λ^0 is an arbitrary function in $H^{\frac{1}{2}}(\Gamma_1)$. Note that, on interface Γ_1, only the value of the normal derivative of the solution of (3) is needed in solving (4). So it is unnecessary to solve (3). Actually, we can obtain $\frac{\partial u_2^k}{\partial n}$ directly from λ^k by making use of the following artificial boundary condition [18]:

$$\frac{\partial u_2^k}{\partial n} = \mathcal{K}\lambda^k, \tag{6}$$

where

$$\mathcal{K}\lambda^k = -\frac{2\pi}{\alpha^2\sqrt{J}} \sum_{n=1}^{+\infty} n \int_0^{\alpha} \lambda^k(\mu_1, \phi) \sin\frac{n\pi\varphi}{\alpha} \sin\frac{n\pi\phi}{\alpha} d\phi. \tag{7}$$

For the second case, we can also construct the Dirichlet-Neumann alternating method. In the following sections, we just consider the discretization and convergence of problem (1), we can obtain corresponding result of problem (2) in the same way.

III. The Weak Form and Discretization

Let

$$V(\Omega_1) = \{v | v \in H^1(\Omega_1), v|_{\Gamma_{01}\cup\Gamma_{\alpha 1}} = 0\},$$

then problem (1) is equivalent to the variational problem: Find $u \in V(\Omega_1)$, such that

$$a(u,v) + b(u,v) = f(v), \quad \forall v \in V(\Omega_1), \tag{8}$$

where

$$a(u,v) = \int_{\Omega_1} \nabla u \nabla v dx, \tag{9}$$

$$b(u,v) = \sum_{n=1}^{+\infty} \frac{2}{n\pi} \int_0^{\alpha}\int_0^{\alpha} \frac{\partial u}{\partial \phi}\frac{\partial v}{\partial \varphi} \cos\frac{n\pi\phi}{\alpha} \cos\frac{n\pi\varphi}{\alpha} d\phi d\varphi, \tag{10}$$

$$f(v) = \int_{\Omega_1} f v dx + \int_{\Gamma} g v ds. \tag{11}$$

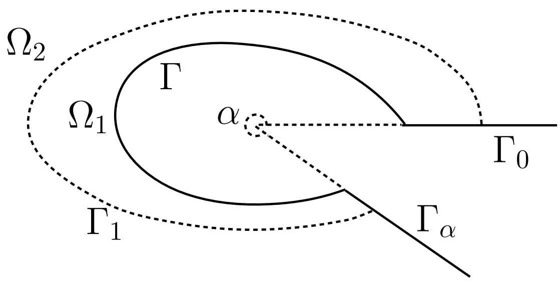

Fig. 2. The Illustration of Domain Ω_1 and Ω_2

Let $S_h(\Omega_1) \subset V(\Omega_1)$ denote the liner finite element space of $V(\Omega_1)$. Then the approximate variational problem of (8) can be written as: Find $u_h \in S_h(\Omega_1)$, such that

$$a(u_h, v_h) + b(u_h, v_h) = f(v_h), \quad \forall v_h \in S_h(\Omega_1). \tag{12}$$

From the problem (12), we can get algebraic equations as follows

$$\begin{pmatrix} A_{11}+B & A_{1i} & A_{10} \\ A_{i1} & A_{ii} & A_{i0} \\ A_{01} & A_{0i} & A_{00} \end{pmatrix} \begin{pmatrix} U_1 \\ U_i \\ U_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ F_0 \end{pmatrix}, \tag{13}$$

where U_1 is a vector whose components are function values at nodes on Γ_1, U_i is a vector whose components are function values at interior nodes in Ω_1 and U_0 is a vector whose components are function values at nodes on Γ. The matrix

$$A = \begin{pmatrix} A_{11} & A_{1i} & A_{10} \\ A_{i1} & A_{ii} & A_{i0} \\ A_{01} & A_{0i} & A_{00} \end{pmatrix}$$

is the stiffness matrix obtained from finite element in Ω_1 while B is gotten from the artificial boundary condition on Γ_1. (13) can also be rewritten as follows

$$\begin{pmatrix} A_{11} & A_{1i} & A_{10} \\ A_{i1} & A_{ii} & A_{i0} \\ A_{01} & A_{0i} & A_{00} \end{pmatrix} \begin{pmatrix} U_1 \\ U_i \\ U_0 \end{pmatrix} = \begin{pmatrix} -BU_1 \\ 0 \\ F_0 \end{pmatrix}. \tag{14}$$

