An Empirical Analysis of the Dependence Structure of International Equity and Bond Markets Using Regime-switching Copula Model

Yuko Otani and Junichi Imai

Abstract—**In this paper, we perform an empirical analysis of the dependence structure of international equity and bond markets using the regime-switching copula model. In equity markets, it is observed that negative returns are more strongly dependent than positive returns. This phenomenon is known as asymmetric dependence. The regime-switching copula model, which includes symmetric and asymmetric regimes, is suitable for describing asymmetry. We apply two kinds of flexible multivariate copulas, a skew t copula and a vine copula, to the asymmetric regime to deal with dependencies between two asset classes. In this paper, we analyze three country pairs: the United Kingdom and United States (UK-US), Japan-US, and Italy-US. We find three implications of our empirical analysis. First, highly dependent regimes are different according to the asset pairs. Second, the strength of the asymmetry of each country pair varies, and that of the Japan-US pair is weak. Third, the skew t copula is a better fit to the data, but is not flexible enough to capture extreme dependencies, while the vine copula fits well in spite of having fewer parameters, but cannot express the different extreme dependencies of each asset pair.**

Index Terms—**International market correlation, Asymmetry, Copulas, Regime-switching model.**

I. Introduction

RESEARCH on the dependence structure of international equity markets has shown that negative returns are more dependent than positive returns. This phenomenon is called “asymmetric dependence”. It has important implications for the risk of international portfolios. If investors neglect increasing dependence in times of crisis, they might overestimate the effects of diversification and lose benefits. To evaluate the full risk of asset allocation, both equity and bond markets must be considered together, because a typical diversification method is to invest both equities and bonds at domestic and international levels. However, most previous research has only focused on dependence in equity markets because of the difficulty of describing the dependence structures of different asset classes.

Many researchers have investigated asymmetric dependence using the exceedance correlation concept, which is defined as the correlation calculated from returns above or below a certain threshold. [1] used extreme value theory to capture the asymptotic dependence of returns in equity markets, and evaluated the asymmetry using exceedance correlation. The advantage of extreme value theory is that the asymptotic property is maintained regardless of the distribution of returns, but its shortcoming is that it cannot determine if the return process from a given model has an asymmetric exceedance correlation. [2] developed a method to test asymmetric correlation. [3] applied a Gaussian regime-switching (RS) model and found two regimes: a bear regime with negative returns, high volatilities, and high dependence, and a bull regime with positive returns, low volatilities, and low dependence. However, [4] analytically showed that multivariate GARCH or RS models with Gaussian innovations cannot reproduce extreme asymmetric dependence. Thus, to correctly investigate asymmetry, we need models that use more complex innovations than the Gaussian.

More flexible models have been proposed by combining copulas with time series models. Copulas are functions that represent dependence structures of multivariate distributions. [5] presented that a nested Archimedean copulas can successfully explained the dependence structures between default probabilities and recovery rates. The advantages of copulas are that they separate dependence structures from marginal models, and that they can express variable dependence structures such as asymmetry. [6] analyzed international correlation in equity markets with a copula-GARCH model. [7] proposed a copula model with time-varying parameters, and investigated asymmetry in foreign exchange markets. [8] examined systemic risk in the U.S. equity market and showed the validity of vine Copula-based ARMA-GARCH model. Other than financial applications, [9] used copula functions in their empirical study to examine monthly precipitation. [10] proposed a reliability assessment model with copula function.

To describe asymmetry more properly, researchers such as [11] and [12] proposed an RS copula model. It allows us to switch copulas depending on two regimes: symmetric and asymmetric. [11] used bivariate copulas to evaluate the asymmetry of equity pairs. [12] investigated asymmetric dependences in equity markets with a multivariate vine copula. They used the flexible multivariate model, which is more complicated than a bivariate copula model. However, they only examined the dependence structure of equities, which is not enough to consider the risk of portfolios composed of equities and bonds.

To analyze international market correlation among different asset classes, we must use a flexible multivariate copula for dependencies in an asymmetric regime. This is because different asset classes exhibit various behavior and dependence structures. To describe them properly, we assume that the following two features are necessary for flexible copulas. First, the strength of the dependence of each pair should be described separately. Second, we should use a model that can express asymmetric tail dependence.

Manuscript received December 06, 2017; revised May 11, 2018.

Y. Otani was studying at Graduate School of Science and Technology, Keio University and she is with JXTG Nippon Oil & Energy Corporation.

J. Imai is with Graduate School of Science and Technology, Keio University.3-14-1 Hiyoshi, Kohoku Yokohama 223-8522, Japan. Tel:+81-45-566-1621. Email:jimai@ae.keio.ac.jp

(Advance online publication: 28 May 2018)

Tail dependence refers to the dependence between extremely large or small pairs. The well-known multivariate copulas only satisfy one of these features. For example, multivariate elliptical copulas such as the Gaussian or t copulas are able to express different dependencies for each pair, but their tail dependencies are symmetric. Other examples belong to the multivariate Archimedean copula family such as the Clayton or Gumbel copulas. Their tail dependences are asymmetric, but their dependence structures are dominated by only one parameter. Thus, constructing flexible multivariate copulas is challenging.

To overcome this problem we need to introduce more complex copulas. When considering copulas that satisfy these two features, a trade-off exists between the power of expression and parsimony. The skew t copula (suggested by [13] and [14]) and vine copula (introduced by [15] and [16]) focus more on the ability to express various dependence structures. The hierarchical Archimedean copula (proposed by [17]) and mixture copula (used in [4]) pay more attention to retaining the number of parameters. [4] incorporated the mixture copula into the RS copula model, and analyzed the dependence structure of international equity and bond markets. This is one of few studies that focused on dependences among different asset classes. However, they neglected the correlation of some pairs to simplify the model, which makes it less flexible. Moreover, as [12] indicated, their model cannot be applied to pairs with strong dependencies. Thus, the model of [4] is not flexible enough to evaluate diversification risks. In this paper, we focus on the former copulas because they are flexible enough to describe the complicated dependence structures among different asset classes, even though the number of parameters becomes large. To the best of our knowledge, our research is the first to analyze dependence structures among different asset classes with a multivariate flexible model.

In this paper, we perform an empirical analysis of the dependence structure of international equity and bond markets using the RS copula model. We use the Gaussian copula in a symmetric regime, while applying the skew t or vine copula in an asymmetric regime. The model itself is the same as in existing research, but it uses flexible multivariate copulas to enable us to properly investigate complicated dependence structures among different asset classes. We use the skew t copula of [14], which is constructed using the skew t distribution expressed in the simple form of [18]. We use the skew t GARCH model of [19] as a marginal distribution that considers skewness. We assume that the marginal distributions are not dependent on regimes. Although this assumption might be less flexible, our empirical study demonstrates that it is reasonable and practical from a parsimonious viewpoint. Furthermore, from a technical viewpoint, it allows us to estimate parameters in a two-step procedure, which makes the estimation procedure possible. In the proposed method, we first estimate the parameters of the marginal models. Next, we infer the parameters of the dependence structure and RS using the results from the first step. We apply the Hamilton filter proposed by [20]. To further investigate the asymmetry and evaluate the model fit, we calculate the exceedance correlation, Value at Risk, and expected shortfall.

We choose three pairs of countries to examine empirical evidence of asymmetry: the United Kingdom and United States (UK-US), Japan-US, and Italy-US. The data are from 2003 to 2013, which covers the credit crunch period and the Greek sovereign crisis. By analyzing the correlation coefficients, we find that the three country pairs have different levels of correlation. The UK and US are strongly correlated, but there is only a small correlation between Japan and the US. In addition, the data for Italy and the US have a particular behavior caused by the Greek sovereign crisis. Our empirical analysis finds that highly dependent regimes are different according to the asset pairs. This is implicitly indicated by [4], but in this paper we explicitly compare all asset pairs. Our findings imply that we should consider the dependence of each pair when constructing international diversified portfolios, to properly estimate the benefits of diversification. If we incorporate asset pairs that are less dependent in asymmetric regimes into a portfolio, we may avoid the risk that all assets in the portfolio have extreme negative returns in the asymmetric regime. Our empirical findings clearly indicate that we should evaluate the dependence structure and asymmetry among all asset and country pairs to correctly capture the benefits of international diversification.

The paper is organized as follows. We introduce the concept of copulas in Section 2, reviewing their definition and features, and the skew t and vine copulas. Section 3 explains the RS copula model. It includes a definition of the model and the marginal models, and introduces the estimation method. Section 4 presents the empirical analysis. First, we explain the data and descriptive statistics. Next, we show the results of the marginal models, and then we present the results of the dependence structure of each country pair. In Section 5, we discuss more implications of the dependence structures using the exceedance correlation, Value at Risk, and expected shortfall for each model. Section 6 concludes the paper.

II. Copulas

In the following section, we introduce the concept of copulas. First, we define them and describe their features. Then, we discuss some specific multivariate copulas: the skew t and vine copulas.

A. Definition and features of copulas

Suppose d is a natural number. A copula is a multivariate joint distribution function, C, with uniform marginal distributions on $[0, 1]$. Hence C is a mapping of the form $C : [0, 1]^d \to [0, 1]$. A copula C has the following properties, and conversely a function with the following properties is a copula.

(1) $C(u_1, \ldots, u_d)$ is increasing in each component u_i.

(2) $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$ for all $i \in \{1, \ldots, d\}, u_i \in [0, 1]$.

(3) For all $(a_1, \ldots, a_d), (b_1, \ldots, b_d) \in [0, 1]^d$ with $a_i \leq b_i$ we have

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\cdots+i_d} C(u_{1i_1}, \ldots, u_{di_d}) \geq 0,$$

where $u_{j1} = a_j$ and $u_{j2} = b_j$ for all $j \in \{1, \ldots, d\}$.

The following theorem of [21] states that copulas are dependence structures of multivariate joint distributions.

Theorem 1: Let F be a joint distribution function with marginal distributions $F_1, \ldots, F_d$. Then there exists a copula $C : [0,1]^d \to [0,1]$ such that, for all $x_1, \ldots, x_d \in \mathbb{R}$,

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)). \tag{1}$$

If the marginal distributions are continuous, then C is unique. Conversely, if C is a copula and $F_1, \ldots, F_d$ are univariate distribution functions, then the function F defined in Equation (1) is a joint distribution function with marginal distributions $F_1, \ldots, F_d$.

Sklar's theorem shows that every joint distribution function can be decomposed into its marginal functions and its copula. We are able to construct various joint distribution functions using copulas with different structures, even though the marginal distributions are the same.

One of the features of a copula is the coefficient of tail dependence. It measures the dependence strength in the tails of a bivariate distribution. Assuming that X_1, X_2 are random variables with distribution functions F_1, F_2, the coefficient of the upper tail dependence of X_1 and X_2 is given by

$$\lambda_u = \lim_{q \to 1-0} P(X_2 > F_2^{\leftarrow}(q) | X_1 > F_1^{\leftarrow}(q)),$$

where $F_i^{\leftarrow}$, $i = 1, 2$, are generalized inverse functions given by $F_i^{\leftarrow}(y) = \inf\{x \in \mathbb{R} | F_i(x) \geq y\}$, provided that a limit $\lambda_u \in [0,1]$ exists. If $\lambda_u = 0$, then X_1 and X_2 are said to be asymptotically independent in the upper tail. If $\lambda_u \in (0,1]$, X_1 and X_2 show upper tail dependence in the upper tail. The coefficient of upper tail dependence can be interpreted as the probability that one variable becomes large when the other becomes large. Analogously, the coefficient of lower tail dependence is

$$\lambda_d = \lim_{q \to 0+} P(X_2 \leq F_2^{\leftarrow}(q) | X_1 \leq F_1^{\leftarrow}(q)),$$

provided that a limit $\lambda_d \in [0,1]$ exists. The coefficient of lower tail dependence can be interpreted as the probability that one variable becomes small when the other becomes small.

