Hued Colorings of Cartesian Products of Square of Cycles with Paths*

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Abstract

A r-hued k-coloring of G is a proper coloring with k colors such that for every vertex v with degree $d(v)$ in G, the color number of the neighbor of v is at least $min\{d(v), r\}$. The smallest integer k such that G has a r-hued k-coloring is called the r-hued chromatic number and denoted by $\chi_r(G)$. In this paper, we study the r-hued coloring of Cartesian products of square of cycles with paths.

Keywords: r-hued coloring; r-hued coloring number; Cartesian product

1 Introduction

In this paper, all graphs that we considered are connected, finite, undirected and simple (i. e., loopless and no multiple edges). For any integers a and b, we use the symbol $[a, b]$ to denote the set $\{a, a+1, \cdots, b\}$ when $a \leq b$, and $[k]$ to denote $[1, k]$ simply. Let $i(mod\ k)$ denote the remainder of i module k and all of them are in $[k]$ unless specified. For a real number x, let $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the smallest integer no less than x and the largest integer no more than x, respectively.

In recent years, many parameters and classes of graphs are studied. For example, in [18], the restricted connectivity of Cartesian product graphs is obtained, and in [19, 20], some results on 3-equitable labeling and the n-dimensional cube-connected complete graph are gained. In [21], some results about resistance distance and Kirchhoff index are obtained.

A vertex k-coloring of G is proper if any two adjacent vertices receive different colors. The smallest integer k such that G has a proper k-coloring is called the chromatic number and denoted by $\chi(G)$. For every $v \in V(G)$, $N(v)$ is denoted the neighborhood of v in G. A r-hued coloring c of G is a vertex proper coloring that for every $v \in V(G)$ with $d(v) \geq 2$ such that $|c(N(v))| \geq min\{d(v), r\}$. The minimum integer k such that G has a r-hued k-coloring is called the r-hued chromatic number of G and denoted by $\chi_r(G)$. The 1-hued chromatic number of G is the chromatic number. The 2-hued chromatic number of G is the dynamic chromatic number. It is obvious that $\chi(G) \leq \chi_2(G)$.

A coloring of a graph in which a typical vertex is adjacent to more than one color class represents a situation in which the typical individual has a greater variety in the type of relations. Thus, the overall interactions would not be so limited but more hued. Recently, the r-hued coloring of graphs has been studied by many researchers, see references [1-10,12-15]. It is shown in [1] that for $n \geq 3$, $\chi_2(C_n) = 3$ if $n \equiv 0(mod\ 3)$, $\chi_2(C_n) = 5$ if $n = 5$, and $\chi_2(C_n) = 4$ otherwise. In [3], it is proved that for every G with maximum degree Δ, if $\Delta \leq 3$, then $\chi_2(G) \leq 4$ unless $\chi_2(G) = 5$ for $G = C_5$, and if $\Delta \geq 4$, then $\chi_2(G) \leq \Delta + 1$.

For given two graphs G and H, the Cartesian product of G and H, denoted by $G \square H$, with the vertex set $V(G) \times V(H)$, and (u, v) and (u', v') are adjacent if and only if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$. Clearly, $\Delta(G \square H) = \Delta(G) + \Delta(H)$. In 2010, Akbari et al. in [2] obtained an upper bound for $\chi_2(G \square H)$ with minimum degree $\delta(G) \geq 2$. In [8], an upper bound is given for $\chi_r(G \square H)$ with $\delta(G) \geq r$. In [9], it is proved that m by n grid has no 3-hued 4-coloring when $mn \equiv 2(mod\ 4)$. In [10], the author studied the r-hued coloring on grids and toroidal grids.

The square of a graph G, denoted by G^2, is a graph with the same vertex set of G such that two vertices are adjacent if and only if their distance is at most 2 in G.

Lemma 1.1. [1] *Let G be a nontrivial graph. Then $\chi_2(K_2) = 2$ and $\chi_d(G) \geq 3$ otherwise.*

Lemma 1.2. [2] *Let G and H be two graphs. If $\delta(G) \geq 2$, then $\chi_2(G \square H) \leq max\{\chi_2(G), \chi(H)\}$.*

Lemma 1.3. [8] *Let G_1, G_2 be graphs. If $\delta(G_1) \geq r$, then $\chi_r(G_1 \square G_2) \leq max\{\chi_r(G_1), \chi(G_2)\}$.*