Then, we have the iterative method

$$\begin{pmatrix} A_{11} & A_{1i} & A_{10} \\ A_{i1} & A_{ii} & A_{i0} \\ A_{01} & A_{0i} & A_{00} \end{pmatrix} \begin{pmatrix} U_1^k \\ U_i^k \\ U_0^k \end{pmatrix} = \begin{pmatrix} -B\Lambda^k \\ 0 \\ F_0 \end{pmatrix}, \tag{15}$$

with

$$\Lambda^{k+1} = \theta_k U_1^k + (1-\theta_k)\Lambda^k, \quad k = 0, 1, 2, \ldots \tag{16}$$

IV. Analysis of Convergence

It is difficult to analyze the convergence of the above alternating method in the general domain. However, the analysis is possible for some special curve Γ. Therefore, we only consider the case where the boundaries Γ and Γ_1 both are elliptical arcs, i.e., $\Gamma = \{(\mu,\varphi) | \mu = \mu_0, 0 < \varphi < \alpha\}$, $\Gamma_1 = \{(\mu,\varphi) | \mu = \mu_1, 0 < \varphi < \alpha\}$, and $\mu_1 > \mu_0$.

We first consider the convergence of the method in continuous case.

Theorem 1. If $0 < \theta_k < 1$, then the Dirichlet-Neumann alternating method (3)-(5) is convergent.

Proof. Let

$$e_2^k = \lambda - \lambda^k = \sum_{n=1}^{+\infty} b_n \sin\frac{n\pi\varphi}{\alpha}, \quad \text{on } \Gamma_1,$$

we have

$$\frac{\partial e_1^k}{\partial n} = -\mathcal{K}(e_2^k) = -\frac{\pi}{\alpha\sqrt{J}}\sum_{n=1}^{+\infty} nb_n \sin\frac{n\pi\varphi}{\alpha}. \quad (17)$$

By the separation of variables, we have

$$e_1^k = -\sum_{n=1}^{+\infty} b_n H_n(\mu)\sin\frac{n\pi\varphi}{\alpha},$$

where

$$H_n(\mu) = \frac{e^{\frac{n\pi}{\alpha}(\mu-\mu_0)} - e^{\frac{n\pi}{\alpha}(\mu_0-\mu)}}{e^{\frac{n\pi}{\alpha}(\mu_1-\mu_0)} - e^{\frac{n\pi}{\alpha}(\mu_0-\mu_1)}}.$$

Hence

$$\mathcal{K}(e_1^k) = -\frac{\pi}{\alpha\sqrt{J}}\sum_{n=1}^{+\infty} nb_n H_n(\mu)\sin\frac{n\pi\varphi}{\alpha}.$$

Then, we have

$$\begin{aligned}&\frac{\partial e_1^{k+1}}{\partial n}\\ &= -\mathcal{K}(\lambda-\lambda^{k+1})\\ &= \mathcal{K}(\theta_k u_1^k + (1-\theta_k)\lambda^k - \lambda)\\ &= \frac{\pi}{\alpha\sqrt{J}}\sum_{n=1}^{+\infty} nb_n(\theta_k H_n(\mu_1) - 1 + \theta_k)\sin\frac{n\pi\varphi}{\alpha}.\end{aligned} \quad (18)$$

If we let

$$E^k = \|\frac{\partial e_1^k}{\partial n}\|^2_{-\frac{1}{2},\Gamma_1},$$

then

$$E^k = \frac{\pi^2}{\alpha^2 J}\sum_{n=1}^{+\infty}(1+n^2)^{-\frac{1}{2}} n^2 b_n^2,$$

and

$$\begin{aligned}&E^{k+1}\\ &= \frac{\pi^2}{\alpha^2 J}\sum_{n=1}^{+\infty}(1+n^2)^{-\frac{1}{2}} n^2 b_n^2(\theta_k H_n(\mu_1) - 1 + \theta_k)^2\\ &= (1-\theta_k)^2 E^k + \frac{\pi^2}{\alpha^2 J}\sum_{n=1}^{+\infty}(1+n^2)^{-\frac{1}{2}} n^2 b_n^2\\ &\quad\cdot \theta_k H_n(\mu_1)(\theta_k H_n(\mu_1) + 2\theta_k - 2).\end{aligned}$$

