Abstract—In this paper, we study the properties of tensor operators on non-commutative residuated lattices. We give some equivalent conditions of (strict) strong tensor non-commutative lattices and investigate the relation between state operators on $\mathcal{L}$ and state operators on $\mathcal{L}^T$. Moreover, we give the representation theory for (strict) strong tensor non-commutative residuated lattices and obtain the one to one correspondence between tense filters in $\mathcal{L}$ and tense congruences on $\mathcal{L}$.

Index Terms—tensor operator, non-commutative residuated lattice, frame, filter, congruence.

I. INTRODUCTION

Residuated lattices are an important algebraic structure in mathematics. The works on residuated lattices were initiated by Krull, Dilworth and Ward etc. ([8], [13], [16], [17]). Also, these structures are closely related to logics. BL-algebra are algebras of basic fuzzy logics. MV-algebras are algebras of Łukasiewicz infinite valued logics and Heyting algebras are algebras of intuitionistic logics. Residuated lattices are a common generalization of these algebras.

Classical tense logic is the propositional logic with two tense operators $G$ which reveals the future and $H$ which expresses the past. Burges [2] studied tensor operators on Boolean algebra. Later, many authors have investigated tensor operators on other algebras. Diaconescu and Georgescu [7] studied the tensor operators for MV-algebra and Łukasiewicz-Moisil algebras. Chajda, Kolářik and Paseka ([5], [6]) studied tense operators for effect algebras for investigating quantum structures dynamically. Recently, Bakhshi [1] studied the algebraic properties of tense operators for non-commutative residuated lattices. The Dedekind-MacNeill completion of non-commutative lattices with involutive is investigated in [1].

In this paper, we will further study the tensor operators on non-commutative residuated lattices. We give some characteristics of tensor non-commutative residuated lattices which extend some results on effect algebras in [6]. The condition $\neg \neg x = x$ is important in studying tense operators for effect algebras. However, this condition is not valid in non-commutative residuated lattices. We have to overcome this difficulty for studying tense operators on non-commutative residuated lattices. In this paper, we get the one to one correspondence between tense filters (not normal tense filters) of $\mathcal{L}$ and tense congruences on $\mathcal{L}$. The paper is constructed as follows: In Section 2, we give some basic properties on tensor non-commutative residuated lattices. In Section 3, we give some equivalent conditions for strong tensor non-commutative residuated lattices and investigate the relation between state operators on $\mathcal{L}$ and state operators on $\mathcal{L}^T$. Finally, we give the representation theory of (strict) strong tensor non-commutative residuated lattices in Section 4. In Section 5, we study the relation between tense filters in $\mathcal{L}$ and tense congruences on $\mathcal{L}$.

II. PRELIMINARIES

In this section, we give some basic notions and properties on non-commutative lattices which is useful in the paper.

A structure $(\mathcal{L}, \wedge, \vee, \rightarrow, \neg, 0, 1)$ is called a non-commutative residuated lattice, if the following conditions are satisfied:

L1) $(\mathcal{L}, \wedge, \rightarrow, \neg, 0, 1)$ is a bounded lattice;

L2) $(\mathcal{L}, \rightarrow, \neg)$ is an involutive residuated lattice;

L3) $x \vee x = 0$.

For $x \in \mathcal{L}$, we denote $x \rightarrow 0$ by $\neg x$ and denote $x \rightarrow 0$ by $x$.

A non-commutative residuated lattice $\mathcal{L}$ is called to be involutive, if $\neg x = \neg x = x$, for all $x \in \mathcal{L}$.

II.1 Proposition. ([11]) Let $\mathcal{L}$ be a non-commutative residuated lattice. For all $x, y, z \in \mathcal{L}$, then

1) $x \leq y$ iff $x \rightarrow y = 1$ iff $x \rightarrow y = 1$.

2) $x \rightarrow y \leq (y \rightarrow z) \sim (x \rightarrow z); x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$.

3) $x \leq y$ implies $y \rightarrow z \leq x \rightarrow z$ and $y \rightarrow z \leq x \rightarrow z$.

Particularly, $x \leq y$ implies $\neg y \leq \neg x$ and $\neg y \leq \sim x$.

