# Dynamics of Almost Periodic Mutualism Model with Time Delays

Yaqin Li, Lijun Xu, and Tianwei Zhang

*Abstract*—**By using some new analytical techniques, modified inequalities and Mawhin's continuous theorem of coincidence degree theory, some new sufficient conditions for the existence of positive almost periodic solutions to a mutualism model with bounded and unbounded delays are obtained. Further, the boundedness and global asymptotic stability have been studied. To the best of the author's knowledge, so far, the result of this paper is completely new. The work of this paper extends and improves some results in recent years. Finally, some examples are given to illustrate the main results in this paper.**

*Index Terms*—**Almost periodicity; Coincidence degree; Mutualism model; Bounded and unbounded delays.**

## I. Introduction

Let $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}^+$ denote the sets of real numbers, integers and positive integers, respectively. Related to a continuous function $f$, we use the following notations:

$$f^l = \inf_{s\in\mathbb{R}} f(s), \quad f^M = \sup_{s\in\mathbb{R}} f(s),$$

$$|f|_\infty = \sup_{s\in\mathbb{R}} |f(s)|, \quad \bar{f} = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(s)\,\mathrm{d}s.$$

Mutualism, or cooperation, is found in many types of communities. For example, some species of Acacia require the ant Pseudomyrmex in order to survive (see [1]), blue-green algae can grow and reproduce in the absence of zooplankton grazers, but growth and reproduction are enhanced by the presence of the zooplankton (see [2]).

In [3], Gopalsamy had proposed the following model to describe the mutualism mechanism:

$$\begin{cases} \frac{\mathrm{d}N_1(t)}{\mathrm{d}t} = r_1(t)N_1(t)\left[\frac{K_1(t)+\alpha_1(t)N_2(t)}{1+N_2(t)} - N_1(t)\right], \\ \frac{\mathrm{d}N_2(t)}{\mathrm{d}t} = r_2(t)N_2(t)\left[\frac{K_2(t)+\alpha_2(t)N_1(t)}{1+N_1(t)} - N_2(t)\right], \end{cases} \tag{1.1}$$

where $r_i$ denotes the intrinsic growth rate of species $N_i$, $i = 1, 2$. The carrying capacity of species $N_i$ is $K_i$ in the absence of other species, while with the help of the other species, the carrying capacity becomes $[K_i + \alpha_i N_{3-i}]/[1 + N_{3-i}]$, $i = 1, 2$. The above mutualism can be classified as facultative, obligate or a combination of both.

Naturally, more realistic and interesting models of population interactions should take into account both the seasonality of the changing environment and the effects of time delay. In system (1.1), the mutualistic or cooperative effects are not realized instantaneously but take place with time delays. Therefore, by way of Mawhin's continuous theorem of coincidence degree theory, Li [4] investigated the existence of positive periodic solutions for a periodic mutualism model with bounded delays:

$$\begin{cases} \frac{\mathrm{d}N_1(t)}{\mathrm{d}t} = r_1(t)N_1(t)\Big[\frac{K_1(t)+\alpha_1(t)N_2(t-\mu_2(t))}{1+N_2(t-\mu_2(t))} \\ \qquad\qquad -N_1(t-\nu_1)\Big], \\ \frac{\mathrm{d}N_2(t)}{\mathrm{d}t} = r_2(t)N_2(t)\Big[\frac{K_2(t)+\alpha_2(t)N_1(t-\mu_1(t))}{1+N_1(t-\mu_1(t))} \\ \qquad\qquad -N_2(t-\nu_2)\Big], \end{cases} \tag{1.2}$$

where $r_i, K_i, \alpha_i \in C(\mathbb{R}, (0,\infty))$, $\alpha_i > K_i$, $\mu_i, \nu_i \in C(\mathbb{R}, [0,\infty))$, $r_i, K_i, \alpha_i, \mu_i, \nu_i$ are functions of period $\omega > 0$, $i = 1, 2$.

Next, Li and Xu [5] studied the following model of two species mutualism with unbounded delays:

$$\begin{cases} \frac{\mathrm{d}N_1(t)}{\mathrm{d}t} = r_1(t)N_1(t)\Big[\frac{H_1(t)+\beta_1(t)\int_0^\infty J_2(s)N_2(t-s)\,\mathrm{d}s}{1+\int_0^\infty J_2(s)N_2(t-s)\,\mathrm{d}s} \\ \qquad\qquad -N_1(t-\nu_1(t))\Big], \\ \frac{\mathrm{d}N_2(t)}{\mathrm{d}t} = r_2(t)N_2(t)\Big[\frac{H_2(t)+\beta_2(t)\int_0^\infty J_1(s)N_1(t-s)\,\mathrm{d}s}{1+\int_0^\infty J_1(s)N_1(t-s)\,\mathrm{d}s} \\ \qquad\qquad -N_2(t-\nu_2(t))\Big]. \end{cases} \tag{1.3}$$

Under the assumption that $r_i, H_i, \beta_i$ and $\nu_i$ are continuous periodic functions with common period $\omega$, $a_i > K_i$, $J_i \in C([0,\infty), [0,\infty))$ and $\int_0^\infty J_i(s)\,\mathrm{d}s = 1$, $i = 1, 2$, they showed that system (1.3) admits at least one positive $\omega$-periodic solution by way of Mawhin's continuous theorem of coincidence degree theory.

In real world phenomenon, the environment varies due to the factors such as seasonal effects of weather, food supplies, mating habits, harvesting. So it is usual to assume the periodicity of parameters in the systems. However, if the various constituent components of the temporally nonuniform environment is with incommensurable (nonintegral multiples, see Example 1.1) periods, then one has to consider the environment to be almost periodic [6-12] since there is no a priori reason to expect the existence of periodic solutions. Hence, if we consider the effects of the environmental factors, almost periodicity is sometimes more realistic and more general than periodicity.

**Example 1.** Let us consider the following simple population model:

$$\dot{N}(t) = N(t)\left[|\sin(\sqrt{2}t)| - |\sin(\sqrt{3}t)|N(t)\right]. \tag{1.4}$$

In Eq. (1.4), $|\sin(\sqrt{2}t)|$ is $\frac{\sqrt{2}\pi}{2}$-periodic function and $|\sin(\sqrt{3}t)|$ is $\frac{\sqrt{3}\pi}{3}$-periodic function, which imply that Eq. (1.4) is with incommensurable periods. Then there is no

Manuscript received September 19, 2017; revised February 18, 2018.

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**(Advance online publication: 28 May 2018)**

a priori reason to expect the existence of positive periodic solutions of Eq. (1.4). Thus, it is significant to study the existence of positive almost periodic solutions of Eq. (1.4).

So, the aim of this paper is to investigate the following almost periodic mutualism model with bounded and unbounded delays:

$$\begin{cases} \frac{\mathrm{d}N_1(t)}{\mathrm{d}t} = r_1(t)N_1(t)\Bigg[\frac{K_1(t)+\alpha_1(t)N_2(t-\mu_2(t))}{1+N_2(t-\mu_2(t))} \\ \qquad +\frac{H_1(t)+\beta_1(t)\int_0^\infty J_2(s)N_2(t-s)\,\mathrm{d}s}{1+\int_0^\infty J_2(s)N_2(t-s)\,\mathrm{d}s} \\ \qquad -N_1(t-\nu_1(t))\Bigg], \\ \frac{\mathrm{d}N_2(t)}{\mathrm{d}t} = r_2(t)N_2(t)\Bigg[\frac{K_2(t)+\alpha_2(t)N_1(t-\mu_1(t))}{1+N_1(t-\mu_1(t))} \\ \qquad +\frac{H_2(t)+\beta_2(t)\int_0^\infty J_1(s)N_1(t-s)\,\mathrm{d}s}{1+\int_0^\infty J_1(s)N_1(t-s)\,\mathrm{d}s} \\ \qquad -N_2(t-\nu_2(t))\Bigg], \end{cases} \tag{1.5}$$

where $r_i, K_i, H_i, \alpha_i, \beta_i, \mu_i$ and $\nu_i$ are continuous nonnegative almost periodic functions, $J_i \in C([0,\infty),[0,\infty))$ and $\int_0^\infty J_i(s)\,\mathrm{d}s = 1$, $i = 1, 2$.

Obviously, systems (1.1)-(1.3) are special cases of system (1.5).