Let

$$\delta = \inf_{n\in Z^+}\frac{2}{2+H_n(\mu_1)}.$$

A computation shows that $\delta = \frac{2}{3}$.

If we let θ_k, $k = 0, 1, 2, \ldots$, satisfy $0 < \theta_k \le \delta$, then

$$E^{k+1} < (1-\theta_k)^2 E^k. \quad (19)$$

By the trace theorem, we have

$$\|e_1^k\|^2_{1,\Omega_1} \le CE^k \to 0, \quad k \to +\infty.$$

This means that the Dirichlet-Neumann alternating method is convergent if $0 < \theta_k \le \delta$.

We also have

$$\begin{aligned}&E^{k+1}\\ &= \frac{\pi^2}{\alpha^2 J}\sum_{n=1}^{+\infty}(1+n^2)^{-\frac{1}{2}} n^2 b_n^2(2\theta_k - 1 - 2\theta_k G_n(\mu_1))^2\\ &= (1-\theta_k)^2 E^k + \frac{\pi^2}{\alpha^2 J}\sum_{n=1}^{+\infty}(1+n^2)^{-\frac{1}{2}} n^2 b_n^2\\ &\quad\cdot \theta_k G_n(\mu_1)(\theta_k G_n(\mu_1) - 2\theta_k + 1),\end{aligned}$$

where

$$G_n(\mu_1) = \frac{1-H_n(\mu_1)}{2}.$$

Let

$$\sigma = \sup_{n\in Z^+}\frac{1}{2-G_n(\mu_1)}.$$

It is easy to get $\sigma = \frac{2}{3}$.

Similar to the above analysis, if we take θ_k, $k = 0, 1, 2, \ldots$, satisfy $\sigma \le \theta_k < 1$, the Dirichlet-Neumann alternating method is also convergent.

Therefore, for $0 < \theta_k < 1$, the Dirichlet-Neumann alternating method is convergent.

In the following, we consider the convergence of the discretization form.

Theorem 2. The discrete Dirichlet-Neumann alternating method (15) and (16) are equivalent to the following preconditioned Richardson iteration:

$$S_h^{(1)}(\Lambda^{k+1} - \Lambda^k) = \theta_k(F_1 - S_h\Lambda^k), \quad (20)$$

where

$$S_h^{(1)} = A_{11} - A_{1i}(A_{ii} - A_{i0}(A_{00})^{-1}A_{0i})^{-1}A_{i1}, \quad (21)$$

$$S_h = S_h^{(1)} + B, \quad (22)$$

$$F_1 = A_{1i}(A_{ii} - A_{i0}(A_{00})^{-1}A_{0i})^{-1}A_{i0}(A_{00})^{-1}F_0. \quad (23)$$

Proof. From (13), we have

$$\begin{aligned}&(A_{11} - A_{1i}(A_{ii} - A_{i0}(A_{00})^{-1}A_{0i})^{-1}A_{i1} + B)U_1\\ &= A_{1i}(A_{ii} - A_{i0}(A_{00})^{-1}A_{0i})^{-1}A_{i0}(A_{00})^{-1}F_0,\end{aligned}$$

namely,

$$S_h U_1 = F_1.$$

From (14) and (15), we obtain

$$A\begin{pmatrix} U_1^k - U_1\\ U_i^k - U_i\\ U_0^k - U_0\end{pmatrix} = \begin{pmatrix} -B(\Lambda^k - U_1)\\ 0\\ 0\end{pmatrix}.$$