B. Skew t copula

We now focus on flexible multivariate copulas, which will be used to describe the dependence structure in an asymmetric regime. Such copulas should hold the following two properties. First, the strength of the dependence of each pair is described separately. Second, the tail dependence can be asymmetric, which means that the coefficients of the upper and lower tails may be different.

It is a challenge to construct copulas with these two features, because basic multivariate copulas cannot handle them both. For instance, famous multivariate elliptical copulas such as the Gaussian or t copula can express different dependences for each pair using correlation coefficient matrices, but their tail dependencies are symmetric. Another example is a copula belonging to the multivariate Archimedean copula family such as the Clayton or Gumbel copula. Its tail dependence is asymmetric, but the dependence structure is dominated by only one parameter.

A skew t copula has the above properties. It was first introduced by [13], who used the skew t distribution in the expression of a special case of a generalized hyperbolic distribution. Because the skew t distribution can be described in many ways (see [22], for example), [14] suggested the skew t copula derived from a simple form of the skew t distribution in [18]. In the following, we use the expression given in [14].

First, we explain a skew normal distribution. Let $\mathbf{Y} = (Y_1, \ldots, Y_d)^{\mathrm{T}}$ denote a d-dimensional random vector. It has mean vector μ, and covariance matrix Σ with components σ_{ij}, $i, j = 1, \ldots, d$. If $\mathbf{Y}$ follows the skew normal distribution, its density function is given by

$$g_d(\mathbf{y}; \mu, \Sigma, \alpha) = 2\phi(\mathbf{y} - \mu; \Sigma)\Phi(\alpha^{\mathrm{T}}\mathbf{W}^{-1}(\mathbf{y} - \mu)), \tag{2}$$

where $\phi(\mathbf{y} - \mu; \Sigma)$ is the density function of the normal distribution $\mathrm{N}_d(\mu, \Sigma)$, $\Phi(\cdot)$ is the distribution function of $\mathrm{N}(0,1)$, $\alpha = (\alpha_1, \ldots, \alpha_d)^{\mathrm{T}}$ is a d-dimensional vector called the shape parameter vector, and $\mathbf{W} = (\delta_{ij}\sqrt{\sigma_{ij}})$, $i, j = 1, \ldots, d$ (δ_{ij} is the Kronecker delta). The notation $\mathbf{Y} \sim \mathrm{SN}_d(\mu, \Sigma, \alpha)$ is used for $\mathbf{Y}$ with the density equation (2).

Next, we describe the skew t distribution. A d-dimensional random vector $\mathbf{X} = (X_1, \ldots, X_d)^{\mathrm{T}}$ that follows the skew t distribution with parameters μ, Σ, α, and the number of degrees of freedom, ν, can be represented as

$$\mathbf{X} = \mu + V^{-1/2}\mathbf{Y},$$

where $\mathbf{Y} \sim \mathrm{SN}_d(\mathbf{0}, \Sigma, \alpha)$ and $\mu V \sim \chi^2_\nu$ independent of $\mathbf{Y}$. The joint density of the skew t distribution with parameters μ, Σ, α, and ν is given by

$$f_{d,\nu}(\mathbf{x}; \mu, \Sigma, \alpha) =$$
$$2 \cdot t_{d,\nu}(\mathbf{x}; \mu, \Sigma) T_{1,\nu+d}\Big\{\alpha^{\mathrm{T}} W^{-1}(\mathbf{x} - \mu)\Big(\frac{\nu + d}{Q + \nu}\Big)^{1/2}\Big\},$$

where $Q = (\mathbf{x} - \mu)^{\mathrm{T}} W^{-1}(\mathbf{x} - \mu)$, $t_{d,\nu}(\cdot; \mu, \Sigma)$ is the joint density function of the d-dimensional t distribution with parameters Σ and ν, and $T_{1,\nu+d}(.)$ is the distribution function of the univariate t-distribution with parameters $\nu + d$.

The skew t copula can be constructed from the skew t distribution. Suppose $\mathbf{u} = (u_1, \ldots, u_d)^{\mathrm{T}}$, where $u_i \in [0,1]$ for $i = 1, \ldots, d$. For the implicit copula of an absolutely continuous joint distribution function F with strictly increasing, continuous marginal distribution functions $F_1, \ldots, F_d$, we may differentiate $C(\mathbf{u}) = F(F_1^{\leftarrow}(u_1), \ldots, F_d^{\leftarrow}(u_d))$ to see that the copula density, c, is given by

$$c(\mathbf{u}) = \frac{f(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))}{f_1(F_1^{-1}(u_1)) \ldots f_d(F_d^{-1}(u_d))},$$

where f is the joint density of F, $f_1, \ldots, f_d$ are the marginal densities, and $F_1^{-1}, \ldots, F_d^{-1}$ are the inverses of the marginal distribution functions. The density of the skew t copula, c_{ST}, is

$$c_{\mathrm{ST}}(\mathbf{u}; R, \alpha, \nu) =$$
$$\frac{f_{d,\nu}(F_{1,\nu}^{-1}(u_1; 0, \rho_{11}, \alpha_1), \ldots, F_{1,\nu}^{-1}(u_d; 0, \rho_{dd}, \alpha_d); \mathbf{0}, R, \alpha)}{f_{1,\nu}(F_{1,\nu}^{-1}(u_1; 0, 1, \alpha_1); 0, 1, \alpha_1) \ldots f_{1,\nu}(F_{1,\nu}^{-1}(u_d; 0, 1, \alpha_d); 0, 1, \alpha_d)},$$

where R is a correlation matrix with coefficients ρ_{ij}, $i, j = 1, \ldots, d$, and $F_{1,\nu}^{-1}(\cdot; \mu, \sigma, \alpha)$ is the inverse of the univariate distribution function of the skew t distribution. In the same way, the density of the skew normal copula, $c_{\rm SN}$, is

$$c_{\rm SN}(\mathbf{u}; R, \alpha, \nu) = \frac{g_d(G_1^{-1}(u_1; 0, \rho_{11}, \alpha_1), \ldots, G_1^{-1}(u_d; 0, \rho_{dd}, \alpha_d); \mathbf{0}, R, \alpha)}{g_1(G_1^{-1}(u_1; 0, 1, \alpha_1); 0, 1, \alpha_1) \ldots g_1(G_1^{-1}(u_d; 0, 1, \alpha_d); 0, 1, \alpha_d)},$$

where $G_1^{-1}(\cdot; \mu, \sigma, \alpha)$ is the inverse of the univariate distribution function of the skew normal distribution.

The skew t copula represents the dependence structure of each pair using its correlation matrix. Furthermore, the strength of the upper and lower tail dependencies can be made different by introducing the shape parameter α. Fig. 1 illustrates 2,000 simulated points from the four-dimensional skew t copula. Its parameters are:

$$\begin{aligned} R &= \begin{bmatrix} 1.0 & 0.3 & 0.5 & 0.7 \\ 0.3 & 1.0 & 0.4 & 0.6 \\ 0.5 & 0.4 & 1.0 & 0.9 \\ 0.7 & 0.6 & 0.9 & 1.0 \end{bmatrix}, \\ \alpha &= [-0.5, -0.8, 0.2, 0.3]^{\rm T}, \\ \nu &= 5. \end{aligned}$$

The two properties stated above are confirmed in Fig. 1.

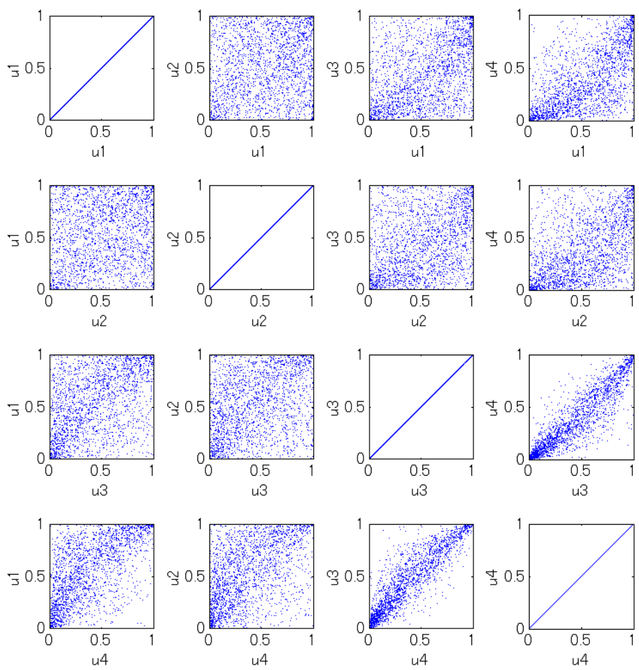

Fig. 1. Simulated points from the four-dimensional skew t copula

C. Vine copula

Another copula that has the desired properties of flexible copulas is a vine (or pair) copula. [15] proposed vine copulas for statistics. [16] introduced vine copulas into finance, and is their expressions that we use here. The vine copula is constructed from marginal densities, bivariate copulas, and their combinations. This construction is based on the idea that a joint density function of d variables $x_1, \ldots, x_d$ can be decomposed to

$$f(x_1, \ldots, f_d) = f(x_1) \cdot f(x_2|x_1) \cdot f(x_3|x_1, x_2) \ldots f(x_d|x_1, \ldots, x_{d-1}). \quad (3)$$

Each factor in Equation (3) can be expressed using bivariate conditional copulas. The first conditional density can be decomposed into

$$f(x_2|x_1) = c_{12}(F_1(x_1), F_2(x_2)),$$

where c_{12} is the copula density, and $F_i(\cdot)$ is the distribution function of x_i. The second conditional density in Equation (3) can be rewritten in the same way. One possible decomposition is

$$f(x_3|x_1, x_2) = c_{23|1}(F_{2|1}(x_2|x_1), F_{3|1}(x_3|x_1))f(x_3|x_1),$$

where $c_{23|1}$ is the conditional copula density of x_2 and x_3, conditioning to x_1, and $F_{i|j}$ is the conditional marginal distribution of x_i, conditioning to x_j. This can be further decomposed into

$$\begin{aligned} &f(x_3|x_1, x_2) = \\ &c_{23|1}(F_{2|1}(x_2|x_1), F_{3|1}(x_3|x_1))c_{13}(F_1(x_1), F_3(x_3))f_3(x_3). \end{aligned}$$

Combining the decomposed expressions, the joint density of the first three variables in Equation (3) can be written

$$\begin{aligned} &f(x_1, x_2, x_3) = \\ &c_{23|1}(F_{2|1}(x_2|x_1), F_{3|1}(x_3|x_1))c_{12}(F_1(x_1), F_2(x_2)) \\ &\times c_{13}(F_1(x_1), F_3(x_3))f_1(x_1)f_2(x_2)f_3(x_3). \end{aligned}$$

The copula density is given by

$$\begin{aligned} &c(x_1, x_2, x_3) = \\ &c_{23|1}(F_{2|1}(x_2|x_1), F_{3|1}(x_3|x_1))c_{13}(F_1(x_1), F_3(x_3)). \end{aligned}$$

According to [23], conditional distribution functions are calculated using

$$F(x|\mathbf{v}) = \frac{\partial C_{x,v_j|\mathbf{v}_{-j}}(F(x|\mathbf{v}_{-j}), F(v_j|\mathbf{v}_{-j}))}{\partial F(v_j|\mathbf{v}_{-j})},$$

where $\mathbf{v}_{-j}$ denotes the vector $\mathbf{v}$ excluding the j-th component.