It is well known that Mawhin's continuation theorem of coincidence degree theory is an important method to investigate the existence of positive periodic solutions of some kinds of non-linear ecosystems (see [13-26]). However, it is difficult to be used to investigate the existence of positive almost periodic solutions of non-linear ecosystems. Therefore, to the best of the author's knowledge, so far, there are scarcely any papers concerning with the existence of positive almost periodic solutions of system (1.5) by using Mawhin's continuation theorem. Motivated by the above reason, our purpose of this paper is to establish some new sufficient conditions on the existence of positive almost periodic solutions for system (1.5) by using Mawhin's continuous theorem.

The paper is organized as follows. In Section 2, we give some basic definitions and necessary lemmas which will be used in later sections. In Section 3, we obtain some new sufficient conditions for the existence of at least one positive almost periodic solution of system (1.5) by way of Mawhin's continuous theorem. Some illustrative examples are given in Section 4.

## II. Preliminaries

**Definition 1.** ([27, 28]) $x \in C(\mathbb{R}, \mathbb{R}^n)$ is called almost periodic, if for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\tau = \tau(\epsilon)$ in this interval such that $\|x(t+\tau) - x(t)\| < \epsilon$, $\forall t \in \mathbb{R}$, where $\|\cdot\|$ is arbitrary norm of $\mathbb{R}^n$. $\tau$ is called to the $\epsilon$-almost period of $x$, $T(x,\epsilon)$ denotes the set of $\epsilon$-almost periods for $x$ and $l(\epsilon)$ is called to the length of the inclusion interval for $T(x,\epsilon)$. The collection of those functions is denoted by $AP(\mathbb{R}, \mathbb{R}^n)$. Let $AP(\mathbb{R}) := AP(\mathbb{R}, \mathbb{R})$.

**Lemma 1.** ([27, 28]) *If $x \in AP(\mathbb{R})$, then $x$ is bounded and uniformly continuous on $\mathbb{R}$.*

**Lemma 2.** ([27, 28]) *If $x \in AP(\mathbb{R})$, then $\int_0^t x(s)\,\mathrm{d}s \in AP(\mathbb{R})$ if and only if $\int_0^t x(s)\,\mathrm{d}s$ is bounded on $\mathbb{R}$.*

Next, we present and prove several useful lemmas which will be used in later section.

**Lemma 3.** ([12]) *Assume that $x \in AP(\mathbb{R}) \cap C^1(\mathbb{R})$ with $\dot{x} \in C(\mathbb{R})$. For arbitrary interval $[a,b]$ with $b - a = \omega > 0$, let $\xi, \eta \in [a,b]$ and*

$$I = \big\{s \in [\xi, b] : \dot{x}(s) \geq 0\big\},$$

*then ones have*

$$x(t) \leq x(\xi) + \int_I \dot{x}(s)\,\mathrm{d}s, \quad \forall t \in [\xi, b].$$

**Lemma 4.** ([12]) *If $x \in AP(\mathbb{R})$, then for arbitrary interval $[a,b]$ with $b-a = \omega > 0$, there exist $\xi \in [a,b]$, $\underline{\xi} \in (-\infty, a]$ and $\bar{\xi} \in [b, +\infty)$ such that*

$$x(\underline{\xi}) = x(\bar{\xi}) \quad \text{and} \quad x(\xi) \leq x(s), \ \forall s \in [\underline{\xi}, \bar{\xi}].$$

**Lemma 5.** ([12]) *If $x \in AP(\mathbb{R})$, then for arbitrary interval $[a,b]$ with $I = b-a = \omega > 0$, there exist $\eta \in [a,b]$, $\underline{\eta} \in (-\infty, a]$ and $\bar{\eta} \in [b, +\infty)$ such that*

$$x(\underline{\eta}) = x(\bar{\eta}) \quad \text{and} \quad x(\eta) \geq x(s), \ \forall s \in [\underline{\eta}, \bar{\eta}].$$

**Lemma 6.** ([12]) *If $x \in AP(\mathbb{R})$, then for $\forall n \in \mathbb{N}^+$, there exists $\alpha_n \in \mathbb{R}$ such that $x(\alpha_n) \in [x^* - \frac{1}{n}, x^*]$, where $x^* = \sup_{s\in\mathbb{R}} x(s)$.*

**Lemma 7.** ([28]) *Assume that $x \in AP(\mathbb{R})$ and $\bar{x} > 0$, then for $\forall t_0 \in \mathbb{R}$, there exists a positive constant $T_0$ independent of $t_0$ such that*

$$\frac{1}{T}\int_{t_0}^{t_0+T} x(s)\,\mathrm{d}s \in \left[\frac{\bar{x}}{2}, \frac{3\bar{x}}{2}\right], \quad \forall T \geq T_0.$$

**Lemma 8.** ([4]) *Let*

$$f(x,y) = \left(a_1 - \frac{a_1 - b_1}{1+e^y} - c_1e^x, a_2 - \frac{a_2 - b_2}{1+e^x} - c_2e^y\right)$$

*and $\Omega = \{(x,y)^T \in \mathbb{R}^2 : |x| + |y| < M\}$, where $M$, $a_i$, $b_i$ and $c_i$ are constants, $a_i \geq b_i$, $i = 1, 2$, and $M > \max\{2|\ln(a_i/c_i)|, 2|\ln(b_i/c_i)|, i = 1,2\}$. Then*

$$\deg\{f(x,y), \Omega, (0,0)^T\} \neq 0.$$

**Lemma 9.** *Let*

$$f(x,y) = \left(a_1 + \frac{b_1 - a_1}{1+e^y} - c_1e^x, a_2 + \frac{b_2 - a_2}{1+e^x} - c_2e^y\right)$$

*and $\Omega = \{(x,y)^T \in \mathbb{R}^2 : |x| + |y| < M\}$, where $M$, $a_i$, $b_i$ and $c_i$ are constants, $b_i \geq a_i$, $i = 1, 2$, and $M > \max\{2|\ln(a_i/c_i)|, 2|\ln(b_i/c_i)|, i = 1,2\}$. Then*

$$\deg\{f(x,y), \Omega, (0,0)^T\} \neq 0.$$

**Proof:** Set

$$\Psi(x,y,\iota) = \left(a_1 + \frac{b_1 - a_1}{1+\iota e^y} - c_1e^x, a_2 + \frac{b_2 - a_2}{1+\iota e^x} - c_2e^y\right),$$

$\iota \in [0,1]$. For $\forall (x,y,\iota) \in \mathbb{R}^2 \times [0,1]$, it is then easy to see that

$$a_1 + \frac{b_1 - a_1}{1+\iota e^y} - c_1e^x < b_1 - c_1e^x < 0, \quad \text{as } x \geq \frac{M}{2},$$

$$a_2 + \frac{b_2 - a_2}{1+\iota e^x} - c_2e^y < b_2 - c_2e^y < 0, \quad \text{as } y \geq \frac{M}{2},$$

$$a_1 + \frac{b_1 - a_1}{1+\iota e^y} - c_1e^x \geq a_1 - c_1e^x > 0, \quad \text{as } x \leq -\frac{M}{2},$$

$$a_2 + \frac{b_2 - a_2}{1 + \iota e^x} - c_2 e^y \geq a_2 - c_2 e^y > 0, \quad \text{as } y \leq -\frac{M}{2}.$$

Hence

$$\Psi(x, y, \iota) \neq 0 \quad \text{for} \quad (x, y, \iota) \in \partial\Omega \times [0, 1].$$

It follows from the property of invariance under a homotopy that

$$\deg\{f(x, y), \Omega, (0, 0)^T\} = \deg\{\Psi(x, y, 0), \Omega, (0, 0)^T\}$$

$$= -1 \neq 0.$$

The proof is complete. ■

## III. Main results

The method to be used in this paper involves the applications of the continuation theorem of coincidence degree. This requires us to introduce a few concepts and results from Gaines and Mawhin [29].