Lemma 1.4. [16] *Let $m \geq 3$ be an integer. Then*

$$\chi_3(C_m^2) = \begin{cases} 3, & m = 3, \\ 5, & m \in \{5, 6, 7, 11\} \\ 4, & otherwise. \end{cases}$$

Lemma 1.5. [17] *Let $n, m \geq 3$ be integers. Then*

$$\chi_3(P_n^2 \square P_m) = \begin{cases} 6, & n \in \{3, 5\}, \\ 4, & otherwise, \\ 5, & n = 7, \end{cases}$$

$$\chi_4(P_n^2 \square P_m) = \begin{cases} 7, & n \in [5, 7] \text{ or } n = 4 \text{ and } m \text{ is odd}, \\ 6, & otherwise, \end{cases}$$

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and

$$\chi_r(P_n^2 \square P_m) = \begin{cases} 6, & n = 3, \\ 7, & otherwise \end{cases}$$

for $r \geq 5$.

Observation 1.1 $\chi_r(G) \geq min\{\Delta(G), r\} + 1$. Equality holds for trees.

Observation 1.2 If $r \geq \Delta(G)$, then $\chi_r(G) = \chi_{\Delta(G)}(G)$.

Observation 1.3 $\chi_{r+1}(G) \geq \chi_r(G)$.

2 The Main results

Theorem 2.1. *Let $n, m \geq 3$ be integers. Then we have*

$$\chi_3(C_n^2 \square P_m) = \begin{cases} 6, & n = 3, \\ 5, & n = 5, \\ 4, & otherwise, \end{cases}$$

$$\chi_4(C_n^2 \square P_m) = \begin{cases} 6, & n = 3, \\ 8, & n = 4, \\ 5, & 5|n, \\ 6, & otherwise, \end{cases}$$

$$\chi_5(C_n^2 \square P_m) = \begin{cases} 8, & n \in \{4, 8, 15, 22, 23, 29\} \\ 6, & n = 3, \ or \ 6 \mid n, \\ 7, & 7 \mid n, \ or \ n \geq 30, \\ & or \ n \in \{13, 19, 20, 25, 26, 27\}, \\ 9 & n \in \{9, 11, 17\}, \\ & or \ n = 10 \ and \ m \ is \ odd, \\ 10, & n = 5, \ or \ n = 10 \ and \ m \ is \ even, \end{cases}$$

and

$$\chi_r(C_n^2 \square P_m) = \begin{cases} 6, & n = 3, \\ 7, & 7|n, \\ 8 & 8 \mid n, \ or \ n \geq 36, \\ & or \ n \in \{4, 15, 22, 23, 29, 30, 31\}, \\ 9, & n \in \{17, 18, 25, 26, 27, 33, 34\} \\ 10, & n \in \{5, 10, 13, 19, 20\} \\ \leq 11, & n \in \{11, 12\}, \end{cases}$$

Proof. Let $G = C_n^2 \square P_m$, and $\{u_1, u_2, \cdots, u_n\}$ and $\{v_1, v_2, \cdots, v_m\}$ be the vertex sets of C_n^2 and P_m, respectively.

Case 1. $n = 3$.

In this case, C_n^2 and P_n^2 both induce a K_3, so we have $\chi_r(C_n^2 \square P_m) = \chi_r(P_n^2 \square P_m)$. Hence we have $\chi_3(G) = \chi_4(G) = \chi_5(G) = \chi_r(G) = 6$ by Lemma 1.5.

Case 2. $n = 4$. Then $\Delta(G) = 5$, so $\chi_3(G) \geq 4$, and $\chi_5(G) = \chi_r(G) \geq 6$ by Observations 1.1-1.2.

For $r = 3$, since $\delta(C_4^2) = 3$, we have $\chi_3(G) \leq max\{\chi_3(C_4^2), \chi(P_m)\}$ by Lemma 1.3, and $\chi_3(C_4^2) = 4$ by Lemma 1.4, thus we obtain that $\chi_3(G) \leq 4$. Therefore, $\chi_3(G) = 4$ in this case.

For $r = 4$, we need at least four colors for the first column, WLOG, assume that $c(u_i, v_1) = i(mod\ 4)$. By the definition of r-hued coloring, we have $c(u_1, v_2) = 5$, $c(u_2, v_2) = 6$, $c(u_3, v_2) = 7$ and $c(u_4, v_2) = 8$. Hence $\chi_4(G) \geq 8$. We consider a 8-coloring c of G with $c(u_i, v_j) \equiv (2(i-1) + j)(mod\ 8)$ as following:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 5 & 6 & 7 & 8 & 1 & 2 & \cdots \\ 7 & 8 & 1 & 2 & 3 & 4 & \cdots \end{pmatrix}.$$

It is obvious that the coloring satisfies $|c(N(u_i, v_j))| = \min\{4, d(u_i, v_j)\}$, so it is a 4-hued 8-coloring of G, thus we have $\chi_4(G) \leq 8$. Hence $\chi_4(G) = 8$.