There are a few possible expressions for decomposing and ordering the data from high dimensional distributions. [15] introduced a graphical model, called the regular vine. In this paper, two special vines are introduced: a canonical vine (a C-vine, for short) and a D-vine. The vines represent the specific way the density is decomposed. Fig. 2 illustrates the structure of a four-dimensional C-vine copula. It consists of three trees, T_j, $j = 1, \ldots, 3$. In the first tree T_1, the dependence is modeled using the bivariate copulas of x_1 with all other variables. In the second tree T_2, all the bivariate copulas conditioned on x_1 represent the dependencies between x_2 and the other variables. Iterating leads to the d-dimensional C-vine copula given by

$$\begin{aligned} &c(x_1, \ldots, x_d) = \\ &\prod_{j=1}^{d-1}\prod_{i=1}^{n-j} c_{j,j+i|1,\ldots,j-1} \\ &\times(F(x_j|x_1, \ldots, x_{j-1}), F(x_j|x_{j+i}, \ldots, x_{j-1})). \end{aligned}$$

A D-vine provides us with a different copula from that of a C-vine. Fig. 3 shows the structure of a four-dimensional D-vine copula. In the D-vine, no node in any tree is connected

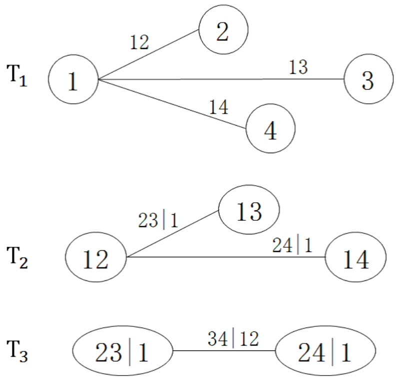

Fig. 2. Dependence structure of a four-dimensional C-vine

to more than two edges. The d-dimensional D-vine copula is

$$c(x_1,\ldots,x_d)=\prod_{j=1}^{d-1}\prod_{i=1}^{n-j}c_{j,i+j|i+1,\ldots,i+j-1}\times(F(x_i|x_{i+1},\ldots,x_{i+j-1}),F(x_{i+j}|x_{i+1},\ldots,x_{i+j-1})).$$

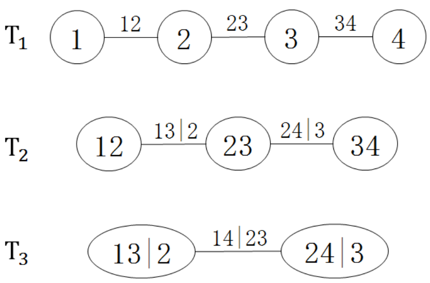

Fig. 3. Dependence structure of a four-dimensional D-vine

Note that the bivariate copulas can be constructed from different types of copulas. Following [16] and [12], we examine five copulas as building blocks for the vine copula: the Gaussian, the t, the Clayton, the Gumbel, and the rotated Gumbel. For an overview of other bivariate copulas, see [24]. The first two are elliptical copulas, which represent the dependence structure of some elliptical distributions (the normal and t distributions). The Gaussian copula has no tail dependence, while the t copula has both upper and lower tail dependence. The rest are called Archimedean copulas, and their distribution functions are explicitly given. The Gumbel copula has the upper tail dependence, while the Clayton and rotated Gumbel have lower tail dependencies. The vine copula can express tail dependencies by combining these copulas.

Fig. 4 shows 2,000 simulated points from the four-dimensional C-vine copula. The building blocks in the first tree are all rotated Gumbel copulas with parameters 1.5, 2.0, and 1.8, and the rest are bivariate Gaussian copulas with parameters 0.3, -0.2, and 0.3. The marginal distributions are assumed to be uniform on $[0,1]$, which leads to $u_i=x_i$, for $i\in[1,4]$. From Fig. 4 it can be seen that the lower tail dependence exists, and the strength of the dependence for each pair differs.

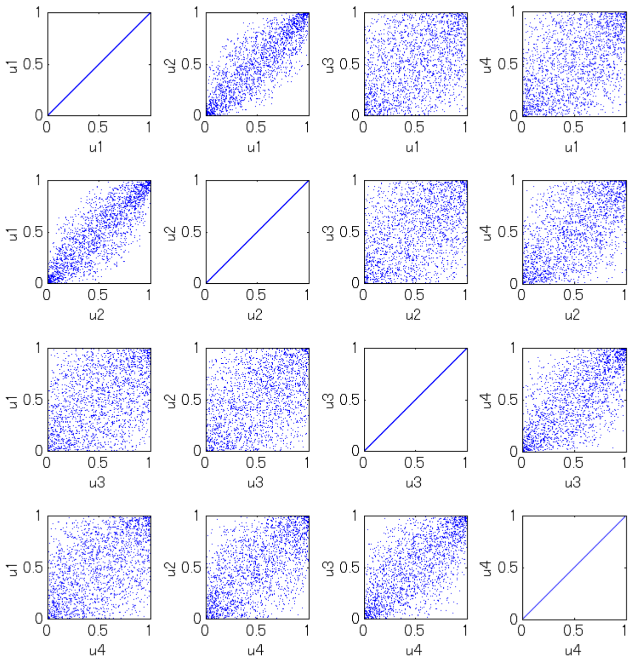

Fig. 4. Simulated points from the four-dimensional C-vine copula

III. Model

This section explains the model that describes the asymmetric dependence structure in equity and bond markets. We first introduce an RS copula model, then discuss a skew t GARCH model of [19] that we use as a marginal model. We also describe the estimation method.

A. Regime-switching copula model

An RS copula can be used to model the dependencies in international market correlations. We follow [12] and [4] whose models have two regimes: symmetric and asymmetric. It is assumed that the d-variate process, X_t, depends on a latent binary variable, which indicates the economy's current regime. In this model, the regime only affects the dependence structure. The density of X_t, conditional on being in regime j, is

$$f(X_t|X_{t-1},s_t=j)=c^{(j)}\left(F_1(x_{1,t}),\ldots,F_d(x_{d,t});\theta_c^{(j)}\right)\prod_{i=1}^{d}f_i(x_{i,t};\theta_{m,i}),$$

where $X_t=(x_{1,t},\ldots,x_{d,t})$, s_t is the latent variable for the regime, $c^{(j)}(\cdot)$ is the copula density in regime j (with parameter $\theta_c^{(j)}$), $f_i(\cdot)$ is the density of the marginal distribution of x_i (with parameter $\theta_{m,i}$), and F_i is the corresponding distribution function. It is assumed that the unobserved latent state variable follows a Markov chain with transition probability

$$P=\begin{pmatrix}p_{11} & 1-p_{11}\\ 1-p_{22} & p_{22}\end{pmatrix},$$

where p_{ij} represents the probability of moving from state i at time t to state j at time $t+1$.

B. Marginal model

The marginal distributions of each of the returns are modeled using the univariate skew t GARCH (1,1) model of [19] to consider the dynamics of the volatility. The skew t distribution introduced here is different from [18]. This system is expressed as

$$x_{i,t} = \sigma_{i,t} \cdot \epsilon_{i,t}, \quad \text{for } i = 1, \ldots, d, \tag{4}$$
$$\sigma_{i,t}^2 = \omega_i + \alpha_i x_{i,t-1}^2 + \beta_i \sigma_{i,t-1}^2, \tag{5}$$
$$\epsilon_{i,t} \sim \text{skew } t(\nu_i, \lambda_i), \quad \lambda_i \in (-1, 1), \tag{6}$$

where ν is the number of degrees of freedom, λ is the skewness parameter, and the skew t density is given by

$$h_{1,\nu}(z;\lambda) = \begin{cases} bc\Big(1 + \frac{1}{\nu-2}\big(\frac{bz+a}{1-\lambda}\big)^2\Big)^{-(\nu+1)/2}, & z < -a/b, \\ bc\Big(1 + \frac{1}{\nu-2}\big(\frac{bz+a}{1+\lambda}\big)^2\Big)^{-(\nu+1)/2}, & z \le -a/b. \end{cases}$$

The contents a, b, and c are defined as

$$a = 4\lambda c\Big(\frac{\nu-2}{\nu-1}\Big), \quad b^2 = 1 + 3\lambda^2 - a^2, \quad c = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi(\nu-2)}\Gamma(\frac{\nu}{2})}.$$

$\theta_{m,i} = (\omega_i, \alpha_i, \beta_i, \nu_i, \lambda_i)^{\mathrm{T}}$ denote all the parameters of a given country.

C. Estimation

The estimation method can be separated into two parts because of the assumption that the marginal models are independent from regimes. Denote the sample of observed data by $\mathbf{X} = (X_1^{\mathrm{T}}, \ldots, X_T^{\mathrm{T}})^{\mathrm{T}}$. The log likelihood function is given by

$$L(\mathbf{X}; \theta_m, \theta_c) = \sum_{t=1}^{T} \log f(X_t | \mathbf{X}^{t-1}; \theta_m, \theta_c),$$

where $\mathbf{X}^{t-1} = (X_1, \ldots, X_{t-1})$ denotes the history of the full process, θ_m denotes the parameters of the marginal, and θ_c denotes the parameters of the RS copula. This likelihood can be decomposed into L_m, which contains the marginal densities, and L_c, which contains the dependence structure:

$$L(\mathbf{X}; \theta_m, \theta_c) = L_m(\mathbf{X}; \theta_m) + L_c(\mathbf{X}; \theta_m, \theta_c), \tag{7}$$

$$L_m(\mathbf{X}; \theta_m) = \sum_{t=1}^{T} \sum_{i=1}^{d} \log f_i(x_{i,t} | x_i^{t-1}; \theta_{m,i}), \tag{8}$$

$$L_c(\mathbf{X}; \theta_m, \theta_c) = \sum_{t=1}^{T} \log c\big(F_1(x_{i,t}|x_1^{t-1}; \theta_{m,1}), \ldots, F_d(x_{d,t}|x_d^{t-1}; \theta_{m,d}); \theta_c\big), \tag{9}$$

where $x_i^{t-1} = (x_{i,1}, \ldots, x_{i,t-1})$ denotes the history of the variable i. The proof can be found in, for example, [4]. The log likelihood of the marginal models L_m is a function of the parameter vector $\theta_m = (\theta_{m,1}, \ldots, \theta_{m,d})$, which contains the parameters of each marginal density f_i. The copula log likelihood depends directly on the vector $\theta_c = (\theta_c^{(1)}, \theta_c^{(2)}, p_{11}, p_{22}, \mathbf{p}_0)$. This vector contains the copula parameters over both regimes, the parameters of the Markov transition probability matrix, and the two-dimensional vector of its initial probabilities, $\mathbf{p}_0$. The function c denotes the density of the RS copula.