II.2 Definition. ([11]) Let $(\mathcal{L}, \wedge, \vee, \rightarrow, \sim, 0, 1)$ be a non-commutative residuated lattice and $G, H$ be maps of $\mathcal{L}$ into itself. We call $(\mathcal{L}, G, H)$ a tensor non-commutative residuated lattice, if the following conditions are satisfied:

TRL1) $G(1) = 1, H(1) = 1$.

TRL2) $G(x \rightarrow y) \leq G(x) \rightarrow G(y), G(x \sim y) \leq G(x) \rightarrow G(y), H(x \rightarrow y) \leq H(x) \rightarrow H(y), H(x \sim y) \leq H(x) \rightarrow H(y)$.

TRL3) $x \leq GP_\sim(x) \wedge GP_\sim(x), x \leq HF_\sim(x) \wedge HF_\sim(x)$, where

$P_\sim(x) = \neg H(\neg x), P_\sim(x) = H(\neg x),$ $F_\sim(x) = G(\sim x), F_\sim(x) = G(\sim x)$.

II.3 Definition. Let $(\mathcal{L}, G, H)$ be a tensor non-commutative residuated lattice. We call $(\mathcal{L}, G, H)$ to be a strong tensor non-commutative residuated lattice, if $G(0) = H(0) = 0$.

In fact, $G$ and $H$ are strong tense operators on $\mathcal{L}$ in Example 1 in [1] (or see [15]).
II.4 Proposition. ([1]) Let \((\mathcal{L}, \cup, \cap, \to, \neg, 0, 1)\) be a non-commutative residuated lattice. For all \(x, y \in \mathcal{L}\), the following conditions are satisfied.

1) \(x \leq y\) implies \(G(x) \leq G(y)\), \(H(x) \leq H(y)\), \(F_{-}(x) \leq F_{-}(y)\), \(F_{-}(x) \leq F_{-}(y)\), \(G_{-}(x) \leq G_{-}(y)\), \(G_{-}(x) \leq G_{-}(y)\).

2) \(G(x) \ast G(y) \leq G(x \ast y)\), \(H(x) \ast H(y) \leq H(x \ast y)\).

II.5 Proposition. Let \((\mathcal{L}, \cup, \cap, \to, \neg, 0, 1)\) be a non-commutative residuated lattice and \(G, H\) be maps of \(\mathcal{L}\) into itself. Then \((\mathcal{L}, G, H)\) is a strong tensor non-commutative residuated lattice if and only if

\(\text{STRL1)}\) \(G(0) = 0\), \(H(0) = 0\), \(G(1) = 1\), \(H(1) = 1\).

\(\text{STRL2)}\) \(x \leq y\) implies \(G(x) \leq G(y)\) and \(H(x) \leq H(y)\);

\(\text{STRL3)}\) \(x \leq G_{P}(x) \land G_{P}(x), x \leq G_{F}(x) \land G_{F}(x)\), where

\(P_{-}(x) = \neg H(\neg x)\), \(P_{+}(x) = \neg H(\neg x)\),

\(F_{-}(x) = \neg G(\neg x)\), \(F_{+}(x) = \neg G(\neg x)\).

Proof: \(\iff\) By Definition II.3 and Proposition II.4, we get the desired result.

\(-\iff\): We only need to prove TRL2). For all \(x, y \in \mathcal{L}\), we have \((x \ast y) \leq x \iff y \leq x\) by Proposition 2.1 4). Hence,

\[G(x) \ast G(y) \leq G((x \ast y) \ast y) \leq G(y)\]

This implies \((x \ast y) \leq G(x) \to G(y)\).

Similarly, we can get \(G(x) \ast H(y) \leq G(x \ast H(y) \leq H(x) \leq H(y)\; H(x) \ast H(y) \leq H(x).

\[\text{II.6 Lemma. Let } (\mathcal{L}, \cup, \cap, \to, \neg, 0, 1) \text{ be a strong tensor non-commutative residuated lattice. For all } x, y \in \mathcal{L}, \text{ the following conditions are satisfied.}

1) \(G(\neg x) \leq \neg G(x)\), \(G(\neg x) \leq \neg G(x)\), \(H(\neg x) \leq\)

\(H(x), \; H(\neg x) \leq H(x)\).

2) \(P_{+}(\neg x) \geq P_{-}(x), \; P_{+}(\neg x) \geq P_{-}(x), \; F_{-}(\neg x) \geq F_{-}(\neg x) \geq F_{+}(\neg x)\).