For $r \geq 5$, by Observations 1.2-1.3, we have $\chi_r(G) = \chi_5(G) \geq \chi_4(G) = 8$. Clearly, the coloring c with $c(u_i, v_j) \equiv (2i + j)(mod\ 8)$ is a 5-hued 8-coloring of G, so we obtain that $\chi_5(G) \leq 8$, hence we have $\chi_r(G) = \chi_5(G) = 8$.

Case 3. $n \geq 5$. Clearly, $\Delta(G) = 6$, so we have $\chi_3(G) \geq 4$, $\chi_4(G) \geq 5$, $\chi_5(G) \geq 6$ and $\chi_r(G) = \chi_6(G) \geq 7$ by Observations 1.1-1.2.

Subcase 3.1 $r = 3$. Since $n \geq 5$, $\delta(C_n^2) = 4$, so we have $\chi_3(G) \geq 4$ by Observation 1.1. But $\chi_3(G) \leq max\{\chi_3(C_n^2), \chi(P_m)\}$ by Lemma 1.3, thus we can obtain that $\chi_3(G) \leq 5$ for $n \in \{5, 6, 7, 11\}$ and $\chi_3(G) = 4$ otherwise by Lemma 1.4.

(1) For $n = 5$, if we use three colors to color the first column, WLOG, assume that $c(u_1, v_1) = 1$, $c(u_2, v_1) = 2$ and $c(u_3, v_1) = 3$, then $c(u_4, v_1) = 2$ and $c(u_5, v_1) = 3$ because any adjacent vertices can not receive the same colors, so we have $c(u_1, v_2) = 4$ and $c(u_2, v_2) = 5$ by the definition of r-hued coloring, hence we need at least five colors in this coloring.

If we use four colors to color the first column, WLOG, assume that $c(u_i, v_1) \equiv i(mod\ 4)$ for $1 \leq i \leq 4$. Then we have $c(u_5, v_1) = 5$ because any adjacent vertices can not receive the same colors, so we need at least five colors in this coloring. Hence $\chi_3(G) = 5$.

(2) For $n = 6$, we define a coloring c of G as following:

$$A = \begin{pmatrix} 1 & 4 & 1 & 4 & \cdots \\ 2 & 3 & 2 & 3 & \cdots \\ 3 & 2 & 3 & 2 & \cdots \\ 4 & 1 & 4 & 1 & \cdots \\ 2 & 3 & 2 & 3 & \cdots \\ 3 & 2 & 3 & 2 & \cdots \end{pmatrix}.$$

Clearly, c is a 3-hued 4-coloring, so we have $\chi_3(G) \leq 4$, hence $\chi_3(G) = 4$.

(3) For $n = 7$, we define a coloring c of G as following:

$$A = \begin{pmatrix} 1 & 4 & 1 & 4 & \cdots \\ 2 & 3 & 2 & 3 & \cdots \\ 3 & 2 & 3 & 2 & \cdots \\ 4 & 1 & 4 & 1 & \cdots \\ 2 & 3 & 2 & 3 & \cdots \\ 3 & 2 & 3 & 2 & \cdots \\ 4 & 1 & 4 & 1 & \cdots \end{pmatrix}.$$

It is not difficult to verify that c is a 3-hued 4-coloring, so we have $\chi_3(G) \leq 4$, hence we obtain that $\chi_3(G) = 4$.

(4) $n = 11$, we define a coloring c of G as following:

$$A = \begin{pmatrix} 1 & 4 & 1 & 4 & \cdots \\ 2 & 3 & 2 & 3 & \cdots \\ 3 & 2 & 3 & 2 & \cdots \\ 4 & 1 & 4 & 1 & \cdots \\ 1 & 4 & 1 & 4 & \cdots \\ 2 & 3 & 2 & 3 & \cdots \\ 3 & 2 & 3 & 2 & \cdots \\ 4 & 1 & 4 & 1 & \cdots \\ 2 & 3 & 2 & 3 & \cdots \\ 3 & 2 & 3 & 2 & \cdots \\ 4 & 1 & 4 & 1 & \cdots \end{pmatrix}.$$

It is obvious that c is a 3-hued 4-coloring which implies that $\chi_3(G) \leq 4$, hence we have $\chi_3(G) = 4$.