The decomposition of the log likelihood function in Equation (7) allows us to use a two-step estimation procedure. The RS copula model contains a large number of parameters, but this method simplifies the estimation. In the first step, we assume that the different series are uncorrelated conditioned on the history. The parameters of the marginal densities are estimated by maximizing Equation (8). This is straightforward, and we estimate each GARCH model separately. In the second step, we calibrate the dependence structure and Markov chain parameters, given the results of the first step. We calculate the parameters by maximizing Equation (9), conditioning on θ_m.

In the second step, we use the EM algorithm of [20]. This is a useful estimation method for an unobservable state variable in the Markov chain. Let

$$\eta_t = \begin{pmatrix} c^{(1)}(F_1(x_{1,t}|x_1^{t-1}), \ldots, F_n(x_{n,t}|x_n^{t-1}); \theta_c^{(1)}) \\ c^{(2)}(F_1(x_{1,t}|x_1^{t-1}), \ldots, F_n(x_{n,t}|x_n^{t-1}); \theta_c^{(2)}) \end{pmatrix},$$

be the two-dimensional vector that contains the copula densities at time t, conditioned on the state variable s_t and the history up to time t. Moreover, let

$$\hat{\xi}_{t|\tau} = \begin{pmatrix} \Pr(s_t = 1 | \mathbf{X}^\tau; \theta_m, \theta_c) \\ \Pr(s_t = 0 | \mathbf{X}^\tau; \theta_m, \theta_c) \end{pmatrix},$$

be the two-dimensional vector containing the conditional probabilities of being in each regime at time t, conditional on observations up to time τ. The log likelihood function can be expressed as

$$L_c(\mathbf{X}; \theta_m, \theta_c) = \sum_{t=1}^{T} \log(\mathbf{1}^{\mathrm{T}}(\hat{\xi}_{t|t-1} \odot \eta_t)),$$

where $\odot$ denotes the Hadamard product (element-by-element multiplication). To evaluate the log likelihood function, we need $\hat{\xi}_{t|t-1}$ for $t = 1, \ldots, T-1$. We are able to calculate these using

$$\hat{\xi}_{t|t} = \frac{\hat{\xi}_{t|t-1} \odot \eta_t}{\mathbf{1}^{\mathrm{T}}(\hat{\xi}_{t|t-1} \odot \eta_t)}, \tag{10}$$
$$\hat{\xi}_{t+1|t} = P^{\mathrm{T}} \cdot \hat{\xi}_{t|t}, \tag{11}$$

where $\mathbf{1}$ is a two-dimensional vector of 1s. We can evaluate the log likelihood by iterating over Equations (10) and (11), from a starting value $\hat{\xi}_{1|0}$, θ_c, and the transition probabilities of the Markov chain.

IV. Empirical analysis

In this section, we present the results of our empirical analysis. First, we discuss the data and their descriptive statistics. Next, we show the estimation results of the marginal distributions. Finally, we explain the dependence structures of three country pairs.

A. Data

In this analysis, we focus on three country pairs: UK-US, Japan-US, and Italy-US. We apply the RS copula model to the weekly returns from investing equities and bonds. The equity returns are calculated from the stock index of each country: the S&P 500 in US, the FTSE 100 in UK, the Nikkei 225 in Japan, and the FTSE MIB in Italy. All indices are expressed in Japanese Yen. The bond returns are computed from the yields of 10-year government bonds. All data are downloaded from Bloomberg for the period from the $8^{\rm th}$ of January 2003, to the $10^{\rm th}$ July 2013, which corresponds to 548 observations.

Table I shows the descriptive statistics. In Table I "eq" refers to an equity and "bn" refers to a bond. "JP" and "IT" are abbreviations for Japan and Italy. All series show clear evidence of non-normality with a kurtosis above 3. The non-zero skewness gives us further signs of non-normality. The skewness of all equity return series are negative, while the bond return series have a positive skewness.

The unconditional correlations are presented in Table II. We make the following five observations. The table indicates that correlations in equity markets are larger than that in bond markets. The dependencies among equity and bond markets are relatively low, even within a country. All the correlations are positive, except in the Italian market. The UK-US pair is strongly correlated, while the correlation between Japan and US is relatively low. The correlation of the equity in the Italy-US pair is high, while that of the bond is low.

TABLE I
SUMMARY STATISTICS

	Mean	Std	Skewness	Kurtosis	Max	Min
eqUS	0.14	2.36	-0.76	8.85	10.12	-15.17
eqUK	0.12	2.43	-0.35	7.34	14.55	-11.95
eqJP	0.15	3.16	-0.43	7.02	15.94	-19.04
eqIT	-0.03	3.24	-0.18	5.59	12.37	-13.70
bnUS	0.01	4.34	0.18	5.37	17.14	-19.62
bnUK	-0.05	3.43	0.19	5.29	15.24	-15.12
bnJP	0.15	5.76	3.26	27.17	54.09	-13.14
bnIT	0.06	3.46	0.10	7.46	18.46	-17.61

Descriptive statistics of the weekly equity index and bond returns for all four countries. All the returns are expressed in Japanese Yen. The data period is from the $8^{\rm th}$ January 2013 to the $10^{\rm th}$ of July 2013, which corresponds to 548 observations.

TABLE II
UNCONDITIONAL CORRELATION

	eqUS	eqUK	eqJP	eqIT	bnUS	bnUK	bnJP	bnIT
eqUS	1.00							
eqUK	0.81	1.00						
eqJP	0.58	0.61	1.00					
eqIT	0.74	0.81	0.60	1.00				
bnUS	0.34	0.37	0.32	0.42	1.00			
bnUK	0.32	0.33	0.35	0.43	0.74	1.00		
bnJP	0.17	0.19	0.39	0.21	0.34	0.39	1.00	
bnIT	0.04	0.08	0.06	-0.12	0.23	0.21	0.16	1.00

The unconditional correlations between the equities and bonds for the US, the UK, Japan (JP), and Italy (IT).

B. Marginal distributions

The estimates of the parameters of the marginal distributions are shown in Table III. The parameters correspond to those in Equations (4), (5), and (6). Table III implies the following. The negative skewness parameter λ in the equity returns, and the positive λ in the bond returns are consistent with the skewness in Table I. The equity markets are more skewed than the bond markets (comparing the absolute values of the skewness parameters), except in Japan. The degrees-of-freedom are less than 10, except for the bond markets of the US and UK. Therefore, it is reasonable to assume that the distributions of the series that have more than 10 degrees of freedom follow the Gaussian laws. The series of the bonds for the US and UK have Gaussian-like distributions.

It is important to determine if the marginal models are well specified, because misspecification in the marginal models leads to biased copula parameter estimates. Therefore, we have performed two kinds of tests. One is the goodness of fit test for the probability integral transformation (PIT) of the marginal models, and includes the Kolomogorov-Smirnov (KS) and Anderson-Darling (AD) tests. If the marginal models are well specified, the PIT samples must follow the uniform distribution on $[0, 1]$. The KS test evaluates the null hypothesis that the PIT samples of cumulative distribution function is equal to the uniform distribution on $[0, 1]$. The AD test is also used to test whether a PIT sample comes from the uniform distribution on $[0, 1]$. The other test is the Ljung-Box test for the residuals of the skew t GARCH models. It evaluates the autocorrelation of the residuals for a fixed number of lags. The residuals should have no autocorrelation for any lags, because of the i.i.d. assumption of the residuals of GARCH models.

Table IV summarizes these results. Panel A contains the statistics and p-values of the uniformity tests for the PIT samples. In both the KS and the AD test, the null hypotheses of all the series cannot be rejected at the 5% level. Panel B contains the statistics of the Ljung-Box test at lags 1, 2, 3, 4, 6, and 12. In all the series, except the bond series of Italy, the null hypotheses of independence cannot be rejected at the 5% level. In Italy's bond series, independence cannot be rejected at the 1% level. If we assume the GARCH model with Gaussian or t innovations, not all of the series pass the tests (see Appendix A). We conclude that each skew t GARCH(1,1) model is specified better than the GARCH model with Gaussian or t innovations.

TABLE III
ESTIMATES OF SKEW t GARCH(1,1) PARAMETERS

	eqUS	eqUK	eqJP	eqIT
ω	0.19	0.32	0.31	0.16
	(0.09)	(0.13)	(0.19)	(0.09)
α	0.15	0.22	0.08	0.14
	(0.04)	(0.06)	(0.03)	(0.04)
β	0.82	0.74	0.89	0.85
	(0.05)	(0.06)	(0.04)	(0.04)
ν	6.89	8.38	8.04	7.87
	(1.93)	(2.95)	(2.35)	(2.66)
λ	-0.31	-0.34	-0.18	-0.25
	(0.05)	(0.06)	(0.06)	(0.06)
logL	-1144.39	-1168.57	-1361.81	-1319.16
	bnUS	bnUK	bnJP	bnIT
ω	0.11	0.07	1.92	0.14
	(0.09)	(0.05)	(0.91)	(0.09)
α	0.10	0.09	0.07	0.07
	(0.03)	(0.02)	(0.03)	(0.03)
β	0.90	0.91	0.84	0.92
	(0.02)	(0.02)	(0.06)	(0.03)
ν	13.15	11.67	5.10	6.88
	(6.45)	(5.26)	(1.04)	(1.78)
λ	0.06	0.02	0.19	0.10
	(0.06)	(0.06)	(0.06)	(0.06)
logL	-1496.34	-1350.36	-1598.15	-1380.71

Estimates of the skew t GARCH(1,1) models of [19], for all the equity and bond returns of four countries. The figures between parentheses represent the standard deviations of the parameters. "logL" represents the log likelihood function.

C. Dependence structures

In the following subsection, we show the estimation results of the dependence structures of each country pair. We apply four models that have a Gaussian copula and non-Gaussian

TABLE IV
GOODNESS OF FIT AND LJUNG-BOX STATISTICS

	eqUS	eqUK	eqJP	eqIT	bnUS	bnUK	bnJP	bnIT
Panel A: Uniformity test								
KS								
Stat	0.04	0.04	0.04	0.02	0.02	0.03	0.02	0.03
p	0.48	0.39	0.49	0.89	0.95	0.81	0.96	0.74
AD								
Stat	1.00	0.76	0.83	0.20	0.26	0.35	0.23	0.42
p	0.36	0.51	0.46	0.99	0.97	0.89	0.98	0.83
Panel B: Test for serial independence								
Ljung-Box								
1	3.54**	1.87**	0.05**	0.64**	0.00**	0.80**	0.81**	6.48*
2	4.38**	1.88**	0.06**	1.52**	1.06**	1.81**	0.84**	9.38**
3	5.20**	1.91**	0.95**	4.18**	1.42**	2.29**	3.88**	9.38*
4	5.40**	2.53**	1.29**	4.58**	1.66**	2.37**	4.47**	11.19*
6	5.74**	2.76**	1.32**	5.49**	2.31**	2.79**	9.95**	14.04*
12	10.96**	9.24**	3.08**	11.73**	5.62**	11.51**	19.59**	23.87*

Panel A contains the KS and AD statistics estimates, with their p-values. "Stat" refers to the statistics, and "p" is the p-value. Panel B contains the Ljung-Box statistics computed at lags 1, 2, 3, 4, 6, and 12. The symbols $*$ and $**$ denote that we cannot reject independence at the 1% and 5% levels, respectively.

regime: the Gaussian, the t, the skew t, and the vine. We refer to them, respectively, as M1, M2, M3, and M4. In addition, we denote the Gaussian copula regime by R1, and the other by R2. If the tail dependence of some pair is weak and the t, or skew t copula, is not suitable, we eliminate the t copula model and introduce the skew normal copula as M3' instead of M3.