Proof: 1) By \((x \to 0) \ast x \leq 0\), we have

\[G(x) \ast G(0) \leq G((x \to 0) \ast x) \leq G(0)\].

Hence, \(G(\neg x) \leq \neg G(\neg x)\).

Similarly, \(G(\neg x) \leq \neg G(x)\), \(H(\neg x) \leq \neg H(x), \; H(\neg x) \leq H(x)\).

3) Using Proposition II.3, we get \(P_{-}(\neg x) = \neg H(\neg x) \geq \neg H(x) \leq \neg P_{+}(x)\).

Similarly, we have \(P_{+}(\neg x) \geq P_{-}(x), \; F_{-}(\neg x) \geq F_{-}(\neg x) \geq F_{+}(\neg x)\)

\[\text{II.7 Proposition. Let } (\mathcal{L}, \cup, \cap, \to, \neg, 0, 1) \text{ be a non-commutative residuated lattice with condition (C). For all }

x, y \in \mathcal{L}, \text{ we have}

\[P_{-}(G(x) \leq x, \; P_{+}(G(x) \leq x, \; F_{-}(H(x) \leq x, \; F_{+}(H(x) \leq x)\].

Proof: By D4) of Definition II.2, we have

\(\neg x \leq H \land F_{-}(\neg x) \leq H(\neg (G(\neg x) \leq H(\neg G(x) \leq \neg P_{+}(G(x)\].

This proves \(P_{-}(G(x) \leq x\). Similarly, we have \(P_{-}(G(x) \leq x, \; F_{+}(H(x) \leq x, \; F_{+}(H(x) \leq x)\).
Let 0 and 1 be the elements in $\mathcal{L}^T$ such that $0(x) = 0$ and $1(x) = 1$, for all $x \in T$.

Similarly to Theorem 3 in [1], we also have the following theorem.

**III.3 Theorem.** Let $(\mathcal{L}, G, H)$ be a strong tensor non-commutative residuated lattice and $(T, R)$ be a frame. For all $p \in \mathcal{L}^T$, we can define $\widehat{G}, \widehat{H} : \mathcal{L}^T \rightarrow \mathcal{L}^T$ as
\[
\widehat{G}(p)(u) = \bigvee \{p(v) \cup uRv\},
\]
\[
\widehat{H}(p)(u) = \bigvee \{p(v) \cup vRu\}.
\]
Then $(\mathcal{L}^T, \widehat{G}, \widehat{H})$ is a strong tensor non-commutative residuated lattice.

**Proof:** By Theorem 3 in [1], we only need to check that
\[
\widehat{G}(0) = 0, \quad H(0) = 0, \quad G(1) = 1 \quad \text{and} \quad H(1) = 1.
\]

**III.4 Definition.** $(\mathcal{L}, G, H)$ is called a strong tensor non-commutative residuated lattice, if for all $x, y \in \mathcal{L}$,
\[
G(x \rightarrow y) = G(x) \rightarrow G(y),
\]
\[
G(x \rightsquigarrow y) = G(x) \rightarrow G(y),
\]
\[
H(x \rightarrow y) = H(x) \rightarrow H(y),
\]
\[
H(x \rightsquigarrow y) = H(x) \rightarrow H(y).
\]

**III.5 Lemma.** Let $(\mathcal{L}, G, H)$ be a strong tensor non-commutative residuated lattice. For all $x \in \mathcal{L}$, we have $G(\sim x) = -G(x)$, $G(\sim x) = \sim G(x)$, $H(\sim x) = -H(x)$, $H(\sim x) = \sim H(x)$.

**Proof:** By Definition III.4, we have $G(x \rightarrow 0) = \mathcal{G}(0) = 0$ and $G(0) = 0$. $G(x \rightarrow 0) = G(x) \rightarrow G(0) = G(x) \rightarrow 0$. So $G(\sim x) = -G(x)$. Similarly, $G(\sim x) = -G(x)$, $H(\sim x) = -H(x)$, $H(\sim x) = \sim H(x)$.

**III.6 Theorem.** If $\mathcal{L}$ is a non-commutative residuated lattice with condition (C), $G, H : \mathcal{L} \rightarrow \mathcal{L}$ are mappings. Then the following conditions are equivalent.