So

$$S_h^{(1)}(U_1^k - U_1) = B(U_1 - \Lambda^k).$$

Therefore

$$\begin{aligned}S_h^{(1)}(\Lambda^{k+1} - \Lambda^k) &= \theta_k S_h^{(1)}(U_1^k - \Lambda^k)\\ &= \theta_k(S_h^{(1)}(U_1^k - U_1) + S_h^{(1)}(U_1 - \Lambda^k))\\ &= \theta_k(B + S_h^{(1)}(U_1 - \Lambda^k))\\ &= \theta_k S_h(U_1 - \Lambda^k)\\ &= \theta_k(F_1 - S_h\Lambda^k).\end{aligned}$$

Theorem 3. Let ρ be spectral radius of $(S_h^{(1)})^{-1}S_h$, which is iterative matrix of preconditioned Richardson iteration. Then, there is a positive constant σ, which is independent of finite element mesh parameter h of subdomain Ω_1, such that $\rho \le \sigma$.

Theorem 4. Put $\theta_k = \theta(k = 0, 1, 2, \ldots)$, then, there exists a constant $\delta(0 < \delta < 1)$, which is independent of finite element mesh parameter h of subdomain Ω_1. For $0 < \theta < \delta$, the preconditioned Richardson iteration, i.e., Dirichlet-Neumann alternating method (15)-(16) converges and the convergence rate is independent of mesh parameter h of subdomain Ω_1.

Proof. From Theorem 2, we have

$$U_1 - \Lambda^{k+1} = (I - \theta(S_h^{(1)})^{-1}S_h)(U_1 - \Lambda^k)$$
$$= (I - \theta(S_h^{(1)})^{-1}S_h)^{k+1}(U_1 - \Lambda^0),$$

it comes that

$$\|U_1 - \Lambda^{k+1}\|_2 \leq \delta^{k+1}\|U_1 - \Lambda^0\|_2,$$

where

$$\delta = \|I - \theta(S_h^{(1)})^{-1}S_h\|_2.$$

Following Theorem 3, there exists a constant $\delta(0 < \delta < 1)$, which is independent of h. For $0 < \theta < \delta$, spectral radius of $I - \theta(S_h^{(1)})^{-1}S_h$ is less than 1, and spectral norm $\delta < 1$; therefore,

$$\lim_{k\to+\infty} \|U_1 - \Lambda^{k+1}\|_2 = 0.$$

It follows that the preconditioned Richardson iteration converges; then, the Dirichlet-Neumann alternating method converges and the convergence rate is independent of mesh parameter h of subdomain Ω_1.

V. Numerical examples

In this section, we give two numerical examples to show the effectiveness of the Dirichlet-Neumann alternating method. In these examples, the exact solutions are known. The purpose of showing these examples is to check the convergence in terms of iteration k and mesh size h. The finite element method with liner elements is used in the computation. u_{1h} is the finite element solution in $\overline{\Omega}_1$, e and e_h denote the maximal error of all node functions in $\overline{\Omega}_1$, respectively, i.e.,

$$e(k) = \sup_{P_i \in \overline{\Omega}_1} |u(P_i) - u_{1h}^k(P_i)|,$$

$$e_h(k) = \sup_{P_i \in \overline{\Omega}_1} |u_{1h}^{k+1}(P_i) - u_{1h}^k(P_i)|.$$

$q_h(k)$ is the approximation of the convergence rate, i.e.,

$$q_h(k) = \frac{e_h(k-1)}{e_h(k)}.$$

Example 1. We consider problem (1), where $\Omega = \{(\mu,\varphi)|\mu > 1, 0 < \varphi < 2\pi\}$, $\Gamma = \{(1,\varphi)|0 < \varphi < 2\pi\}$, $\Gamma_0 = \{(\mu,0)|\mu > 1\}$, $\Gamma_\alpha = \{(\mu,2\pi)|\mu > 1\}$ and $f_0 = 2$. Let $u(\mu,\varphi) = \frac{\sin\frac{\varphi}{2}}{\cosh\frac{\mu}{2}+\sinh\frac{\mu}{2}}$ be the exact solution of original problem and $g = \frac{\partial u}{\partial n}|_\Gamma$. Let $\Gamma_{\mu_1} = \{(3,\varphi)|0 < \varphi < 2\pi\}$ be the artificial boundary. Fig. 3 shows the mesh h of subdomain Ω_1, Table 1 shows the relation between convergence rate and mesh ($\theta = 0.5$), Table 2 shows the relation between convergence rate and relaxation factor (mesh $h/4$), Fig. 4 shows $L^\infty(\Omega_1)$ errors with iteration k.