Let $\mathbb{X}$ and $\mathbb{Y}$ be real Banach spaces, $L : \mathrm{Dom}L \subseteq \mathbb{X} \to \mathbb{Y}$ be a linear mapping and $N : \mathbb{X} \to \mathbb{Y}$ be a continuous mapping. The mapping $L$ is called a Fredholm mapping of index zero if $\mathrm{Im}L$ is closed in $\mathbb{Y}$ and $\mathrm{dimKer}L = \mathrm{codimIm}L < +\infty$. If $L$ is a Fredholm mapping of index zero and there exist continuous projectors $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L = \mathrm{Im}(I - Q)$. It follows that $L|_{\mathrm{Dom}L \cap \mathrm{Ker}P} : (I - P)\mathbb{X} \to \mathrm{Im}L$ is invertible and its inverse is denoted by $K_P$. If $\Omega$ is an open bounded subset of $\mathbb{X}$, the mapping $N$ will be called $L$-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to \mathbb{X}$ is compact. Since $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$, there exists an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$.

**Lemma 10.** ([29]) *Let $\Omega \subseteq \mathbb{X}$ be an open bounded set, $L$ be a Fredholm mapping of index zero and $N$ be $L$-compact on $\bar{\Omega}$. If all the following conditions hold:*

(a) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap \mathrm{Dom}L$, $\lambda \in (0, 1)$;
(b) $QNx \neq 0$, $\forall x \in \partial\Omega \cap \mathrm{Ker}L$;
(c) $\deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0$, *where* $J : \mathrm{Im}Q \to \mathrm{Ker}L$ *is an isomorphism.*

*Then $Lx = Nx$ has a solution on $\bar{\Omega} \cap \mathrm{Dom}L$.*

For $f \in AP(\mathbb{R})$, we denote by

$$\Lambda(f) = \left\{\varpi \in \mathbb{R} : \lim_{T \to \infty} \frac{1}{T} \int_0^T f(s) e^{-\mathrm{i}\varpi s} \mathrm{d}s \neq 0\right\}$$

the set of Fourier exponents of $f$.

Now we are in the position to present and prove our result on the existence of at least one positive almost periodic solution for system (1.5).

Let

$$\nu_0 := \max\{\nu_1^M, \nu_2^M\} = \max\{\sup_{s \in \mathbb{R}} \nu_1(s), \sup_{s \in \mathbb{R}} \nu_2(s)\}.$$

**Theorem 1.** *Assume that*

$(F_1)$ $\inf_{s \in \mathbb{R}}[K_i(s) - \alpha_i(s)] \leq 0$, $i = 1, 2$.
$(F_2)$ $\inf_{s \in \mathbb{R}}[H_i(s) - \beta_i(s)] \leq 0$, $i = 1, 2$.
$(F_3)$ $\bar{\Phi}_i > 0$, *where* $\Phi_i := r_i(s)\big[K_i(s) + H_i(s)\big]$, $i = 1, 2$.

*Then system* (1.5) *admits at least one positive almost periodic solution.*

**Proof:** Under the invariant transformation $(N_1, N_2)^T = (e^u, e^v)^T$, system (1.5) reduces to

$$\begin{cases} \frac{\mathrm{d}u(t)}{\mathrm{d}t} &= r_1(t)\Bigg[\frac{K_1(t)+\alpha_1(t)e^{v(t-\mu_2(t))}}{1+e^{v(t-\mu_2(t))}} \\ &\quad + \frac{H_1(t)+\beta_1(t)\int_0^\infty J_2(s)e^{v(t-s)}\,\mathrm{d}s}{1+\int_0^\infty J_2(s)e^{v(t-s)}\,\mathrm{d}s} - e^{u(t-\nu_1(t))}\Bigg] \\ &:= F_1(t), \\ \frac{\mathrm{d}v(t)}{\mathrm{d}t} &= r_2(t)\Bigg[\frac{K_2(t)+\alpha_2(t)e^{u(t-\mu_1(t))}}{1+e^{u(t-\mu_1(t))}} \\ &\quad + \frac{H_2(t)+\beta_2(t)\int_0^\infty J_1(s)e^{u(t-s)}\,\mathrm{d}s}{1+\int_0^\infty J_1(s)e^{u(t-s)}\,\mathrm{d}s} - e^{v(t-\nu_2(t))}\Bigg] \\ &:= F_2(t). \end{cases} \tag{3.0}$$

It is easy to see that if system (3.0) has one almost periodic solution $(u, v)^T$, then $(N_1, N_2)^T = (e^u, e^v)^T$ is a positive almost periodic solution of system (1.5). Therefore, to completes the proof it suffices to show that system (3.0) has one almost periodic solution.

Take $\mathbb{X} = \mathbb{Y} = \mathbb{V}_1 \bigoplus \mathbb{V}_2$, where

$$\mathbb{V}_1 = \Big\{z = (u, v)^T \in AP(\mathbb{R}, \mathbb{R}^2) :$$
$$\forall \varpi \in \Lambda(u) \cup \Lambda(v), |\varpi| \geq \theta_0\Big\},$$

$$\mathbb{V}_2 = \big\{z = (u, v)^T \equiv (k_1, k_2)^T, k_1, k_2 \in \mathbb{R}\big\},$$

where $\theta_0$ is a given positive constant. Define the norm

$$\|z\| = \max\left\{\sup_{s \in \mathbb{R}} |u(s)|, \sup_{s \in \mathbb{R}} |v(s)|\right\}, \quad \forall z \in \mathbb{X} = \mathbb{Y},$$

then $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces with the norm $\|\cdot\|$. Set

$$L : \mathrm{Dom}L \subseteq \mathbb{X} \to \mathbb{Y}, \quad Lz = L(u, v)^T = (u', v')^T,$$

where $\mathrm{Dom}L = \{z = (u, v)^T \in \mathbb{X} : u, v \in C^1(\mathbb{R}), u', v' \in C(\mathbb{R})\}$ and

$$N : \mathbb{X} \to \mathbb{Y}, \quad Nz = N\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}.$$

With these notations system (3.0) can be written in the form

$$Lz = Nz, \quad \forall z \in \mathbb{X}.$$

It is not difficult to verify that $\mathrm{Ker}L = \mathbb{V}_2$, $\mathrm{Im}L = \mathbb{V}_1$ is closed in $\mathbb{Y}$ and $\dim\mathrm{Ker}L = 2 = \mathrm{codim}\,\mathrm{Im}L$. Therefore, $L$ is a Fredholm mapping of index zero (see Lemma 3.2 in [31]). Now define two projectors $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ as

$$Pz = P\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} m(u) \\ m(v) \end{bmatrix} = Qz, \quad \forall z = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{X} = \mathbb{Y}.$$

Then $P$ and $Q$ are continuous projectors such that $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I - Q)$. Furthermore, through an easy computation we find that the inverse $K_P : \mathrm{Im}L \to \mathrm{Ker}P \cap \mathrm{Dom}L$ of $L_P$ has the form

$$K_P z = K_P\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \int_0^t u(s)\,\mathrm{d}s - m\left[\int_0^t u(s)\,\mathrm{d}s\right] \\ \int_0^t v(s)\,\mathrm{d}s - m\left[\int_0^t v(s)\,\mathrm{d}s\right] \end{bmatrix}.$$

Then $QN : \mathbb{X} \to \mathbb{Y}$ and $K_P(I - Q)N : \mathbb{X} \to \mathbb{X}$ read

$$QNz = QN\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} m(F_1) \\ m(F_2) \end{bmatrix},$$

$$K_P(I-Q)Nz=\begin{bmatrix} f[u(t)]-Qf[u(t)] \\ f[v(t)]-Qf[v(t)] \end{bmatrix}, \quad \forall z\in \mathrm{Im}L,$$

where $f(x)$ is defined by $f[x(t)] = \int_0^t \big[Nx(s) - QNx(s)\big]\,\mathrm{d}s$. Then $N$ is $L$-compact on $\bar{\Omega}$ (see Lemma 3.3 in [31]).

In order to apply Lemma 10, we need to search for an appropriate open-bounded subset $\Omega$.