Subcase 3.2 $r = 4$. We have $\chi_4(G) \geq 5$ by Observation 1.2, and $\chi_4(G) \leq max\{\chi_4(C_n^2), \chi(P_m)\}$ by Lemma 1.3.

When $n \equiv 0(mod\, 5)$, by Lemma 1.4, we have $\chi_4(C_n^2) = 5$, so $\chi_4(G) \leq 5$, hence we obtain that $\chi_4(G) = 5$ in this case.

Suppose that $n > 5$ and $5 \nmid n$ in the following.

(1) If we use three colors for the first column, then we have $|c(N(u_1, v_1))| \leq 3$, a contradiction. Hence we need at least four colors to the first column, without loss of generality, assume that $c(u_i, v_1) = i(mod\ 4)$ for $i \leq 4(\lceil \frac{n}{4} \rceil - 1)$.

(i) When $n \equiv 1(mod\ 4)$, i. e., $n = 4k + 1$ for some $k \geq 2$, we obtain that $c(u_n, v_1) = 5$ because any adjacent vertices can not receive the same colors. By the definition of r-hued coloring, we have $c(u_3, v_2) = 5$, and $c(u_4, v_2) = 6$, so we obtain that $\chi_4(G) \geq 6$ in this case.

(ii) When $n \equiv 2(mod\ 4)$, i. e., $n = 4k + 2$ for some $k \geq 1$, then we have $c(u_{n-1}, v_1) = 2$ and $c(u_n, v_1) = 3$ because any adjacent vertices can not receive the same colors, but $|c(N(u_1, v_1))| \leq 3$, a contradiction, so this coloring is false. Hence we need at least five colors in the first column. If $k = 1$, then $n = 6$, so we have $c(u_5, v_1) = 2$, $c(u_6, v_1) = 5$, $c(u_1, v_2) = 4$, and $c(u_2, v_2) \in \{1, 3, 5\}$ by the definition of r-hued coloring. When $c(u_2, v_2) = 1$, we have $c(u_3, v_2) = 5$ and $c(u_4, v_2) = 6$. When $c(u_2, v_2) \in \{3, 5\}$, we obtain that the coloring needs at least six colors similarly. If $k \geq 2$, then we have $c(u_3, v_2) = 5$ and $c(u_4, v_2) = 6$. Hence $\chi_4(G) \geq 6$.

(iii) For $n \equiv i(mod\ 4)$ with $i \in [3, 4]$, it is not difficult to show that $\chi_4(G) \geq 6$ similarly.

(2) If we use five colors to first column, WLOG, assume that $c(u_i, v_1) \equiv i(mod\ 5)\ for\ i \leq 5(\lceil \frac{n}{5} \rceil - 1)$.

(i) when $n \equiv 1(mod\ 5)$, i. e., $n = 5k+1$ for some $k \geq 1$, we have $c(u_{5k+1}, v_1) = 3$ because any adjacent vertices can not receive the same colors. By the definition of r-hued coloring, we obtain that $c(u_1, v_2) = 4$, $c(u_2, v_2) = 5$, $c(u_{5k-1}, v_2) = 1$, and $c(u_{5k}, v_2) = 2$, so $c(u_{5k+1}, v_2) \notin [5]$, hence we have $\chi_4(G) \geq 6$. We may consider a 6-coloring c of G as following:

$$A = \begin{pmatrix} 1 & 4 & 1 & 4 & \cdots \\ 2 & 5 & 2 & 5 & \cdots \\ 3 & 6 & 3 & 6 & \cdots \\ 4 & 1 & 4 & 1 & \cdots \\ 5 & 2 & 5 & 2 & \cdots \end{pmatrix},$$

$$B = \begin{pmatrix} 6 & 3 & 6 & 3 & \cdots \end{pmatrix},$$

and

$$c = \begin{pmatrix} A \\ A \\ \vdots \\ A \\ B \end{pmatrix}.$$

Clearly, c is a 4-hued 6-coloring of G, then we have $\chi_4(G) \leq 6$, hence we obtain that $\chi_4(G) = 6$.