Example 2. We consider problem (1), where $\Omega = \{(\mu,\varphi)|\mu > 1, 0 < \varphi < \frac{3\pi}{2}\}$, $\Gamma = \{(1,\varphi)|0 < \varphi < \frac{3\pi}{2}\}$, $\Gamma_0 = \{(\mu,0)|\mu > 1\}$, $\Gamma_\alpha = \{(\mu,\frac{3\pi}{2})|\mu > 1\}$ and $f_0 = 2$. Let $u(\mu,\varphi) = \frac{\sin\frac{2\varphi}{3}}{\cosh\frac{2\mu}{3}+\sinh\frac{2\mu}{3}}$ be the exact solution of original problem and $g = \frac{\partial u}{\partial n}|_\Gamma$. Let $\Gamma_{\mu_1} = \{(3,\varphi)|0 < \varphi < 2\pi\}$ be the artificial boundaries. Fig. 5 shows the mesh h of subdomain Ω_1, Table 3 shows the relation between convergence rate and mesh ($\theta = 0.6$), Table 4 shows the relation between convergence rate and relaxation factor (mesh $h/4$), Fig. 6 shows $L^\infty(\Omega_1)$ errors with iteration k.

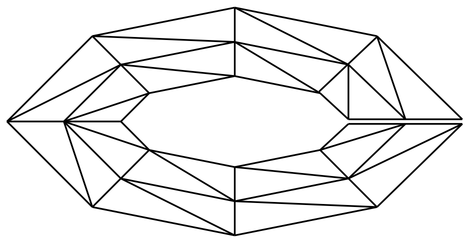

Fig. 3. Mesh h of Subdomain Ω_1 for Example 1

TABLE I
THE RELATION BETWEEN CONVERGENCE RATE AND MESH FOR EXAMPLE 1 ($\theta = 0.5$)

Mesh	k	1	3	5	7	9
$h/2$	$e(k)$	0.0805	0.0582	0.0577	0.0577	0.0577
	$e_h(k)$	0.3054	0.0057	0.0001	0.0000	0.0000
	$q_h(k)$		7.3175	7.3180	7.3181	7.3182
$h/4$	$e(k)$	0.0465	0.0163	0.0157	0.0156	0.0156
	$e_h(k)$	0.3299	0.0076	0.0002	0.0000	0.0000
	$q_h(k)$		6.6071	6.6072	6.6072	6.6071
$h/8$	$e(k)$	0.0460	0.0047	0.0040	0.0040	0.0040
	$e_h(k)$	0.3366	0.0081	0.0002	0.0000	0.0000
	$q_h(k)$		6.4428	6.4428	6.4428	6.4428

TABLE II
THE RELATION BETWEEN CONVERGENCE RATE AND RELAXATION FACTOR FOR EXAMPLE 1 (MESH $h/4$)

θ	k	1	3	5	7	9
0.1	$e(k)$	0.2174	0.1275	0.0743	0.0427	0.0240
	$e_h(k)$	0.0660	0.0391	0.0232	0.0137	0.0081
	$q_h(k)$		1.2992	1.2992	1.2992	1.2992
0.3	$e(k)$	0.0854	0.0102	0.0151	0.0156	0.0156
	$e_h(k)$	0.1980	0.0189	0.0018	0.0002	0.0000
	$q_h(k)$		3.2343	3.2344	3.2345	3.2346
0.5	$e(k)$	0.0465	0.0163	0.0157	0.0156	0.0156
	$e_h(k)$	0.3299	0.0076	0.0002	0.0000	0.0000
	$q_h(k)$		6.6071	6.6072	6.6072	6.6071
0.7	$e(k)$	0.1785	0.0688	0.0314	0.0215	0.0178
	$e_h(k)$	0.4619	0.1729	0.0648	0.0242	0.0091
	$q_h(k)$		1.6343	1.6225	1.6343	1.6343