Corresponding to the operator equation $Lz=\lambda z$, $\lambda\in(0,1)$, we have

$$\begin{cases} \dot{u}(t) = \lambda r_1(t)\Big[\frac{K_1(t)+\alpha_1(t)e^{v(t-\mu_2(t))}}{1+e^{v(t-\mu_2(t))}} \\ \qquad +\frac{H_1(t)+\beta_1(t)\int_0^\infty J_2(s)e^{v(t-s)}\,\mathrm{d}s}{1+\int_0^\infty J_2(s)e^{v(t-s)}\,\mathrm{d}s} - e^{u(t-\nu_1(t))}\Big], \\ \dot{v}(t) = \lambda r_2(t)\Big[\frac{K_2(t)+\alpha_2(t)e^{u(t-\mu_1(t))}}{1+e^{u(t-\mu_1(t))}} \\ \qquad +\frac{H_2(t)+\beta_2(t)\int_0^\infty J_1(s)e^{u(t-s)}\,\mathrm{d}s}{1+\int_0^\infty J_1(s)e^{u(t-s)}\,\mathrm{d}s} - e^{v(t-\nu_2(t))}\Big]. \end{cases} \tag{3.1}$$

Suppose that $(u,v)^T\in \mathrm{Dom}L\subseteq \mathbb{X}$ is a solution of system (3.1) for some $\lambda\in(0,1)$, where $\mathrm{Dom}L=\{z=(u,v)^T\in \mathbb{X}: u,v\in C^1(\mathbb{R}), \dot{u},\dot{v}\in C(\mathbb{R})\}$. By Lemma 6, there exist two sequences $\{T_n: n\in\mathbb{N}^+\}$ and $\{P_n:n\in\mathbb{N}^+\}$ such that

$$u(T_n)\in\left[u^*-\frac{1}{n},u^*\right],\ u^*=\sup_{s\in\mathbb{R}}u(s),\ n\in\mathbb{N}^+, \tag{3.2}$$

$$v(P_n)\in\left[v^*-\frac{1}{n},v^*\right],\ v^*=\sup_{s\in\mathbb{R}}v(s),\ n\in\mathbb{N}^+. \tag{3.3}$$

From $\bar{\Phi}_i>0 (i=1,2)$ and Lemma 7, for $\forall a\in\mathbb{R}$, there exists a constant $\omega_0\in(2\nu_0,+\infty)$ independent of $a$ such that

$$\frac{1}{T}\int_a^{a+T} r_i(s)\,\mathrm{d}s\in\left[\frac{\bar{r}_i}{2},\frac{3\bar{r}_i}{2}\right],$$
$$\frac{1}{T}\int_a^{a+T}\Phi_i(s)\,\mathrm{d}s\in\left[\frac{\bar{\Phi}_i}{2},\frac{3\bar{\Phi}_i}{2}\right],$$
$$\forall T\geq\frac{\omega_0}{2},\ i=1,2. \tag{$\Box$}$$

For $\forall n_0\in\mathbb{N}^+$, we consider $[T_{n_0}-\omega_0,T_{n_0}]$ and $[P_{n_0}-\omega_0,P_{n_0}]$, where $\omega_0$ is defined as that in ($\Box$). By Lemma 4, there exist $\xi\in[T_{n_0}-\omega_0,T_{n_0}]$, $\underline{\xi}\in(-\infty,T_{n_0}-\omega_0]$ and $\bar{\xi}\in[T_{n_0},+\infty)$ such that

$$u(\underline{\xi})=u(\bar{\xi}) \quad\text{and}\quad u(\xi)\leq u(s),\ \forall s\in[\underline{\xi},\bar{\xi}]. \tag{3.4}$$

Integrating the first equation of system (3.1) from $\underline{\xi}$ to $\bar{\xi}$ leads to

$$\int_{\underline{\xi}}^{\bar{\xi}} r_1(s)\Big[\frac{K_1(s)+\alpha_1(s)e^{v(s-\mu_2(s))}}{1+e^{v(s-\mu_2(s))}} +\frac{H_1(s)+\beta_1(s)\int_0^\infty J_2(z)e^{v(s-z)}\,\mathrm{d}z}{1+\int_0^\infty J_2(z)e^{v(s-z)}\,\mathrm{d}z} -e^{u(s-\nu_1(s))}\Big]\,\mathrm{d}s=0,$$

which yields that

$$\int_{\underline{\xi}}^{\bar{\xi}} r_1(s)\Big[\frac{\alpha_1(s)-K_1(s)}{1+e^{v(s-\mu_2(s))}}+\frac{\beta_1(s)-H_1(s)}{1+\int_0^\infty J_2(z)e^{v(s-z)}\,\mathrm{d}z} +e^{u(s-\nu_1(s))}-(\alpha_1(s)+\beta_1(s))\Big]\,\mathrm{d}s=0,$$

which implies from $(F_1)$-$(F_2)$ that

$$\int_{\underline{\xi}+\nu_0}^{\bar{\xi}} r_1(s)e^{u(s-\nu_1(s))}\,\mathrm{d}s\leq\int_{\underline{\xi}}^{\bar{\xi}} r_1(s)e^{u(s-\nu_1(s))}\,\mathrm{d}s \leq\int_{\underline{\xi}}^{\bar{\xi}} r_1(s)(\alpha_1(s)+\beta_1(s))\,\mathrm{d}s.$$

By the integral mean value theorem and ($\Box$), there exists $s_0\in[\underline{\xi}+\nu_0,\bar{\xi}]$ $(s_0-\nu_1(s_0)\in[\underline{\xi},\bar{\xi}])$ such that

$$\begin{aligned}
&\frac{\bar{r}_1}{4}e^{u(s_0-\nu_1(s_0))}\\
&\leq \frac{\bar{\xi}-\underline{\xi}-\nu_0}{\bar{\xi}-\underline{\xi}}\frac{\bar{r}_1}{2}e^{u(s_0-\nu_1(s_0))}\\
&\leq \frac{\bar{\xi}-\underline{\xi}-\nu_0}{\bar{\xi}-\underline{\xi}}e^{u(s_0-\nu_1(s_0))}\frac{1}{\bar{\xi}-\underline{\xi}-\nu_0}\int_{\underline{\xi}+\nu_0}^{\bar{\xi}} r_1(s)\,\mathrm{d}s\\
&= \frac{1}{\bar{\xi}-\underline{\xi}}\int_{\underline{\xi}+\nu_0}^{\bar{\xi}} r_1(s)e^{u(s-\nu_1(s))}\,\mathrm{d}s\\
&\leq \frac{1}{\bar{\xi}-\underline{\xi}}\int_{\underline{\xi}}^{\bar{\xi}} r_1(s)(\alpha_1(s)+\beta_1(s))\,\mathrm{d}s\\
&\leq \frac{3\bar{r}_1}{2}(\alpha_1^M+\beta_1^M),
\end{aligned}$$

which implies from (3.4) that

$$u(\xi)\leq\ln[6(\alpha_1^M+\beta_1^M)]. \tag{3.5}$$

Let $I=\{s\in[\xi,T_{n_0}]:\dot{u}(s)\geq 0\}$. It follows from system (3.1) that

$$\begin{aligned}
&\int_I\dot{u}(s)\,\mathrm{d}s=\int_I\frac{\mathrm{d}u(t)}{\mathrm{d}t}\,\mathrm{d}s\\
&=\int_I\lambda r_1(s)\Big[\frac{K_1(s)+\alpha_1(s)e^{v(s-\mu_2(s))}}{1+e^{v(s-\mu_2(s))}}\\
&\quad+\frac{H_1(s)+\beta_1(s)\int_0^\infty J_2(z)e^{v(s-z)}\,\mathrm{d}z}{1+\int_0^\infty J_2(z)e^{v(s-z)}\,\mathrm{d}z}-e^{u(s-\nu_1(s))}\Big]\,\mathrm{d}s\\
&=\int_I\lambda r_1(s)\Big[(\alpha_1(s)+\beta_1(s))-\frac{\alpha_1(s)-K_1(s)}{1+e^{v(s-\mu_2(s))}}\\
&\quad-\frac{\beta_1(s)-H_1(s)}{1+\int_0^\infty J_2(z)e^{v(s-z)}\,\mathrm{d}z}-e^{u(s-\nu_1(s))}\Big]\,\mathrm{d}s\\
&\leq\int_I\lambda r_1(s)[\alpha_1(s)+\beta_1(s)]\,\mathrm{d}s\\
&\leq\int_{T_{n_0}-\omega_0}^{T_{n_0}} r_1(s)[\alpha_1(s)+\beta_1(s)]\,\mathrm{d}s\\
&\leq r_1^M(\alpha_1^M+\beta_1^M)\omega_0.
\end{aligned} \tag{3.6}$$