The analysis in Subsections IV-C1, IV-C2, and IV-C3 can be summarized into the following three findings. First, highly dependent regimes are different according to the asset pairs. Second, the strength of the asymmetry of each country pair varies, and that of the Japan-US pair is weak. Third, the comparison of the skew t and the vine copulas showed that the skew t copula is a better fit to the data, but it is not flexible enough to capture extreme dependencies. The vine copula fits well in spite of having fewer parameters, but cannot express the extreme dependencies of each asset pair.

1) UK-US dependence structure: Table V shows the estimated parameter values for each model of the UK-US pair. The specification of the vine copula follows [16] and [12], which we will explain below. First, the variables are sorted in descending order according to their correlations. Strongly correlated pairs are chosen as the components of the first tree in the vine structure. Next, we select each pair copula in the first tree that has the best AIC for the unconditional estimation of each pair. The pair copulas in the second and the third trees are set so that they maximize the log likelihood. We choose the C-vine or D-vine structure that results in the larger log likelihood. The building blocks are the Gaussian, the t, the Clayton, the Gumbel, and the rotated Gumbel (rGu), as discussed in Subsection II-C.

The results in Table III lead to the following conclusions. First, the correlation coefficients are larger in R2, except for the bond markets. This means that not all asset pairs become more dependent in the asymmetric regime. Note that our results include analyses of the dependence structures between the UK equity and the US bond, and between the UK bond and the US equity, which was neglected in [4]. An important implication of these findings is that we can construct an equities and bonds portfolio for the UK and US with a lower risk than that invested under the assumption that all assets are more dependent in times of crisis. Second, the shape parameters of M3 indicate that the equities have negative skewness, while the bonds have positive. Third, the building blocks for the first tree of the vine copula for M4 are rotated Gumbel copulas. This means that most asset pairs have the lower tail dependence. Fourth, in terms of fit, M3 and M4 are superior to M1 and M2, with respect to the log likelihood. M3 has the highest log likelihood, while M4 is best in terms of AIC. Thus, it is difficult to state which of M3 and M4 is superior. Furthermore, the transition probabilities show high persistence in both regimes for all models. This is consistent with [12] and [4].

The smoothed probabilities of being in R2 are obtained as a by-product of the estimation. They provide a probabilistic assessment of being in R2 at time t, conditional on the information available at the end of the period. The changes of the probabilities of the hidden states are evident from the smoothed probabilities. Fig. 5 shows the smoothed probabilities of being in R2, calculated from each model. The shapes of M1, M2, and M3 are similar. R2 is dominant from the middle of 2006 to the middle of 2007, and from the middle of 2009 to the middle of 2012. These periods correspond to the credit crunch and the Greek sovereign crisis. Thus, R2 can be regarded as the crisis regime. M4 has a similar pattern to the other models, but it has higher estimated probabilities for R2 from the middle of 2012 to the middle of 2013. This is a result of the lower correlations of R2 in M4 than the correlations of R2 of the other models. From Table V the correlation coefficients of R2 in M4 are smaller than those of other models, except that of the bond markets. R2 in M4 corresponded to times of crisis, but the features of the crisis are not emphasized. This can be interpreted as the trade-off of vine copulas. They are so focused on describing the tail dependence, that they cannot model the strength of the dependence itself. Thus, the regime transitions become a little ambiguous.

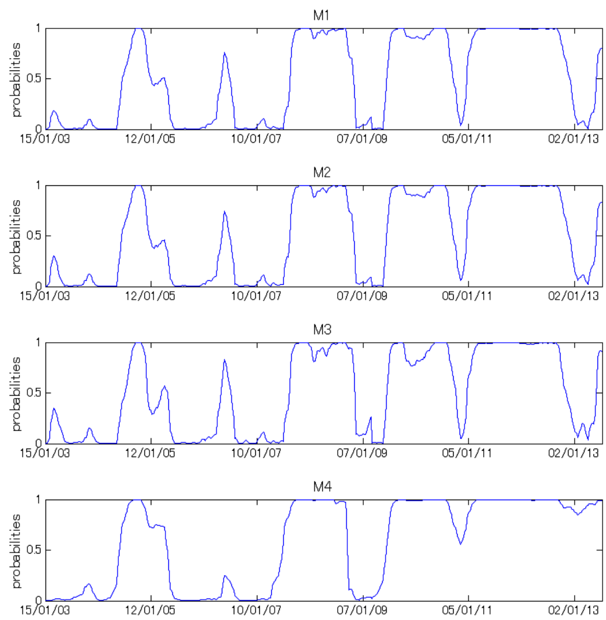

Fig. 5. Smoothed probabilities of R2 for the UK-US pair

2) Japan-US dependence structure: Table VI shows the estimation results for the Japan-US pair. The t and the skew t copulas are not suitable, because the degrees-of-freedom parameters become too large. Thus, we eliminate M2 and instead use M3'. The vine copula in M4 is chosen in the same way as the UK-US pair. One pair copula in the first tree is the rotated Gumbel, and the others are Gaussian. This means that the tail dependence is weaker in the Japan-US pair than in the UK-US pair, which supports the use of the skew normal copulas.

The results in Table VI lead to three findings. First, for M1 and M3', all the correlation coefficients are higher for R2 than R1. However, R1 has a stronger dependence for M4, except for the correlation between the US equity and the Japanese bond. These results are not consistent with each other. This may be caused by the weak asymmetry in the Japan-US pair. The absolute values of the shape parameters are smaller than those in the UK-US pair. Moreover, the building blocks in the vine copula represent the weak tail dependence stated in the previous paragraph. The symmetry-like dependence structure makes it difficult to detect an asymmetric regime. Weak asymmetry is seldom reported in existing research, but it is meaningful because we may decrease the risk of portfolios if we incorporate assets from countries with weak asymmetry. It is important to note that the log likelihood of M3' or M4 is still larger than that of M1 (the symmetric model). Thus, we should use asymmetric copula models, even when analyzing countries with weak asymmetry. Second, M4 is superior to M3' in terms of both log likelihood and AIC. This is because M3' totally neglects asymmetry. Furthermore, the transition probabilities show high persistence in both regimes for all models.

The transition probabilities of being in R2 are shown in Fig. 6. The shapes of the figures of M1 and M3' are similar to each other, but the figure of M4 is almost upside down. In M1 and M3', R2 is dominant around 2008 and from the middle of 2010 to the middle of 2013. In M4, R1 is dominant in the same periods. Thus, the highly dependent regime can be interpreted as the crisis regime. Comparing these results to the UK-US pair, the period from the middle of 2009 to the middle of 2010 is a low dependency regime. This means that the credit crunch has a smaller effect on the Japan-US pair.

3) Italy-US dependence structure: The estimation results for the Italy-US pair are reported in Table VII. The vine copula is specified in the same way as the UK-US pair. The results in Table VII lead to five findings. First, all the building blocks for the first tree are rotated Gumbel copulas, which represents the strong lower tail dependence. The degrees-of-freedom parameters for M2 and M3 are small enough to express the tail dependence. These are evidence of asymmetry in the Italy-US pair. Second, in all models the correlation coefficients of the pairs related to the Italian bond are higher for R1, while those for the rest of the pairs are larger for R2. This indicates that not all asset pairs become more dependent in R2. Our flexible model enables us to analyze the dependencies between the US equity and Italian bond, which was neglected in [4]. Third, some of the asset pairs related to the Italian bond have a negative correlation in R2, which is not the case in the UK-US or Japan-US pair. This phenomenon can be interpreted as the effect of the Greek sovereign crisis. Furthermore, considering the fit, the log likelihoods of M3 and M4 are larger than those of M1 and M2. M3 has the highest log likelihood, while M4 has the lowest AIC. This coincides with the results of the UK-US pair. We again conclude that both M3 and M4 have their merits and demerits. Finally, the transition probabilities show high persistence in both regimes in all models.

Fig. 7 shows the transition probabilities for R2. All models have similar patterns. The probabilities of R2 are sometimes larger from the beginning of 2007 to the middle of 2008, and R2 is dominant after 2009. R2 can be considered as the crisis regime, which corresponds to the UK-US pair. It is notable that after the credit crunch the crisis regime is always dominant when compared with the usual regime. This is due to the Greek sovereign crisis. Although the Italian governmental bond yields were not low from the middle of 2008 to the middle of 2011, the RS model captures the potential risk.

V. Further Investigation for Asymmetry

We make two additional analyses to find more implications of the international dependence structure. We calculate the exceedance correlation to investigate asymmetry in view of existing work. We also compute risk measures, VaR, and expected shortfall (ES), to examine the risk of portfolios investing in international equities and bonds.

A. Exceedance correlation

Exceedance correlation is defined as the correlation calculated from returns above or below a certain threshold. It has been used in existing work such as [1] to measure the asymmetry of dependence structures. The exceedance correlation of variables X and Y at thresholds θ_1 and θ_2 is defined by

$$\mathrm{Excorr}(Y, X; \theta_1, \theta_2) = \begin{cases} \mathrm{corr}(X, Y | X \leq \theta_1, Y \leq \theta_2), & \text{for } \theta_1 \leq 0 \text{ and } \theta_2 \leq 0, \\ \mathrm{corr}(X, Y | X \geq \theta_1, Y \geq \theta_2), & \text{for } \theta_1 \geq 0 \text{ and } \theta_2 \geq 0. \end{cases}$$

We calculate the exceedance correlation using the method of [1]. We use the 100,000 PIT samples generated from