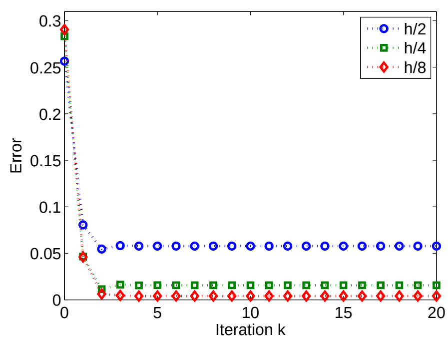

Fig. 4. $L^\infty(\Omega_1)$ Errors with Iteration k for Example 1

The numerical results show that this method is feasible and convergent quickly. Its convergence rate is independent of finite element mesh parameter h. The convergence of the method is the best when the relaxation factor θ_k approaches to 0.5.

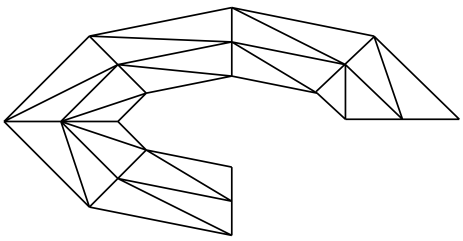

Fig. 5. Mesh h of Subdomain Ω_1 for Example 2

TABLE III
THE RELATION BETWEEN CONVERGENCE RATE AND MESH FOR EXAMPLE 2 ($\theta = 0.6$)

Mesh	k	1	3	5	7	9
$h/2$	$e(k)$	0.0775	0.0611	0.0598	0.0597	0.0597
	$e_h(k)$	0.1826	0.0141	0.0011	0.0001	0.0000
	$q_h(k)$		3.5962	3.5962	3.5962	3.5962
$h/4$	$e(k)$	0.0460	0.0181	0.0166	0.0164	0.0164
	$e_h(k)$	0.1957	0.0161	0.0013	0.0001	0.0000
	$q_h(k)$		3.4897	3.4897	3.4897	3.4897
$h/8$	$e(k)$	0.0453	0.0060	0.0044	0.0042	0.0042
	$e_h(k)$	0.1993	0.0166	0.0014	0.0001	0.0000
	$q_h(k)$		3.4622	3.4622	3.4622	3.4622

TABLE IV
THE RELATION BETWEEN CONVERGENCE RATE AND RELAXATION FACTOR FOR EXAMPLE 2 (MESH $h/4$)

θ	k	1	3	5	7	9
0.2	$e(k)$	0.0845	0.0260	0.0121	0.0150	0.0160
	$e_h(k)$	0.0652	0.0213	0.0069	0.0023	0.0007
	$q_h(k)$		1.7509	1.7509	1.7509	1.7509
0.4	$e(k)$	0.0193	0.0162	0.0164	0.0164	0.0164
	$e_h(k)$	0.1305	0.0026	0.0001	0.0000	0.0000
	$q_h(k)$		7.0276	7.0276	7.0276	7.0277
0.6	$e(k)$	0.0460	0.0181	0.0166	0.0164	0.0164
	$e_h(k)$	0.1957	0.0161	0.0013	0.0001	0.0000
	$q_h(k)$		3.4897	3.4897	3.4897	3.4897
0.8	$e(k)$	0.1112	0.0581	0.0309	0.0232	0.0199
	$e_h(k)$	0.2610	0.1336	0.0684	0.0350	0.0179
	$q_h(k)$		1.3978	1.3978	1.3978	1.3978

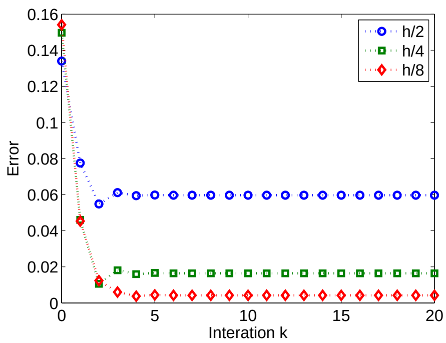

Fig. 6. $L^\infty(\Omega_1)$ Errors with Different Iteration k for Example 2

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