1) $(\mathcal{L}, G, H)$ is a strong tensor non-commutative residuated lattice.

2) $G$ and $H$ satisfy the following properties:
   i) $G(0) = H(0) = 0$.
   ii) $G$ is both a left adjoint and a right adjoint to $H$.
   iii) $G(x \rightarrow y) = G(x) \rightarrow G(y)$, $G(x \rightsquigarrow y) = G(x) \rightsquigarrow G(y)$.
   iv) $H(x \rightarrow y) = H(x) \rightarrow H(y)$, $H(x \rightsquigarrow y) = H(x) \rightsquigarrow H(y)$.

**Proof:** 1) $\implies$ 2): i) By Definition III.3, we obviously have $G(0) = H(0) = 0$.

ii) For all $x, y \in \mathcal{L}$, if $x \leq G(y)$, By Proposition II.7 we get $P_+(x) \leq y$, i.e. $\sim H(\sim x) \leq y \leq \sim \sim y$. This implies $\sim y \leq H(\sim x) \leq \sim H(x)$. Hence $H(x) \leq y$. Conversely, if $H(x) \leq y$, we have $\sim y \leq H(\sim x) \leq \sim H(x) \leq \sim H(x) \leq \sim H(x) = \sim P_+(x)$.

Then $P_+(x) \leq y$. By Theorem III.1 again, $x \leq G(y)$ holds. Hence, $G$ is a right adjoint to $H$. Similarly, we can prove that $H$ is also a right adjoint to $G$.

iii and iv) are obvious.

2) $\implies$ 1): Since $G$ is a right adjoint, $G$ preserves infima and $G(1) = 1$. For all $x, y \in \mathcal{L}$, we have $G(x) \leq G(y) \rightarrow x \leq y \rightarrow x \leq y$. This implies $G(x) \leq G(x) \rightarrow \leq G(y) \leq G(y)$.

Similarly, $H$ preserves order and $H(x) \leq H(x) \rightarrow \leq H(y) \leq H(y)$. Then STRL2 holds.

Now, we prove STRL3. For all $x \in \mathcal{L}$, we have
\[
H(x) \leq H(\sim x) = H(\sim x) \leq H(\sim x) = P_+(x).
\]
By $H(x) \leq H(x)$, then $x \leq H(\sim x) \leq GP_-(x)$ and $x \leq H(\sim x) \leq HF_-(x)$. Similarly, $x \leq GP_-(x)$, $x \leq H(\sim x) \leq H(\sim x)$.

In the following, we discuss the relation between state operators on commutative residuated lattice $\mathcal{L}$ and state operators on $\mathcal{L}^T$.

**III.7 Definition.** $(\mathcal{L}, \bigcap, \bigcup, \ast, \rightarrow, 0, 1)$ be a residuated lattice and $\tau : \mathcal{L} \rightarrow \mathcal{L}$ a map. If the following conditions are satisfied
1) $\tau(0) = 0$, $\tau(1) = 1$.
2) $x \leq y$ implies $\tau(x) \leq \tau(y)$.

For $x, y \in \mathcal{L}$, we have $x \leq y$ $\iff$ $x \rightarrow y = 1$.

**III.8 Proposition.** $(\mathcal{L}, \bigcap, \bigcup, \ast, \rightarrow, 0, 1)$ be a residuated lattice and $\tau : \mathcal{L} \rightarrow \mathcal{L}$ a state operator on $\mathcal{L}$, the following propositions are satisfied.

1) $\tau(1) = 1$.
2) $x \leq y$ implies $\tau(x) \leq \tau(y)$.

For $x, y \in \mathcal{L}$, we have $x \leq y$ $\iff$ $x \rightarrow y = 1$.

**III.9 Theorem.** If $\tau : \mathcal{L} \rightarrow \mathcal{L}$ is a state operator on $\mathcal{L}$, the mapping $\check{\tau} : \mathcal{L}^T \rightarrow \mathcal{L}^T$ defined by $\check{\tau}(f) = \tau f$ is also a state operator on $\mathcal{L}^T$.

**Proof:** 1) Obviously, $\check{\tau}(0) = 0$.

2) For $f, g \in \mathcal{L}^T$, if $f \rightarrow g = 1$, we have $f \leq g$. For every $x \in \mathcal{L}$, for $\tau f(\ast) = f(x) \leq \tau(\ast)(x)$ by Proposition III.8. It concludes that $\check{\tau}(f) \leq \check{\tau}(g)$. That is, $\check{\tau}(f) \leq \check{\tau}(g) = 1$.