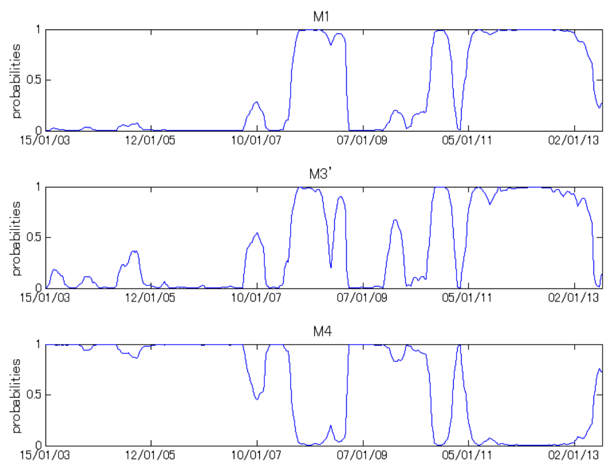

Fig. 6. Smoothed probabilities of R2 for the Japan-US pair

TABLE V
ESTIMATION RESULTS FOR THE UK-US PAIR

	M1		M2		M3			M4				
	Ga	Ga	Ga	t	Ga	skew t	(ρ)		Ga	C-vine	(ρ)	
eqUK-eqUS	0.65	0.83	0.65	0.83	0.64	0.86	0.82	eqUK-bnUK	0.32	1.32	0.34	rGu
	(0.009)	(0.004)	(0.010)	(0.005)	(0.010)	(0.073)			(0.018)	(0.017)		
eqUK-bnUK	0.34	0.40	0.33	0.41	0.34	0.21	0.39	eqUK-bnUS	0.23	1.41	0.35	rGu
	(0.014)	(0.014)	(0.014)	(0.015)	(0.016)	(0.066)			(0.021)	(0.020)		
eqUK-bnUS	0.27	0.47	0.26	0.48	0.27	0.34	0.46	eqUK-eqUS	0.66	2.46	0.69	rGu
	(0.020)	(0.019)	(0.021)	(0.020)	(0.023)	(0.079)			(0.006)	(0.025)		
eqUS-bnUK	0.28	0.42	0.28	0.43	0.29	0.25	0.41	bnUK-bnUS	0.77	0.65	0.71	Ga
	(0.015)	(0.012)	(0.016)	(0.013)	(0.018)	(0.066)			(0.009)	(0.009)		
eqUS-bnUS	0.14	0.58	0.14	0.59	0.15	0.45	0.56	bnUK-eqUS	0.26	0.16	0.36	Ga
	(0.018)	(0.011)	(0.019)	(0.012)	(0.019)	(0.006)			(0.022)	(0.021)		
bnUK-bnUS	0.80	0.69	0.79	0.70	0.80	0.72	0.69	bnUS-eqUS	0.10	1.24	0.38	rGu
	(0.006)	(0.010)	(0.006)	(0.012)	(0.007)	(0.045)			(0.020)	(0.021)		
ν			27.35 (8.441)		25.95 (10.208)							
α					-0.65 (0.046)							
					-0.57 (0.041)							
					0.70 (0.035)							
					0.42 (0.016)							
p_{11}	0.97 (0.003)		0.97 (0.003)		0.97 (0.003)				0.98 (0.003)			
p_{22}	0.97 (0.003)		0.97 (0.003)		0.97 (0.003)				0.99 (0.002)			
logL	524.93		526.02		532.15				528.96			
AIC	-1019.86		-1020.04		-1024.30				-1027.92			

Dependence structure between the UK and US equity and bond markets. Correlation coefficients are shown for the Gaussian, the t, and the skew t copulas. The parameters of the Archimedean copulas their parameters are shown. “Ga” is the Gaussian copula. ν represents the degrees-of-freedom parameter of the t and the skew t copula, and α is the shape parameter of the skew t copula. p_{11} and p_{22} are the transition probabilities of the Markov chain, and denote the probability of staying in the same regime. “logL” refers to the log likelihood, and “AIC” is the Akaike information criteria. “(ρ)” represents the unconditional correlation coefficients calculated from 100,000 samples generated from the skew t or vine copula in R2. Standard deviations of the parameters are shown in parentheses.

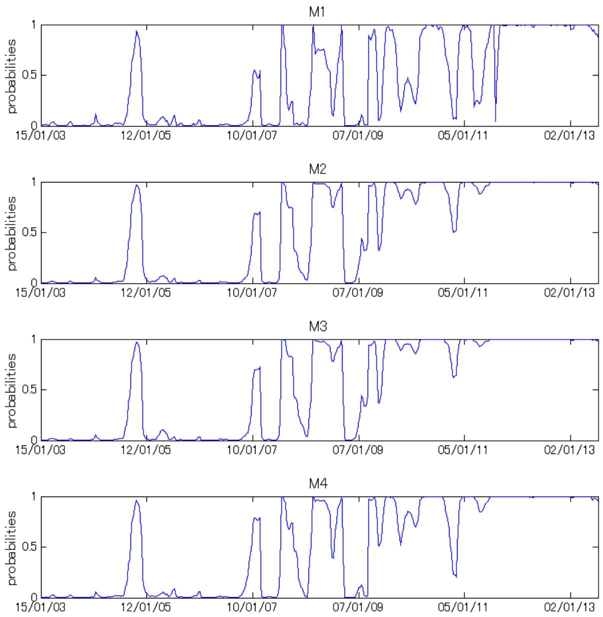

Fig. 7. Smoothed probabilities of R2 for the Italy-US pair

each RS model with the estimated parameters to calculate the exceedance correlation. The thresholds are specified in terms of quantiles, from 10 to 90% in increments of 10%. For thresholds less than the 50% quantile, the correlation is calculated for the left tail, while the right tail is used for thresholds greater than the 50% quantile.

Fig. 8 to 10 illustrate the exceedance correlation of the data and each model in the UK-US, Japan-US, and Italy-US pairs. Each figure shows the pairwise exceedance correlation for thresholds from 10% to 90% in 10% increments. The vertical axes represent the exceedance correlation, and the horizontal axes represent the thresholds. From Fig. 8, 9, and 10, we can see that only the equity markets have clear asymmetric dependence. Other asset pairs do not express significant asymmetry in terms of exceedance correlation. Moreover, the exceedance correlation differs amongst country pairs. For example, we compare three pairs (US equity to bonds from the UK, Japan, and Italy). The shapes of the figures are not similar, even though they are the same kind of asset classes. This indicates the difficulty in expressing asymmetry for different asset classes. Thus, we conclude that we cannot generalize dependence structure features for international asset pairs, except equity pairs, and that flexible models are necessary if we wish to treat each pair differently.

When comparing the power of expression of the different models, M1, M2, and M3 (or M3’) have similar exceedance correlations and fail to describe the asymmetry of the data samples. M3 (or M3’) has slightly more similar patterns to the data than M1 and M2, but not enough to reproduce the asymmetry. M4 succeeds in expressing the asymmetry of the equity markets. The correlation in the left tail tends to be more similar to the data than other models, while the right tail is not as similar. Furthermore, it is not flexible enough to reproduce the various types of asymmetry in each asset pair. These results indicate that both the skew t and vine copula have advantages and shortcomings in terms of the power of expression. The skew t copula is not flexible enough to capture extreme asymmetric dependencies, while the vine copula cannot express the different extreme dependencies of each asset pair.

TABLE VI
ESTIMATION RESULTS FOR THE JAPAN-US PAIR

	M1		M3'				M4			
	Ga	Ga	Ga	skew normal	(ρ)		Ga	C-vine	(ρ)	
eqJP-eqUS	0.49	0.68	0.49	0.68	0.66	eqJP-eqUS	0.68	1.47	0.54	rGu
	(0.013)	(0.013)	(0.015)	(0.060)			(0.013)	(0.020)		
eqJP-bnJP	0.44	0.45	0.44	0.44	0.45	eqJP-bnUS	0.54	0.21	0.45	Gau
	(0.012)	(0.013)	(0.013)	(0.074)			(0.020)	(0.021)		
eqJP-bnUS	0.22	0.54	0.20	0.56	0.56	eqJP-bnJP	0.44	0.45	0.34	Ga
	(0.015)	(0.015)	(0.018)	(0.099)			(0.018)	(0.013)		
eqUS-bnJP	0.19	0.31	0.17	0.31	0.33	eqUS-bnUS	0.66	0.04	0.21	t
	(0.022)	(0.020)	(0.028)	(0.060)			(0.013)	(0.015)		
eqUS-bnUS	0.16	0.66	0.14	0.69	0.68	eqUS-bnJP	0.02	-0.04	0.34	Ga
	(0.019)	(0.020)	(0.027)	(0.013)			(0.015)	(0.018)		
bnJP-bnUS	0.42	0.51	0.41	0.51	0.50	bnUS-bnJP	0.51	0.37	0.44	Ga
	(0.016)	(0.015)	(0.019)	(0.021)			(0.015)	(0.018)		
α			-0.20 (0.027)				ν	15.89 (5.399)		
			-0.30 (0.025)							
			0.27 (0.021)							
			0.12 (0.019)							
p_{11}	0.99 (0.001)		0.98 (0.002)				0.98 (0.002)			
p_{22}	0.98 (0.002)		0.97 (0.003)				0.99 (0.002)			
logL	277.19		280.88				282.51			
AIC	-524.38		-523.76				-533.02			

Dependence structure between the Japanese and US equity and bond markets. Correlation coefficients are shown for the Gaussian, the t, and the skew normal copula. The parameters of the Archimedean copulas are shown. "Ga" is the Gaussian copula. ν represents the degrees-of-freedom parameter of the t copula, and α is the shape parameter of the skew normal copula. p_{11} and p_{22} are the transition probabilities of the Markov chain, and denote the probability of staying in the same regime. "logL" refers to log likelihood, and "AIC" is the Akaike information criteria. (ρ) are the unconditional correlation coefficients calculated from 100,000 samples generated from the skew t or vine copula in R2. Standard deviations of the parameters are shown in parentheses.

TABLE VII
ESTIMATION RESULTS FOR THE ITALY-US PAIR

	M1		M2		M3				M4			
	Ga	Ga	Ga	t	Ga	skew t	(ρ)		Ga	D-vine	(ρ)	
eqIT-eqUS	0.66	0.72	0.63	0.73	0.63	0.74	0.72	eqUS-eqIT	0.64	2.05	0.66	rGu
	(0.006)	(0.009)	(0.009)	(0.007)	(0.009)	(0.049)			(0.009)	(0.026)		
eqIT-bnIT	0.26	-0.41	0.25	-0.29	0.25	-0.28	-0.26	eqIT-bnUS	0.19	1.51	0.34	rGu
	(0.018)	(0.012)	(0.019)	(0.014)	(0.020)	(0.039)			(0.019)	(0.024)		
eqIT-bnUS	0.25	0.51	0.21	0.48	0.20	0.48	0.46	bnUS-bnIT	0.75	1.07	0.39	rGu
	(0.020)	(0.018)	(0.024)	(0.013)	(0.026)	(0.221)			(0.019)	(0.011)		
eqUS-bnIT	0.29	-0.21	0.27	-0.11	0.27	-0.11	-0.10	eqUS-bnUS	0.15	0.36	0.37	Ga
	(0.017)	(0.015)	(0.019)	(0.014)	(0.020)	(0.120)			(0.024)	(0.020)		
eqUS-bnUS	0.22	0.60	0.17	0.56	0.16	0.56	0.54	eqIT-bnIT	0.10	-0.39	-0.05	Ga
	(0.015)	(0.017)	(0.019)	(0.013)	(0.020)	(0.007)			(0.019)	(0.013)		
bnIT-bnUS	0.74	-0.03	0.76	0.09	0.76	0.11	0.10	eqUS-bnIT	0.27	0.06	0.07	Ga
	(0.005)	(0.017)	(0.006)	(0.016)	(0.006)	(0.019)			(0.006)	(0.012)		
ν			9.85 (0.893)		10.12 (0.974)							
α					0.00 (0.024)							
					-0.13 (0.022)							
					0.15 (0.015)							
					0.16 (0.021)							
p_{11}	0.96 (0.003)		0.97 (0.003)		0.97 (0.003)				0.97 (0.003)			
p_{22}	0.94 (0.004)		0.98 (0.002)		0.98 (0.002)				0.97 (0.003)			
logL	352.94		359.02		362.96				361.97			
AIC	-675.88		-686.04		-685.92				-693.94			

Dependence structure between Italy and US equity and bond markets. Correlation coefficients are shown for the Gaussian, the t, and the skew t copulas. The parameters of the Archimedean copulas are shown. "Ga" is the Gaussian copula. ν represents the degrees-of-freedom of the t and skew t copula, and α is the shape parameter of the skew t copula. p_{11} and p_{22} are the transition probabilities of the Markov chain, denoting the probability of staying in the same regime. "logL" refers the log likelihood, and "AIC" is the Akaike information criteria. "(ρ)" represents the unconditional correlation coefficients calculated from 100,000 samples generated from the skew t or vine copula in R2. Standard deviations of the parameters are shown in parentheses.