By Lemma 3, it follows from (3.5)-(3.6) that

$$\begin{aligned}
u(t)&\leq u(\xi)+\int_{I_1}\dot{u}(s)\,\mathrm{d}s\\
&\leq \ln[6(\alpha_1^M+\beta_1^M)]+r_1^M(\alpha_1^M+\beta_1^M)\omega_0\\
&:=\rho_1,\quad\forall t\in[\xi_u^{n_0},T_{n_0}],
\end{aligned}$$

which implies that

$$u(T_{n_0})\leq\rho_1.$$

In view of (3.2), letting $n_0\to+\infty$ in the above inequality leads to

$$u^*=\lim_{n_0\to+\infty}u(T_{n_0})\leq\rho_1. \tag{3.7}$$

Similar to the argument as that in (3.7), we can obtain that

$$v^* \le \ln[6(\alpha_2^M + \beta_2^M)] + r_2^M(\alpha_2^M + \beta_2^M)\omega_0 := \rho_2. \quad (3.8)$$

Taking

$$\pi_0 = \max\left\{\omega_0, \frac{4r_1^M e^{\rho_1}\nu_0(1+e^{\rho_1})}{\bar{\Phi}_1}, \frac{4r_2^M e^{\rho_2}\nu_0(1+e^{\rho_2})}{\bar{\Phi}_2}\right\}.$$

For $\forall n_0 \in \mathbb{Z}$, by Lemma 5, we can conclude that there exist $\eta \in [n_0\pi_0, n_0\pi_0 + \pi_0]$, $\underline{\eta} \in (-\infty, n_0\pi_0]$ and $\bar{\eta} \in [n_0\pi_0 + \pi_0, +\infty)$ such that

$$u(\underline{\eta}) = u(\bar{\eta}) \quad \text{and} \quad u(\eta) \ge u(s), \quad \forall s \in [\underline{\eta}, \bar{\eta}]. \quad (3.9)$$

Integrating the first equation of system (3.1) from $\underline{\eta}$ to $\bar{\eta}$ leads to

$$\int_{\underline{\eta}}^{\bar{\eta}} r_1(s)\bigg[\frac{K_1(s) + \alpha_1(s)e^{v(s-\mu_2(s))}}{1+e^{v(s-\mu_2(s))}} + \frac{H_1(s) + \beta_1(s)\int_0^\infty J_2(z)e^{v(s-z)}\,\mathrm{d}z}{1+\int_0^\infty J_2(z)e^{v(s-z)}\,\mathrm{d}z} - e^{u(s-\nu_1(s))}\bigg]\,\mathrm{d}s = 0,$$

which yields from (3.7)-(3.8) that

$$\begin{aligned}
&\frac{1}{1+e^{\rho_2}}\int_{\underline{\eta}}^{\bar{\eta}} r_1(s)\big[K_1(s) + H_1(s)\big]\,\mathrm{d}s \\
\le& \int_{\underline{\eta}}^{\bar{\eta}} r_1(s)\bigg[\frac{K_1(s) + \alpha_1(s)e^{v(s-\mu_2(s))}}{1+e^{v(s-\mu_2(s))}} + \frac{H_1(s) + \beta_1(s)\int_0^\infty J_2(z)e^{v(s-z)}\,\mathrm{d}z}{1+\int_0^\infty J_2(z)e^{v(s-z)}\,\mathrm{d}z}\bigg]\,\mathrm{d}s \\
=& \int_{\underline{\eta}}^{\bar{\eta}} r_1(s)e^{u(s-\nu_1(s))}\,\mathrm{d}s \\
=& \int_{\underline{\eta}+\nu_0}^{\bar{\eta}} r_1(s)e^{u(s-\nu_1(s))}\,\mathrm{d}s + \int_{\underline{\eta}}^{\underline{\eta}+\nu_0} r_1(s)e^{u(s-\nu_1(s))}\,\mathrm{d}s \\
\le& \int_{\underline{\eta}+\nu_0}^{\bar{\eta}} r_1(s)e^{u(s-\nu_1(s))}\,\mathrm{d}s + r_1^M e^{\rho_1}\nu_0,
\end{aligned}$$

which implies from $(F_3)$ and $(\square)$ that

$$\begin{aligned}
&\frac{\bar{\Phi}_1}{2+2e^{\rho_2}} \\
\le& \frac{1}{1+e^{\rho_2}}\frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}}^{\bar{\eta}} r_1(s)\big[K_1(s)+H_1(s)\big]\,\mathrm{d}s \\
\le& \frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}+\nu_0}^{\bar{\eta}} r_1(s)e^{u(s-\nu_1(s))}\,\mathrm{d}s + \frac{r_1^M e^{\rho_1}\nu_0}{\bar{\eta}-\underline{\eta}} \\
\le& \frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}+\nu_0}^{\bar{\eta}} r_1(s)e^{u(s-\nu_1(s))}\,\mathrm{d}s + \frac{r_1^M e^{\rho_1}\nu_0}{\pi_0} \\
\le& \frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}+\nu_0}^{\bar{\eta}} r_1(s)e^{u(s-\nu_1(s))}\,\mathrm{d}s + \frac{\bar{\Phi}_1}{4+4e^{\rho_2}}. \quad (3.10)
\end{aligned}$$

In view of (3.10), by the integral mean value theorem and (3.9), there exists $s_1 \in [\underline{\eta}+\nu_0, \bar{\eta}]$ $(s_1 - \nu_1(s_1) \in [\underline{\eta}, \bar{\eta}])$ such that

$$\begin{aligned}
\frac{\bar{\Phi}_1}{4+4e^{\rho_2}} &\le \frac{e^{u(s_1-\nu_1(s_1))}}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}+\nu_0}^{\bar{\eta}} r_1(s)\,\mathrm{d}s \\
&\le r_1^M e^{u(\eta)}\frac{\bar{\eta}-\underline{\eta}-\nu_0}{\bar{\eta}-\underline{\eta}} \le r_1^M e^{u(\eta)},
\end{aligned}$$

which implies that

$$u(\eta) \ge \ln\frac{\bar{\Phi}_1}{4r_1^M(1+e^{\rho_2})}. \quad (3.11)$$

Further, we obtain from system (3.1) that

$$\begin{aligned}
&\int_{n_0\pi_0}^{n_0\pi_0+\pi_0} |\dot{u}(s)|\,\mathrm{d}s \\
=& \int_{n_0\pi_0}^{n_0\pi_0+\pi_0} \lambda r_1(s)\bigg|\frac{K_1(s)+\alpha_1(s)e^{v(s-\mu_2(s))}}{1+e^{v(s-\mu_2(s))}} + \frac{H_1(s)+\beta_1(s)\int_0^\infty J_2(z)e^{v(s-z)}\,\mathrm{d}z}{1+\int_0^\infty J_2(z)e^{v(s-z)}\,\mathrm{d}z} - e^{u(s-\nu_1(s))}\bigg|\,\mathrm{d}s \\
\le& r_1^M\big[K_1^M + \alpha_1^M e^{\rho_2} + H_1^M + \beta_1^M e^{\rho_2} + e^{\rho_1}\big]\pi_0 \\
:=& \Theta_1. \quad (3.12)
\end{aligned}$$

It follows from (3.11)-(3.12) that

$$\begin{aligned}
u(t) &\ge u(\eta) - \int_{n_0\pi_0}^{n_0\pi_0+\pi_0} |\dot{u}(s)|\,\mathrm{d}s \\
&\ge \ln\frac{\bar{\Phi}_1}{4r_1^M(1+e^{\rho_2})} - \Theta_1 \\
&:= \rho_3, \quad \forall t \in [n_0\pi_0, n_0\pi_0+\pi_0]. \quad (3.13)
\end{aligned}$$

Obviously, $\rho_3$ is a constant independent of $n_0$. So it follows from (3.13) that

$$u_* = \inf_{s\in\mathbb{R}} u(s) \ge \inf_{n_0\in\mathbb{Z}}\{\rho_3\} = \rho_3. \quad (3.14)$$

Similar to the argument as that in (3.14), we can obtain that

$$\begin{aligned}
v_* \ge& \ln\frac{\bar{\Phi}_1}{4r_1^M(1+e^{\rho_2})} - r_2^M\big[K_2^M + \alpha_2^M e^{\rho_1} \\
&+ H_2^M + \beta_2^M e^{\rho_1} + e^{\rho_2}\big]\pi_0 := \rho_4.
\end{aligned}$$

Set $C = |\rho_1| + |\rho_2| + |\rho_3| + |\rho_4| + C_0 + 1$, where $C_0$ is taken sufficiently large such that

$$C > \max_{i=1,2}\left\{2\ln\frac{m[r_i(\alpha_i+\beta_i)]}{m(r_i)}, 2\ln\frac{m[r_i(K_i+H_i)]}{m(r_i)}\right\}.$$