3) $c \check{\tau}(f \rightarrow g)(x) =$ $\tau(f \rightarrow g)(x)$
\[
\tau(f \rightarrow g)(x) = \tau(f \rightarrow g)(x) \wedge g(x)
\]$
\tau(f \rightarrow g)(x) = \tau(f \rightarrow g)(x) \wedge g(x)$
\[
\tau(f \rightarrow g)(x) = (\check{\tau} \rightarrow(\check{\tau} \vee g))(x).
\]
Therefore, $\check{\tau}(f \rightarrow g) = \tau(f) \rightarrow \tau(g)$. Similarly, we have
\[
\tau(f) \ast g = \tau(f) \ast g,
\]
\[
\tau(\tau(f) \ast \tau(g)) = \tau(f) \ast \tau(g),
\]
\[
\tau(\tau(f) \rightarrow \tau(g)) = \tau(f) \rightarrow \tau(g),
\]
\[
\tau(\tau(f) \vee \tau(g)) = \tau(f) \vee \tau(g),
\]
\[
\tau(\tau(f) \vee \tau(g)) = \tau(f) \vee \tau(g).
\]

Hence, $\check{\tau}$ is a state operator on $\mathcal{L}^T$.
IV. REPRESENTATIONS OF STRONG TENSOR NON-COMMUTATIVE RESIDUATED LATTICES

In this section, we shall give representation theorems for strong tensor non-commutative residuated lattices and strict tensor non-commutative residuated lattices. Some proofs are similar to those in [6].

Let $P$ and $P'$ be two bounded posets. A map $f : P \rightarrow P'$ is called to be morphism, if $f$ preserves order, top element and bottom element. A map $f : P \rightarrow P'$ is called to be order reflecting, if $f$ is a morphism and $f(x) \leq f(y) \iff x \leq y, \forall x, y \in P$.

IV.1 Definition. Let $\mathcal{L}$ and $\mathcal{L}'$ be two non-commutative residuated lattices. A map $f : \mathcal{L} \rightarrow \mathcal{L}'$ is called a semi-morphism from $\mathcal{L}$ into $\mathcal{L}'$, if $f$ satisfies the following:

1) $f$ preserves order.
2) $f(x) \ast f(y) \leq f(x \ast y), \forall x, y \in \mathcal{L}$.
3) $f(0) = 0, f(1) = 1$.

A semi-morphism $f : \mathcal{L} \rightarrow \mathcal{L}'$ is called to be strict, if for all $x, y \in \mathcal{L}$,

$$f(x \rightarrow y) = f(x) \rightarrow f(y), f(x \leftarrow y) = f(x) \leftarrow f(y).$$

If $f$ is a strict semi-morphism, for all $x \in \mathcal{L}$, we have

$$f(\neg x) = \neg f(x), f(x) \iff f(x).$$

Let $S$ be a set of semi-morphisms from $\mathcal{L}$ into $\mathcal{L}'$. A subset $T \subseteq S$ is full, if for $x, y \in \mathcal{L}$,

$$x \leq y \iff f(x) \leq f(y), \forall x, y \in T.$$

IV.2 Theorem. Let $(\mathcal{L}, G, H)$ be a dynamic non-commutative residuated lattice with a full set $S$ of semi-morphisms into a non-commutative residuated lattice $\mathcal{C}$. Then

1) There exists a semi-morphisms set $T$ satisfies the following conditions:
   i) $S \subseteq T$;
   ii) the map $i^*_T : (\mathcal{L}, C, H) \rightarrow (\mathcal{L}^T, \widehat{G}, \widehat{H})$ which sends $x$ to $i^*_T(x)$ is order reflecting, where $i^*_T(x)(t) = t(x)$, for all $x \in \mathcal{T}, t \in T$.