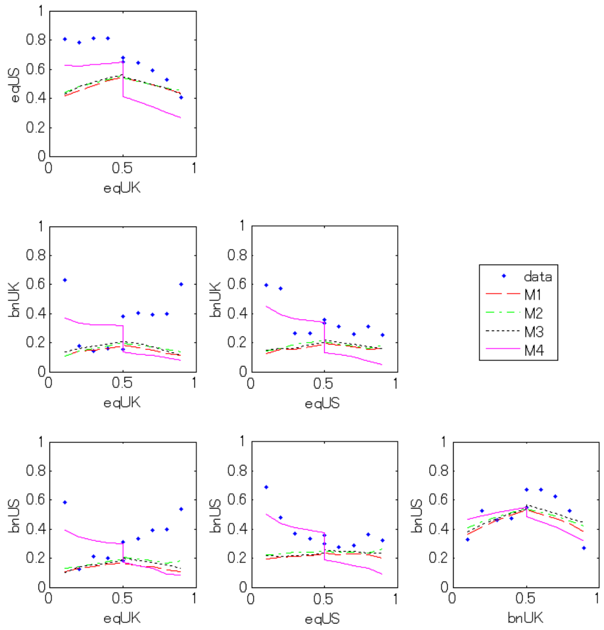

Fig. 8. Exceedance correlation for the UK-US pair

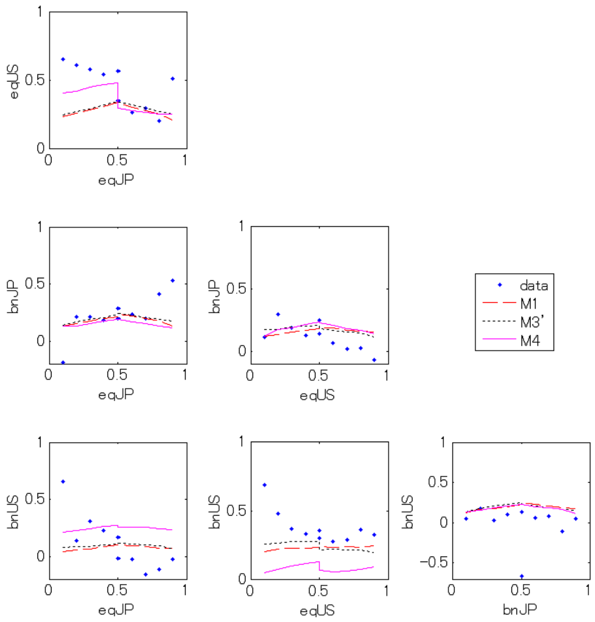

Fig. 9. Exceedance correlation for the Japan-US pair

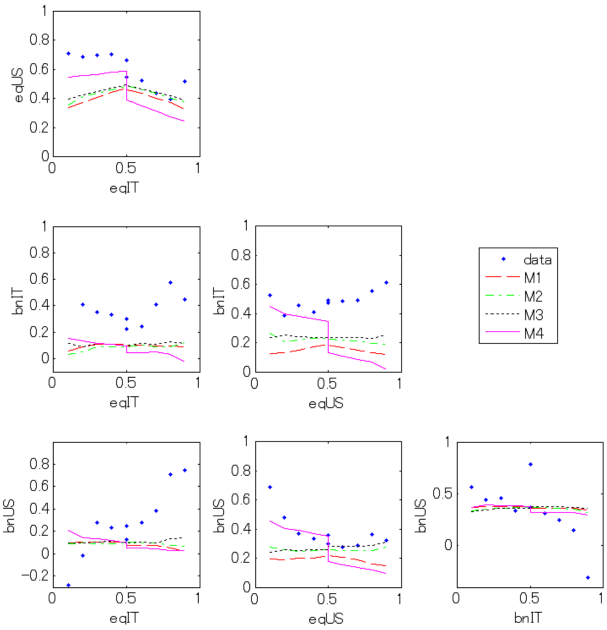

Fig. 10. Exceedance correlation for the Italy-US pair

B. Value at Risk and expected shortfall

VaR and ES are commonly used risk measures for risk management. Let α denote a confidence level, then the VaR at α is defined by

$$\mathrm{VaR}(\alpha) = \inf\{l; \Pr(L > l) \leq 1 - \alpha\},$$

and the ES at α is given by

$$\mathrm{ES}(\alpha) = \mathrm{E}[L | L \geq \mathrm{VaR}(\alpha)],$$

where L is the loss of the portfolio. We can use these measures to evaluate the risk of portfolios, computed from each model. In our analysis, the VaR and the ES are calculated using the Monte Carlo method with 100,000 iterations. We assume an equally weighted portfolio. The confidence levels are set between 90% and 99%, in 1% increments.

Fig. 11 to 13 illustrate the VaR and ES ratios (ratios of the values from each model compared with M1) for the UK-US, Japan-US, and Italy-US pairs. Fig. 11 shows that for the UK-US pair, the risk measures calculated from M2 are similar to those from M1. M3 and M4 have higher values of ES compared with M1 and M2. This coincides with the intuitive understanding that asymmetric models have larger risks than symmetric models. M3 has higher values regardless of the thresholds. M4 has larger values as the threshold become larger. This demonstrates each copula's abilities to express the tails. The skew t copula estimates the heavy right tail in the loss distribution, while the vine copula stresses the extreme values in the right tail.

In Fig. 12 we can see that, for the Japan-US pair, M3' and M4 have higher values than M1. This indicates that it is better to use asymmetric models, even if a country pair has a symmetry-like dependence structure. Otherwise, we might underestimate the risk of portfolios. M3' has a higher VaR than M4 for $\alpha \in [0.90, 0.95]$, and was as high as

M4 otherwise. With respect to the ES, M3' has a higher ES regardless of the confidence level. These results demonstrate that the loss distribution of M3' has a longer right tail, but the skewness is not large. On the other hand, M4 estimates the right-skewed loss distribution but its right tail is not as heavy as M3, because the asymmetric regime corresponds to the usual (not crisis) regime.

Fig. 13 shows that, for the Italy-US pair, M2 and M3 have similar values to M1, while M4 has higher values. As stated in Subsection IV-C3, it is notable that some asset pairs related to the Italian bond have negative correlation coefficients. M3 evaluates a stronger negative correlation, which leads to portfolio diversification. M4 only focuses the right tail of the loss distribution and has larger risk measure values. We find that the skew t copula is more suitable for describing negative dependence than the vine copula.

From these three figures, we can conclude that the vine copula emphasizes the right tail of the loss distribution more than the skew t copula. Moreover, we also find that the skew t copula better described the dependence structure of each asset pair, including the negative correlation. This finding indicates that we should pay attention to the choice of copulas when calculating risk measures. If we neglect the features of the copulas, the computed risk measures may be underestimated or overestimated.

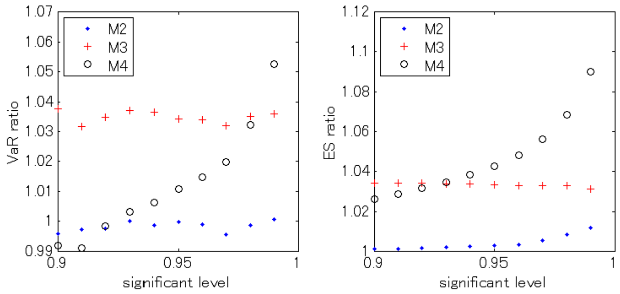

Fig. 11. VaR and ES for the UK-US pair

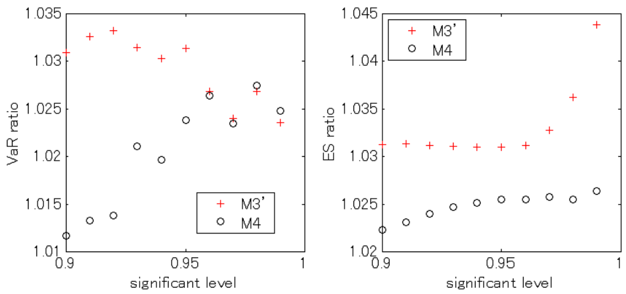

Fig. 12. VaR and ES for the Japan-US pair

VI. Conclusion

In this paper, we perform an empirical analysis of the dependence structures of international equity and bond markets using the RS copula model. We use the Gaussian copula in a symmetric regime, and the skew t or vine copula

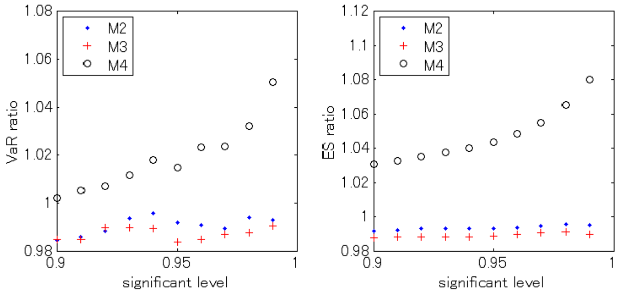

Fig. 13. VaR and ES for the Italy-US pair

in an asymmetric regime. The advantage of using the two asymmetric copulas is that they can express various dependence structures. The choice of copula in the asymmetric regime is significant, because they have desirable features for capturing dependencies among different asset classes. We use the skew t copula that is constructed from the skew t distribution. We describe the marginal models using the skew t GARCH models, and we assume that they were independent from the regimes. This assumption allows us to estimate parameters using a two-step procedure. In this two-step estimation, the parameters of the marginal models and those of the dependence structure are calculated separately. The Hamilton filter is used to estimate the parameters of the dependence structure. To find further implications for the dependence structure, we also compute the exceedance correlation, Value at Risk, and expected shortfall.

We apply the RS model to three country pairs: UK-US, Japan-US, and Italy-US. We analyze four models using different copulas: the Gaussian, the t, the skew t, and the vine. Our empirical analysis leads to the following conclusions. First, highly dependent regimes are different according to the asset pairs. We can determine this using our flexible multivariate model, which enables us to compare all the asset pairs. This implies that we should pay attention to the dependencies of each pair when constructing international diversified portfolios, to properly evaluate diversification benefits. Second, the strength of the asymmetry of each country pair varies, and that of the Japan-US pair is weak. This indicates that we should also consider weak asymmetry when calculating the risk of portfolios. Third, the skew t copula fits better to the data, but is not flexible enough to capture extreme dependencies, while the vine copula fits well in spite of having fewer parameters, but cannot express extreme dependencies. These empirical findings indicate that the dependence structure and asymmetry among all asset pairs and country pairs should be evaluated to correctly capture the benefits of international diversification.

In closing, we mention some future research topics. The first is the efficient estimation of the skew t copula model. The estimation of the skew t copula is computationally more intensive than that of the vine copula. Efficiency is crucial when the RS model is applied to a high dimensional case. The second is a better construction of the vine copula. In this paper, we only consider two types of vines and five building blocks. If the range of the vine copulas is expanded, some of their disadvantages might be overcome. Solving these issues will make the RS model more sophisticated and tractable, and will enable us to consider more than two countries.