Clearly, $C$ is independent of $\lambda \in (0,1)$. Consider the algebraic equations $QNz_0 = 0$ for $z_0 = (u_0, v_0)^T \in \mathbb{R}^2$ as follows:

$$\begin{cases}
m[r_1(\alpha_1+\beta_1)] - \frac{m[r_1(\alpha_1+\beta_1)] - m[r_1(K_1+H_1)]}{1+e^{v_0}} - m(r_1)e^{u_0} = 0, \\
m[r_2(\alpha_2+\beta_2)] - \frac{m[r_2(\alpha_2+\beta_2)] - m[r_2(K_2+H_2)]}{1+e^{u_0}} - m(r_2)e^{v_0} = 0.
\end{cases}$$

Similar to the arguments as that in (3.7)-(3.8) and (3.14)-(3.15), we can easily obtain that

$$\rho_3 \le u_0 \le \rho_1, \quad \rho_4 \le v_0 \le \rho_2.$$

Then $\|z_0\|_{\mathbb{X}} = |u_0| + |v_0| < C$. Let $\Omega = \{z \in \mathbb{X} : \|z\|_{\mathbb{X}} < C\}$, then $\Omega$ satisfies conditions $(a)$ and $(b)$ of Lemma 10.

Finally, we will show that condition $(c)$ of Lemma 10 is satisfied. By Lemma 8, we have

$$\deg\big(JQN, \Omega\cap\mathrm{Ker}L, 0\big) = \deg\big(QN, \Omega\cap\mathrm{Ker}L, 0\big) \ne 0,$$

where $\deg(\cdot,\cdot,\cdot)$ is the Brouwer degree and $J$ is the identity mapping since $\mathrm{Im}Q = \mathrm{Ker}L$. Obviously, all the conditions of Lemma 10 are satisfied. Therefore, system (3.0) has one almost periodic solution, that is, system (1.5) has at least one positive almost periodic solution. This completes the proof. ■

**Remark 1.** By Theorem 1, it is easy to obtain the existence of at least one positive almost periodic solution of Eq. (1.4) in Example 1, although there is no a priori reason to expect the existence of positive periodic solutions of Eq. (1.4).

Assume that all the coefficients of system (1.5) are $\omega$-periodic functions, we have

**Corollary 1.** *Assume that $(F_1)$-$(F_3)$ hold. Suppose further that all the coefficients of system (1.5) are $\omega$-periodic functions. Then system (1.5) admits at least one positive $\omega$-periodic solution.*

**Theorem 2.** *Assume that $(F_3)$ holds. Suppose further that*

$(F_4)$ $\inf_{s\in\mathbb{R}}[K_i(s)-\alpha_i(s)]\geq 0,\ i=1,2.$

$(F_5)$ $\inf_{s\in\mathbb{R}}[H_i(s)-\beta_i(s)]\geq 0,\ i=1,2.$

*Then system (1.5) admits at least one positive almost periodic solution.*

**Proof:** Proceeding as in the proof of Theorem 1, we see that (3.4) holds. Integrating the first equation of system (3.1) from $\underline{\xi}$ to $\bar{\xi}$ leads to

$$\int_{\underline{\xi}}^{\bar{\xi}} r_1(s)\Big[\frac{K_1(s)+\alpha_1(s)e^{v(s-\mu_2(s))}}{1+e^{v(s-\mu_2(s))}}$$

$$+\frac{H_1(s)+\beta_1(s)\int_0^\infty J_2(z)e^{v(s-z)}\,\mathrm{d}z}{1+\int_0^\infty J_2(z)e^{v(s-z)}\,\mathrm{d}z}-e^{u(s-\nu_1(s))}\Big]\,\mathrm{d}s=0,$$

which yields from $(F_4)$-$(F_5)$ that

$$\begin{aligned}&\int_{\underline{\xi}}^{\bar{\xi}} r_1(s)e^{u(s-\nu_1(s))}\,\mathrm{d}s\\ =&\int_{\underline{\xi}}^{\bar{\xi}} r_1(s)\Big[\frac{K_1(s)-\alpha_1(s)}{1+e^{v(s-\mu_2(s))}}\\ &+\frac{H_1(s)-\beta_1(s)}{1+\int_0^\infty J_2(z)e^{v(s-z)}\,\mathrm{d}z}+(\alpha_1(s)+\beta_1(s))\Big],\\ \leq&\int_{\underline{\xi}}^{\bar{\xi}} r_1(s)\Big([K_1(s)-\alpha_1(s)]\\ &+[H_1(s)-\beta_1(s)]+(\alpha_1(s)+\beta_1(s))\Big),\\ =&\int_{\underline{\xi}}^{\bar{\xi}} r_1(s)[K_1(s)+H_1(s)].\end{aligned}$$

Similar to the argument as that in (3.7)-(3.8), we can obtain that

$$u^*\leq\ln[6(K_1^M+H_1^M)]+r_1^M(K_1^M+H_1^M)\omega_0:=\rho_1',\quad(3.15)$$

and

$$v^*\leq\ln[6(K_2^M+H_2^M)]+r_2^M(K_2^M+H_2^M)\omega_0:=\rho_2'.\quad(3.16)$$

By a parallel arguments as that in (3.14)-(3.15), there exist $\rho_3'$ and $\rho_4'$ such that

$$u_*\geq\rho_3',\quad v_*\geq\rho_4'.\qquad(3.17)$$

Set $C'=\rho_1'+\rho_2'+\rho_3'+\rho_4'+C_0'+1$, where $C_0'$ is taken sufficiently large such that

$$C'>\max_{i=1,2}\Big\{2\ln\frac{m[r_i(\alpha_i+\beta_i)]}{m(r_i)},2\ln\frac{m[r_i(K_i+H_i)]}{m(r_i)}\Big\}.$$

Clearly, $C'$ is independent of $\lambda\in(0,1)$. Consider the algebraic equations $QNz_0=0$ for $z_0=(u_0,v_0)^T\in\mathbb{R}^2$ as follows:

$$\begin{cases}m[r_1(\alpha_1+\beta_1)]\\ +\frac{m[r_1(K_1+H_1)]-m[r_1(\alpha_1+\beta_1)]}{1+e^{v_0}}-m(r_1)e^{u_0}=0,\\ m[r_2(\alpha_2+\beta_2)]\\ +\frac{m[r_2(K_2+H_2)]-m[r_2(\alpha_2+\beta_2)]}{1+e^{u_0}}-m(r_2)e^{v_0}=0.\end{cases}$$

Let $\Omega'=\{z\in\mathbb{X}:\|z\|_{\mathbb{X}}<C'\}$. From the proof in Theorem 1, it is easy to see that $\Omega'$ satisfies conditions $(a)$ and $(b)$ of Lemma 10. Further, by Lemma 9, condition $(c)$ of Lemma 10 is also satisfied. Obviously, all the conditions of Lemma 10 are satisfied. Therefore, system (3.0) has one almost periodic solution, that is, system (1.5) has at least one positive almost periodic solution. This completes the proof. ■

From Theorem 2 and Corollary 1, we also have the following corollary:

**Corollary 2.** *Assume that $(F_3)$-$(F_5)$ hold. Suppose further that all the coefficients of system (1.5) are $\omega$-periodic functions. Then system (1.5) admits at least one $\omega$-positive periodic solution.*

Now we give some assumptions:

$(F_6)$ $\inf_{s\in\mathbb{R}}[K_1(s)-\alpha_1(s)]\geq 0$ and $\inf_{s\in\mathbb{R}}[K_2(s)-\alpha_2(s)]\leq 0$.

$(F_7)$ $\inf_{s\in\mathbb{R}}[K_1(s)-\alpha_1(s)]\leq 0$ and $\inf_{s\in\mathbb{R}}[K_2(s)-\alpha_2(s)]\geq 0$.

$(F_8)$ $\inf_{s\in\mathbb{R}}[H_1(s)-\beta_1(s)]\geq 0$ and $\inf_{s\in\mathbb{R}}[H_2(s)-\beta_2(s)]\leq 0$.

$(F_9)$ $\inf_{s\in\mathbb{R}}[H_1(s)-\beta_1(s)]\leq 0$ and $\inf_{s\in\mathbb{R}}[H_2(s)-\beta_2(s)]\geq 0$.