2) There exists a frame $(T, R)$ satisfies:
   for all $s, t \in T$, $(s, t) \in R$ iff $\forall x \in \mathcal{L}, s(G(x)) \leq t(x)$.
   Moreover,
   $$s(G(x)) = \bigwedge \{t(x) \mid sRt\}.$$
V. Congruences on Commutative Residuated Lattices

There is a bijection correspondence between normal filters of \( \mathcal{L} \) and congruences on \( \mathcal{L} \). In [1], the author proves that there is a bijection correspondence between tense normal filters of \( \mathcal{L} \) and tense congruences on \( \mathcal{L} \). In this section, we will prove that there is a bijection correspondence between filters of \( \mathcal{L} \) and congruences on \( \mathcal{L} \) when \( G(x * y) = G(x) * G(y), H(x * y) = H(x) * H(y) \), for \( x, y \in \mathcal{L} \).

Recall that a filter \( F \) of \( (\mathcal{L}, G, H) \) is called to be a tense filter, if \( G(x), H(x) \in F \), for all \( x \in F \).

A congruence \( \theta \) on \( (\mathcal{L}, G, H) \) is called to be a tense congruence, if \( x \theta y \), then \( G(x) \theta G(y) \) and \( H(x) \theta H(y) \), for \( x, y \in \mathcal{L} \).

In paper [14], the author gives the one to one correspondence between the ideals in quasi-Mv algebras and ideal congruences on quasi-Mv algebras. Inspired this fact, we will define a relation on \( \mathcal{L} \), which can be used to construct congruences on \( \mathcal{L} \). Further, we can give the one to one correspondence between tense filters in \( \mathcal{L} \) and tense congruences on \( \mathcal{L} \) under certain conditions.

Let \( F \) be a subset of \( \mathcal{L} \). The relation \( \mathcal{C}(F) \) on \( \mathcal{L} \) is defined as following: for \( x, y \in \mathcal{L} \),

\[
x \mathcal{C}(F) y \iff G(x) \wedge G(y) \wedge H(x) \wedge H(y) \in F \cdot (R)
\]

V.1 Proposition. Let \( (\mathcal{L}, G, H) \) be a tense commutative residuated lattice such that \( G(x * y) = G(x) * G(y), H(x * y) = H(x) * H(y) \) and \( F \) be a tense filter of \( \mathcal{L} \). The relation \( \mathcal{C}(F) \) is a tense congruence on \( \mathcal{L} \).

Proof: 1) Since \( F \) is a filter, we have \( 1 \in F \). For all \( x \in \mathcal{L} \),

\[
G(x) \rightarrow G(x) = 1 \in F
\]

holds. This concludes that \( \mathcal{C}(F) \) is reflexive.

2) The symmetry is obvious.

3) Suppose \( x \mathcal{C}(F) y \) and \( y \mathcal{C}(F) z \). We have

\[
G(x \rightarrow y), G(y \rightarrow x), G(y \rightarrow z), G(z \rightarrow y) \in F.
\]

Hence,

\[
cG(x \rightarrow z) \leq G((x \rightarrow y) \rightarrow (z \rightarrow y)) \\
\leq G(x \rightarrow y) \rightarrow G(z \rightarrow y) \in F.
\]

Similarly, \( G(z \rightarrow x) \in F \). So \( \mathcal{C}(F) \) is transitive.

4) Suppose \( x \mathcal{C}(F) y \) and \( a \mathcal{C}(F) b \). We have

\[
x \ast (x \rightarrow y) \leq y, \quad a \ast (a \rightarrow b) \leq b.
\]

Hence,

\[
x \ast a \ast (x \rightarrow y) \ast (a \rightarrow b) \leq y \ast b.
\]

It concludes that

\[
(x \rightarrow y) \ast (a \rightarrow b) \leq x \ast a \rightarrow y \ast b.
\]

Then

\[
G(x \rightarrow y) \ast G(a \rightarrow b) = G((x \rightarrow y) \ast (a \rightarrow b)) \leq G((x \ast a) \rightarrow (y \ast b)).
\]

By \( x \rightarrow y \leq (x \rightarrow z) \rightarrow (y \rightarrow z) \), we get

\[
G(x \rightarrow y) \leq G(x \rightarrow z) \rightarrow (y \rightarrow z).
\]

Then \( (x \rightarrow z) \mathcal{C}(F) (y \rightarrow z) \).

5) Suppose \( x \mathcal{C}(F) y \). Then \( G(x \rightarrow y), G(y \rightarrow x) \in F \).