APPENDIX

The estimation results of GARCH(1,1) models with Gaussian and t innovations are shown in Tables VIII and IX. Tables X and XI represent the results of the KS, the AD, and the Ljung-Box tests.

TABLE VIII
ESTIMATES OF NORMAL GARCH(1,1) PARAMETERS

	eqUS	eqUK	eqJP	eqIT
ω	0.28	0.30	0.33	0.14
	(0.08)	(0.08)	(0.19)	(0.06)
α	0.21	0.23	0.07	0.14
	(0.03)	(0.03)	(0.01)	(0.02)
β	0.75	0.74	0.90	0.85
	(0.04)	(0.03)	(0.03)	(0.02)
logL	-1172.59	-1193.51	-1381.42	-1338.14
	bnUS	bnUK	bnJP	bnIT
ω	0.11	0.06	0.49	0.15
	(0.07)	(0.04)	(0.19)	(0.08)
α	0.11	0.09	0.12	0.08
	(0.02)	(0.02)	(0.01)	(0.01)
β	0.89	0.91	0.88	0.91
	(0.02)	(0.02)	(0.01)	(0.02)
logL	-1499.33	-1354.00	-1642.99	-1397.06

Estimates of normal GARCH(1,1) models for all equity and bond returns, for four countries. The figures between parentheses represent the standard deviations of the parameters. "logL" is the value of the log likelihood function.

TABLE IX
ESTIMATES OF t GARCH(1,1) PARAMETERS

	eqUS	eqUK	eqJP	eqIT
ω	0.17	0.29	0.35	0.15
	(0.08)	(0.12)	(0.22)	(0.09)
α	0.12	0.19	0.08	0.13
	(0.04)	(0.05)	(0.03)	(0.03)
β	0.84	0.76	0.88	0.86
	(0.04)	(0.05)	(0.04)	(0.03)
ν	7.19	8.29	8.35	7.04
	(1.72)	(2.66)	(1.78)	(2.21)
logL	-1158.11	-1185.33	-1365.79	-1328.37
	bnUS	bnUK	bnJP	bnIT
ω	0.11	0.07	1.83	0.14
	(0.09)	(0.06)	(0.82)	(0.11)
α	0.10	0.09	0.08	0.07
	(0.03)	(0.03)	(0.03)	(0.02)
β	0.90	0.91	0.84	0.92
	(0.02)	(0.02)	(0.05)	(0.03)
ν	13.64	11.57	4.95	6.72
	(7.61)	(5.35)	(0.82)	(1.65)
logL	-1496.80	-1350.41	-1603.63	-1382.00

Estimates of t GARCH(1,1) models for all equity and bond returns, for four countries. The figures between parentheses represent the standard deviations of the parameters. "logL" is the value of the log likelihood function.

ACKNOWLEDGMENT

Both authors are very grateful to anonymous reviewers for their useful suggestions. This research was supported by JSPS KAKENHI Grant Number YYK5B02.

REFERENCES

[1] F. Longin and B. Solnik, "Extreme correlation of international equity markets," *The Journal of Finance*, vol. 56, no. 2, pp. 649–676, 2001.

TABLE X
GOODNESS OF FIT AND THE LJUNG-BOX STATISTICS FROM THE NORMAL GARCH(1,1) MODEL

	eqUS	eqUK	eqJP	eqIT	bnUS	bnUK	bnJP	bnIT
Panel A: Uniformity test								
KS								
Stat	0.09	0.08	0.06	0.06	0.03	0.03	0.07	0.04
p	0.00	0.00	0.07	0.05	0.58	0.61	0.01	0.48
AD								
Stat	6.04	5.89	2.71	2.77	0.55	0.65	3.78	1.31
p	0.00	0.00	0.04	0.04	0.70	0.60	0.01	0.23
Panel B: Test for serial independence								
Ljung-Box								
1	3.44^{**}	1.82^{**}	0.03^{**}	0.68^{**}	0.00^{**}	0.83^{**}	0.44^{**}	6.01^{*}
2	4.24^{**}	1.84^{**}	0.03^{**}	1.56^{**}	1.05^{**}	1.88^{**}	0.44^{**}	8.77^{*}
3	5.25^{**}	1.86^{**}	1.00^{**}	4.19^{**}	1.41^{**}	2.35^{**}	2.71^{**}	8.77^{*}
4	5.37^{**}	2.49^{**}	1.22^{**}	4.61^{**}	1.64^{**}	2.44^{**}	3.74^{**}	10.67^{*}
6	5.83^{**}	2.75^{**}	1.28^{**}	5.51^{**}	2.31^{**}	2.86^{**}	8.00^{**}	13.23^{*}
12	10.66^{**}	9.21^{**}	3.18^{**}	11.70^{**}	5.62^{**}	11.53^{**}	16.20^{**}	23.00^{*}

Panel A shows the Kolomogorov-Smirnov (KS) and Anderson-Darling (AD) statistics estimates with their p-values. "Stat" represents the statistics, while "p" is the p-value. Panel B represents the Ljung-Box statistics computed at lags 1, 2, 3, 4, 6, and 12. The symbol $*$ implies that independence at the 1% level cannot be rejected, and $**$ refers to the 5% level.

TABLE XI
GOODNESS OF FIT AND LJUNG-BOX STATISTICS FROM THE t GARCH(1,1) MODEL

	eqUS	eqUK	eqJP	eqIT	bnUS	bnUK	bnJP	bnIT
Panel A: Uniformity test								
KS								
Stat	0.09	0.08	0.06	0.06	0.04	0.04	0.07	0.05
p	0.00	0.00	0.06	0.02	0.45	0.39	0.01	0.08
AD								
Stat	7.43	6.91	3.97	4.48	1.14	1.45	7.42	3.60
p	0.00	0.00	0.01	0.01	0.29	0.19	0.00	0.01
Panel B: Test for serial independence								
Ljung-Box								
1	3.60^{**}	1.96^{**}	0.05^{**}	0.79^{**}	0.00^{**}	0.80^{**}	0.70^{**}	6.58^{*}
2	4.41^{**}	1.97^{**}	0.05^{**}	1.67^{**}	1.06^{**}	1.81^{**}	0.71^{**}	9.51
3	5.18^{**}	2.02^{**}	0.95^{**}	4.29^{**}	1.42^{**}	2.29^{**}	3.60^{**}	9.51^{*}
4	5.41^{**}	2.61^{**}	1.26^{**}	4.72^{**}	1.66^{**}	2.37^{**}	4.19^{**}	11.29^{*}
6	5.70^{**}	2.82^{**}	1.30^{**}	5.65^{**}	2.32^{**}	2.80^{**}	9.44^{**}	14.20^{*}
12	11.20^{**}	9.38^{**}	3.11^{**}	11.97^{**}	5.62^{**}	11.51^{**}	18.95^{**}	24.05^{*}

Panel A shows the Kolomogorov-Smirnov (KS) and Anderson-Darling (AD) statistics estimates with their p-values. "Stat" represents the statistics, while "p" is the p-value. Panel B represents the Ljung-Box statistics computed at lags 1, 2, 3, 4, 6, and 12. The symbol $*$ implies that independence at the 1% level cannot be rejected, and $**$ refers to the 5% level.

[2] A. Ang and J. Chen, "Asymmetric correlations of equity portfolios," *Journal of Financial Economics*, vol. 63, no. 3, pp. 443–494, 2002.

[3] A. Ang and G. Bekaert, "International asset allocation with regime shifts," *The Review of Financial Studies*, vol. 15, no. 4, pp. 1137–1187, 2002.

[4] R. Garcia and G. Tsafack, "Dependence structure and extreme co-movements in international equity and bond markets," *Journal of Banking & Finance*, vol. 35, no. 8, pp. 1954–1970, 2011.

[5] Y. Otani and J. Imai, "Pricing portfolio credit derivatives with stochastic recovery and systematic factor," *IAENG International Journal of Applied Mathematics*, vol. 43, no. 4, pp. 176–184, 2013.

[6] E. Jondeau and M. Rockinger, "The copula-GARCH model of conditional dependencies: An international stock market application," *Journal of International Money and Finance*, vol. 25, no. 5, pp. 827–853, 2006.

[7] A. J. Patton, "Modelling asymmetric exchange rate dependence," *International Economic Review*, vol. 47, no. 2, pp. 527–556, 2006.

[8] K. H. Chen and K. Khashanah, "Analysis of systemic risk: A vine copula-based ARMA-GARCH model," *Engineering Letters*, vol. 24, no. 3, pp. 268–273, 2016.

[9] Y. Li, Y. Zhang, and L. Zhou, "Variations detection of bivariate dependence based on copulas model," *IAENG International Journal of Applied Mathematics*, vol. 47, no. 3, pp. 255–260, 2017.

[10] C. Li and H. Hao, "A copula-based degradation modeling and reliability assessment," *Engineering Letters*, vol. 24, no. 3, pp. 295–300, 2016.

[11] T. Okimoto, "New evidence of asymmetric dependence structures in international equity markets," *Journal of Financial and Quantitative Analysis*, vol. 43, no. 3, pp. 787–815, 2008.

[12] L. Chollete, A. Heinen, and A. Valdesogo, "Modeling international financial returns with a multivariate regime-switching copula," *Journal of Financial Econometrics*, vol. 7, no. 4, pp. 437–480, 2009.
[13] S. Demarta and A. J. McNeil, "The t copula and related copulas," *International Statistical Review*, vol. 73, no. 1, pp. 111–129, 2005.
[14] T. Kollo and G. Pettere, "Parameter estimation and application of the multivariate skew t-copula," in *Copula Theory and Its Applications*. Springer, 2010, pp. 289–298.
[15] T. Bedford and R. M. Cooke, "Vines: a new graphical model for dependent random variables," *The Annals of Statistics*, vol. 30, no. 4, pp. 1031–1068, 2002.
[16] K. Aas, C. Czado, A. Frigessi, and H. Bakken, "Pair-copula constructions of multiple dependence," *Insurance: Mathematics and Economics*, vol. 44, no. 2, pp. 182–198, 2009.
[17] C. Savu and M. Trede, "Hierarchical archimedean copulas," in *International Conference on High Frequency Finance, Konstanz, Germany, May*, 2006.
[18] A. Azzalini and A. Capitanio, "Distributions generated by perturbation of symmetry with emphasis on a multivariate skew t-distribution," *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, vol. 65, no. 2, pp. 367–389, 2003.
[19] B. E. Hansen, "Autoregressive conditional density estimation," *International Economic Review*, pp. 705–730, 1994.
[20] J. D. Hamilton, "A new approach to the economic analysis of nonstationary time series and the business cycle," *Econometrica: Journal of the Econometric Society*, pp. 357–384, 1989.
[21] A. Sklar, *Fonctions de répartition à n dimensions et leurs marges*. Université Paris 8, 1959.
[22] M. G. Genton, *Skew-elliptical distributions and their applications: a journey beyond normality*. CRC Press, 2004.
[23] H. Joe, "Families of m-variate distributions with given margins and $m(m-1)/2$ bivariate dependence parameters," *Lecture Notes-Monograph Series*, vol. 28, pp. 120–141, 1996.
[24] ——, *Multivariate models and dependence concepts: Monographs on Statistics and Applied Probability, 73*. Chapmann & Hall, London, 1997.