From the proof of Theorems 1-2, we can easily show that

**Theorem 3.** *Assume that $(F_2)$-$(F_3)$ hold, suppose further that $(F_6)$ or $(F_7)$ is satisfied. Then system (1.5) admits at least one positive almost periodic solution.*

**Theorem 4.** *Assume that $(F_1)$ and $(F_3)$ hold, suppose further that $(F_8)$ or $(F_9)$ is satisfied. Then system (1.5) admits at least one positive almost periodic solution.*

**Theorem 5.** *Assume that $(F_3)$ and $(F_6)$ hold, suppose further that $(F_8)$ or $(F_9)$ is satisfied. Then system (1.5) admits at least one positive almost periodic solution.*

**Theorem 6.** *Assume that $(F_3)$ and $(F_7)$ hold, suppose further that $(F_8)$ or $(F_9)$ is satisfied. Then system (1.5) admits at least one positive almost periodic solution.*

From Theorems 3-6, we can easily show that

**Corollary 3.** *Assume that all the coefficients of system (1.5) are $\omega$-periodic functions and $(F_2)$-$(F_3)$ hold, suppose further that $(F_6)$ or $(F_7)$ is satisfied. Then system (1.5) admits at least one positive $\omega$-periodic solution.*

**Corollary 4.** *Assume that all the coefficients of system (1.5) are $\omega$-periodic functions and $(F_1)$ and $(F_3)$ hold, suppose further that $(F_8)$ or $(F_9)$ is satisfied. Then system (1.5) admits at least one positive $\omega$-periodic solution.*

**Corollary 5.** *Assume that all the coefficients of system (1.5) are $\omega$-periodic functions and $(F_3)$ and $(F_6)$ hold, suppose* *further that $(F_8)$ or $(F_9)$ is satisfied. Then system (1.5) admits at least one positive $\omega$-periodic solution.*

**Corollary 6.** *Assume that all the coefficients of system (1.5) are $\omega$-periodic functions and $(F_3)$ and $(F_7)$ hold, suppose further that $(F_8)$ or $(F_9)$ is satisfied. Then system (1.5) admits at least one positive $\omega$-periodic solution.*

In [4], Li obtained that

**Corollary 7.** ([4]) *Assume that $(F_1)$-$(F_3)$ hold. Suppose further that all the coefficients of system (1.2) are $\omega$-periodic functions. Then system (1.2) admits at least one positive $\omega$-periodic solution.*

**Remark 2.** Obviously, the works in this paper extend and improve the result in [4].

## IV. Boundedness

**Theorem 7.** *In system (1.5), assume that*

$(S_1)$ $H_i^M = \beta_i^M = \nu_i^M = 0$, $\mu_i(t) \equiv \mu_i$ *for all* $t \in \mathbb{R}$, $i = 1, 2$.

*Then every solution of system (1.5) satisfies* $\limsup_{t\to\infty} N_i(t) \le N_i^* = K_i^M + \alpha_i^M$, $i = 1, 2$, $\forall t \in \mathbb{R}$.

*Proof:* From system (1.5), it leads

$$\begin{cases} \frac{\mathrm{d}N_1(t)}{\mathrm{d}t} \le r_1^M N_1(t)\left[K_1^M + \alpha_1^M - N_1(t)\right], \\ \frac{\mathrm{d}N_2(t)}{\mathrm{d}t} \le r_2^M N_2(t)\left[K_2^M + \alpha_2^M - N_2(t)\right]. \end{cases}$$

By the comparison theorem of differential equations, we have

$$\limsup_{t\to\infty} N_i(t) \le K_i^M + \alpha_i^M,\ i = 1, 2.$$

This completes the proof. ■

## V. Global asymptotic stability

Let

$$k_i := \sup_{s\in\mathbb{R}} |K_i(s) - \alpha_i(s)|,\ i = 1, 2.$$

**Theorem 8.** *Assume that $(S_1)$ holds, then system (1.5) is globally asymptotically stable.*

*Proof:* From Theorem 1, we know that system (1.5) has at least one positive almost periodic solution $(N_1, N_2)^T$. Suppose that $(\bar{N}_1, \bar{N}_2)^T$ is another solution of system (1.5). Let $(x_1, x_2)^T = (\ln N_1, \ln N_2)^T$ and $(\bar{x}_1, \bar{x}_2)^T = (\ln \bar{N}_1, \ln \bar{N}_2)^T$, then system (1.5) is transformed into

$$\begin{cases} \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = r_1(t)\left[\frac{K_1(t)+\alpha_1(t)N_2(t-\mu_2)}{1+N_2(t-\mu_2)} - N_1(t)\right], \\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = r_2(t)\left[\frac{K_2(t)+\alpha_2(t)N_1(t-\mu_1)}{1+N_1(t-\mu_1)} - N_2(t)\right], \\ \frac{\mathrm{d}\bar{x}_1(t)}{\mathrm{d}t} = r_1(t)\left[\frac{K_1(t)+\alpha_1(t)\bar{N}_2(t-\mu_2)}{1+\bar{N}_2(t-\mu_2)} - \bar{N}_1(t)\right], \\ \frac{\mathrm{d}\bar{x}_2(t)}{\mathrm{d}t} = r_2(t)\left[\frac{K_2(t)+\alpha_2(t)\bar{N}_1(t-\mu_1)}{1+\bar{N}_1(t-\mu_1)} - \bar{N}_2(t)\right]. \end{cases} \tag{5.1}$$

Define

$$V(t) = V_0(t) + V_1(t) + V_2(t),$$

where

$$V_0(t) = |x_1(t) - \bar{x}_1(t)| + |x_2(t) - \bar{x}_2(t)|,$$

$$V_1(t) = r_1^M k_1 \int_{t-\mu_2}^{t} |N_2(s) - \bar{N}_2(s)|\,\mathrm{d}s,$$

$$V_2(t) = r_2^M k_2 \int_{t-\mu_1}^{t} |N_1(s) - \bar{N}_1(s)|\,\mathrm{d}s.$$

By calculating the upper right derivative of $V_0$ along system (5.1), it follows that

$$\begin{aligned} & D^+V_0(t) \\ = {} & \mathrm{sgn}[x_1(t) - \bar{x}_1(t)][x_1'(t) - \bar{x}_1'(t)] \\ & +\mathrm{sgn}[x_2(t) - \bar{x}_2(t)][x_2'(t) - \bar{x}_2'(t)] \\ \le {} & -r_1^l|N_1(t) - \bar{N}_1(t)| - r_2^l|N_2(t) - \bar{N}_2(t)| \\ & +r_1^M k_1|N_2(t-\mu_2) - \bar{N}_2(t-\mu_2)| \\ & +r_2^M k_2|N_1(t-\mu_1) - \bar{N}_1(t-\mu_1)|. \end{aligned} \tag{5.2}$$

Further, by calculating the upper right derivative of $V_1$, $V_2$ and $V_3$ along system (5.1), it follows that

$$\begin{aligned} D^+V_1(t) = {} & r_1^M k_1|N_2(t) - \bar{N}_2(t)| \\ & -r_1^M k_1|N_2(t-\mu_2) - \bar{N}_2(t-\mu_2)|, \end{aligned} \tag{5.3}$$

$$\begin{aligned} D^+V_2(t) = {} & r_2^M k_2|N_1(t) - \bar{N}_1(t)| \\ & -r_2^M k_2|N_1(t-\mu_1) - \bar{N}_1(t-\mu_1)|. \end{aligned} \tag{5.4}$$

Together with (5.2)-(5.4), it follows that

$$\begin{aligned} D^+V(t) \le {} & -(r_1^l - r_2^M k_2)|N_1(t) - \bar{N}_1(t)| \\ & -(r_2^l - r_1^M k_1)|N_2(t) - \bar{N}_2(t)|,\ \ \forall t \ge T. \end{aligned}$$