By \( G(x \rightarrow y) \leq G(x) \rightarrow G(y) \), we have \( G(G(x \rightarrow y)) \leq G(G(x) \rightarrow G(y)) \). Since \( F \) is a tensor filter, we concludes \( G(G(x \rightarrow y)) \in F \) and so \( G(G(x) \rightarrow G(y)) \in F \). Similarly, \( G(G(y) \rightarrow G(x)) \in F \). This proves that \( G(x) \mathcal{C}(F) G(y) \).

By above, we get that \( \mathcal{C}(F) \) is a tense congruence on \( \mathcal{L} \).


V.2 Proposition. Let \( (\mathcal{L}, G, H) \) be a tense commutative residuated lattice and \( F \) be a subset of \( \mathcal{L} \). If \( \mathcal{C}(F) \) be a tensor congruence on \( \mathcal{L} \), then \( \mathcal{C}(F)(1) \) is a filter of \( \mathcal{L} \).

Proof: For \( x, y \in \mathcal{C}(F)(1) \), we have

\[
G(1) = G(x \rightarrow 1), G(x) = G(1 \rightarrow x) \in F.
\]

Hence,

\[
G((x \rightarrow y) \rightarrow 1) = G((x \rightarrow y) \rightarrow 1) = G(1) \in F.
\]

This implies that \( x \ast y \in \mathcal{C}(F)(1) \).

If \( x \in \mathcal{C}(F)(1) \) and \( x \leq y \), we get \( (1 \rightarrow x) \leq (1 \rightarrow y) \) and so \( G(1 \rightarrow x) \leq G(1 \rightarrow y) \in F \). Since \( G(x \rightarrow 1) = G(y \rightarrow 1) = 1 \in F \), we get \( y \in \mathcal{C}(F)(1) \).


V.3 Proposition. Let \( \theta \) be a tense congruence on \( (\mathcal{L}, G, H) \), then \( \theta(1) \) is a tensor filter.

Proof: Let \( x, y \in \theta(1) \). We have \( G(x), G(y) \in \theta(1) \).

By \( x \theta y, y \theta 1 \), we concludes that \( x \ast y \theta 1 \), i.e. \( x \ast y \in \theta(1) \).

If \( x \leq y \) and \( x \theta 1 \), then \( x \rightarrow y \theta 1 \rightarrow y \), i.e. \( 1 \theta y \). Hence, \( \theta(1) \) is a tensor filter.


V.4 Theorem. Let \( (\mathcal{L}, G, H) \) be a tense commutative residuated lattice. There is a bijection between the tense filters of \( \mathcal{L} \) and tense congruences on \( \mathcal{L} \).

Let \( A \) be a subset of \( \mathcal{L} \). Denote by \( \text{Fil}(A) \) the filter generated by \( A \). Ciung [3] proved that

\[
\text{Fil}(A) = \{ x \in \mathcal{L} \mid x \geq a_1 \ast \cdots \ast a_n, n \in N, a_1, \ldots, a_n \in A \}.
\]

If \( F \) is a filter of \( \mathcal{L} \) and \( a \in \mathcal{L} \), then

\[
\text{Fil}(F, a) = \{ x \in \mathcal{L} \mid x \geq (f_1 \ast a_n) \ast (f_2 \ast a_n) \ast \cdots \ast (f_m \ast a_n), m \in N, a_1, a_2, \ldots, a_m \in N^+ \}.
\]

Similar to Proposition 5.1 of [12], we have the following proposition.

V.5 Proposition. Let \( \mathcal{L} \) be a tense residuated lattice and \( a \in \mathcal{L} \) such that \( G(a) = H(a) = a \). Then \( \text{Fil}(F, a) \) is a tense filter of \( \mathcal{L} \).

Proof: For \( x \in \text{Fil}(F, a) \), there exist \( y_1, y_2, \ldots, y_t \in F \), \( m_1, m_2, \ldots, m_t \in N^+ \) such that \( x \geq y_1 \ast a^{m_1} \ast y_2 \ast a^{m_2} \ast \cdots \ast y_t \ast a^{m_t} \). Thus

\[
cG(x) \geq G(y_1) \ast a^{m_1} \ast y_2 \ast a^{m_2} \ast \cdots \ast y_t \ast a^{m_t} \ast G(a) \ast G(a) \ast \cdots \ast G(a) \ast G(a) \ast G(a).
\]

This proves that \( G(x) \in \text{Fil}(F, a) \).

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References


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