Therefore, $V$ is non-increasing. Integrating of the last inequality from $T$ to $t$ leads to

$$\begin{aligned} & V(t) + (r_1^l - r_2^M k_2)\int_T^t |N_1(s) - \bar{N}_1(s)|\,\mathrm{d}s \\ & +(r_2^l - r_1^M k_1)\int_T^t |N_2(s) - \bar{N}_2(s)|\,\mathrm{d}s \le V(0) \\ < {} & +\infty,\ \ \forall t \ge T, \end{aligned}$$

that is,

$$\int_0^{+\infty} |N_1(s) - \bar{N}_1(s)|\big|\,\mathrm{d}s < +\infty,$$

$$\int_0^{+\infty} |N_2(s) - \bar{N}_2(s)|\,\mathrm{d}s < +\infty,$$

which implies that

$$\lim_{s\to+\infty} |N_1(s) - \bar{N}_1(s)| = \lim_{s\to+\infty} |N_2(s) - \bar{N}_2(s)| = 0.$$

This completes the proof. ■

**Theorem 9.** *Assume that $(F_1)$-$(F_3)$ and $(S_1)$ hold. Then the almost periodic solution of system (1.5) is globally asymptotic stable.*

## VI. Some Examples and Simulations

**Example 2.** Consider the following almost periodic system:

$$\begin{cases} \frac{\mathrm{d}N_1(t)}{\mathrm{d}t} = N_1(t)\Bigg[\frac{K_1(t)+\alpha_1(t)N_2(t-1)}{1+N_2(t-1)} \\ \qquad + \frac{H_1(t)+\beta_1(t)\int_0^\infty e^{-s}N_2(t-s)\,\mathrm{d}s}{1+\int_0^\infty J_2(s)N_2(t-s)\,\mathrm{d}s} \\ \qquad -N_1(t-\sin^2(\sqrt{3}t))\Bigg], \\ \frac{\mathrm{d}N_2(t)}{\mathrm{d}t} = N_2(t)\Bigg[\frac{K_2(t)+\alpha_2(t)N_1(t-2)}{1+N_1(t-2)} \\ \qquad + \frac{H_2(t)+\beta_2(t)\int_0^\infty e^{-s}N_1(t-s)\,\mathrm{d}s}{1+\int_0^\infty J_1(s)N_1(t-s)\,\mathrm{d}s} \\ \qquad -N_2(t-0.5)\Bigg], \end{cases} \tag{6.1}$$

where

$$\begin{pmatrix} K_1(t) \\ K_2(t) \end{pmatrix} = \begin{pmatrix} |\sin\sqrt{10}t| \\ |\cos\sqrt{10}t| \end{pmatrix},$$

$$\begin{pmatrix} H_1(t) \\ H_2(t) \end{pmatrix} = \begin{pmatrix} \sin^2(\sqrt{2}t) \\ \cos^2(\sqrt{2}t) \end{pmatrix},$$

$$\begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix} = \begin{pmatrix} 2|\sin\sqrt{10}t| \\ 2|\cos\sqrt{10}t| \end{pmatrix},$$

$$\begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix} = \begin{pmatrix} \sin^2(\sqrt{2}t)+0.5 \\ \cos^2(\sqrt{2}t)+0.5 \end{pmatrix}.$$

By a easy computation, it is not difficult to verify that $(F_1)$-$(F_3)$ in Theorem 1 are satisfied. By Theorem 1, system (6.1) admits at least one positive almost periodic solution.

**Example 3.** In system (6.1), let

$$\begin{pmatrix} K_1(t) \\ K_2(t) \end{pmatrix} = \begin{pmatrix} |\sin t| \\ |\cos t| \end{pmatrix}, \quad \begin{pmatrix} H_1(t) \\ H_2(t) \end{pmatrix} = \begin{pmatrix} \sin^2 t \\ \cos^2 t \end{pmatrix},$$

$$\begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix} = \begin{pmatrix} 2|\sin t| \\ 2|\cos t| \end{pmatrix},$$

$$\begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix} = \begin{pmatrix} \sin^2 t+0.5 \\ \cos^2 t+0.5 \end{pmatrix}.$$

By a easy computation, it is not difficult to verify that $(F_1)$-$(F_3)$ in Corollary 1 are satisfied. By Corollary 1, system (6.1) admits at least one positive $\pi$-periodic solution.

**Example 4.** In system (6.1), let

$$\begin{pmatrix} K_1(t) \\ K_2(t) \end{pmatrix} = \begin{pmatrix} 1.5+|\sin\sqrt{10}t| \\ 2.5+|\cos\sqrt{10}t| \end{pmatrix},$$

$$\begin{pmatrix} H_1(t) \\ H_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix} = \begin{pmatrix} |\cos\sqrt{5}t| \\ |\sin\sqrt{5}t| \end{pmatrix}, \quad \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By a easy computation, it is not difficult to verify that $(F_3)$-$(F_5)$ in Theorem 3 and $(S_1)$ in Theorem 8 are satisfied. By Theorem 3 and Theorem 8, system (6.1) admits at least one positive almost periodic solution (see Figure 4.1), which is globally asymptotic stable.

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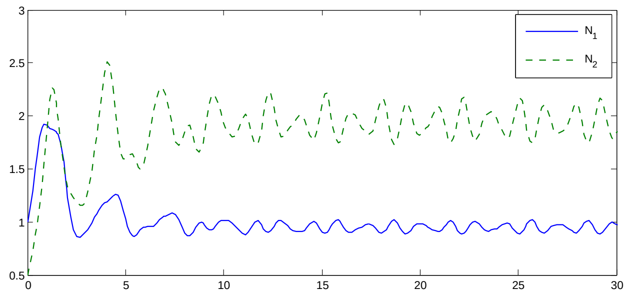


Fig. 1 State variables $N_1$ and $N_2$ of Example 2

**Remark 3.** In Example 4, $|\sin\sqrt{10}t|$ is $\frac{\sqrt{10}\pi}{10}$-periodic function and $|\cos(\sqrt{5}t)|$ is $\frac{\sqrt{5}\pi}{5}$-periodic function. So system (6.1) is with incommensurable periods. Through all the coefficients of system (6.1) are periodic functions, the positive periodic solutions of system (6.1) could not possibly exist. However, by Theorem 3, the positive almost periodic solutions of system (6.1) exactly exist (see Figure 1).

**Example 5.** In system (6.1), let

$$\begin{pmatrix} K_1(t) \\ K_2(t) \end{pmatrix} = \begin{pmatrix} |\sin\sqrt{5}t|+|\cos\sqrt{10}t| \\ |\cos\sqrt{10}t| \end{pmatrix},$$

$$\begin{pmatrix} H_1(t) \\ H_2(t) \end{pmatrix} = \begin{pmatrix} \sin^2(\sqrt{2}t) \\ \cos^2(\sqrt{2}t) \end{pmatrix},$$

$$\begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix} = \begin{pmatrix} 2|\sin\sqrt{5}t|+2|\cos\sqrt{10}t| \\ 2|\cos\sqrt{10}t| \end{pmatrix},$$

$$\begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix} = \begin{pmatrix} \sin^2(\sqrt{2}t)+0.5 \\ \cos^2(\sqrt{2}t)+0.5 \end{pmatrix}.$$

In Example 4, $|\sin\sqrt{5}t|+|\cos\sqrt{10}t|$ is almost periodic function, which is not periodic function. Similar to the argument as that in Example 2, it is easy to obtain that system (6.1) admits at least one positive almost periodic solution.

**Example 6.** In system (6.1), let

$$\begin{pmatrix} K_1(t) \\ K_2(t) \end{pmatrix} = \begin{pmatrix} 1.5+|\sin t| \\ 2.5+|\cos t| \end{pmatrix},$$

$$\begin{pmatrix} H_1(t) \\ H_2(t) \end{pmatrix} = \begin{pmatrix} 0.5+\sin^2 t \\ 0.5+\cos^2 t \end{pmatrix},$$

$$\begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix} = \begin{pmatrix} \sin^2 t \\ \cos^2 t \end{pmatrix}, \quad \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix} = \begin{pmatrix} \sin^2 t \\ \cos^2 t \end{pmatrix}.$$

By a easy computation, it is not difficult to verify that $(F_3)$-$(F_5)$ in Corollary 5 are satisfied. By Corollary 5, system (6.1) admits at least one positive $\pi$-periodic solution.

**Remark 4.** It is obvious that the results in papers [4,5,30] couldn't obtain the existence of positive (almost) periodic solutions for the models in Example 2 and Examples 4-5. Therefore, our works extend the results in papers [4,5,30].

## Acknowledgements

This work is supported by the Scientific Research Fund of Yunnan Provincial Education Department (2014Y388) and the Research Program of Kunming University in China (XJL15010).

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