(ii) When $n \equiv 2(mod\ 5)$, i. e., $n = 5k + 2\ for\ k \geq 1$, we obtain that $c(u_{5k+2}, v_1) \in [3, 4]$ similarly. If $c(u_{5k+2}, v_1) = 3$, then $c(u_{5k+1}, v_1) = 2$ because any adjacent vertices can not receive the same colors, but we have $|c(N(u_1, v_1))| \leq 3$, a contradiction. If $c(u_{5k+2}, v_1) = 4$, then $c(u_{5k+1}, v_1) \in [2, 3]$, so $c(u_{5k+2}, v_2) = 3$ and $c(u_{5k+1}, v_2) \notin [5]$ if $c(u_{5k+1}, v_1) = 2$, and $c(u_1, v_2) = 5$ and $c(u_2, v_2) \notin [5]$ if $c(u_{5k+1}, v_1) = 3$, hence $\chi_4(G) \geq 6$. We define the coloring c of G as following:

$$A = \begin{pmatrix} 1 & 4 & 1 & 4 & \cdots \\ 2 & 5 & 2 & 5 & \cdots \\ 3 & 6 & 3 & 6 & \cdots \\ 4 & 1 & 4 & 1 & \cdots \\ 5 & 2 & 5 & 2 & \cdots \end{pmatrix},$$

$$B = \begin{pmatrix} 6 & 3 & 6 & 3 & \cdots \\ 4 & 6 & 4 & 6 & \cdots \end{pmatrix},$$

and

$$c = \begin{pmatrix} A \\ A \\ \vdots \\ A \\ B \end{pmatrix}.$$

Clearly, c is a 4-hued 6-coloring of G, then we have $\chi_4(G) \leq 6$. Hence $\chi_4(G) = 6$.

(iii) Suppose that $n \equiv 3(mod\ 5)$, i. e., $n = 5k + 3\ for\ k \geq 1$. Similar as (ii), we can obtain that $c(u_{5k+3}, v_1) \in [3, 5]$. When $c(u_{5k+3}, v_1) = 3$, we have $c(u_{5k+2}, v_1) \in \{2, 4\}$ because any adjacent vertices can not receive the same colors. If $c(u_{5k+2}, v_1) = 2$, then $|c(N(u_1, v_1))| \leq 3$, a contradiction. Hence we have $c(u_{5k+2}, v_1) = 4$, so $c(u_1, v_2) = 5$ and $c(u_2, v_2) \notin [5]$ by the definition of r-hued coloring. Hence $\chi_4(G) \geq 6$.

If $c(u_{5k+3}, v_1) = 4$, then $c(u_{5k+2}, v_1) \in [2, 3]$, so $c(u_1, v_2) = 5$ and $c(u_2, v_2) \notin [5]$ if $c(u_{5k+2}, v_1) = 2$, and $c(u_1, v_2) = 5$ and $c(u_2, v_2) \notin [5]$ if $c(u_{5k+2}, v_1) = 3$ by the definition of r-hued coloring. If $c(u_{5k+3}, v_1) = 5$, then $c(u_{5k+2}, v_1) \in [2, 4]$, so we have $c(u_{5k+1}, v_1) \in \{1, 3\}$ when $c(u_{5k+2}, v_1) = 2$. If $c(u_{5k+1}, v_1) = 1$, then $|c(N(u_{5k+2}, v_1))| \leq 3$, a contradiction. If $c(u_{5k+1}, v_1) = 3$, then we have $c(u_{5k+1}, v_2) = 1$ and $c(u_{5k}, v_2) \notin [5]$ by the definition. Hence we need at least 6 colors in this case.

If $c(u_{5k+2}, v_1) = 3$, then we have $c(u_{5k+1}, v_1) \in [1, 2]$, so $|c(N(u_{5k+2}, v_1))| \leq 3$ when $c(u_{5k+1}, v_1) = 1$, and $c(u_{5k+1}, v_2) = 1$ and $c(u_{5k}, v_2) \notin [5]$ when $c(u_{5k+1}, v_1) = 2$. If $c(u_{5k+2}, v_1) = 4$, then $|c(N(u_{5k+1}, v_1))| \leq 3$, a contradiction. Hence we have $\chi_4(G) \geq 6$.

We define the coloring c of G as following:

$$A = \begin{pmatrix} 1 & 5 & 1 & 5 & \cdots \\ 2 & 6 & 2 & 6 & \cdots \\ 3 & 4 & 3 & 4 & \cdots \\ 4 & 3 & 4 & 3 & \cdots \\ 5 & 2 & 5 & 2 & \cdots \end{pmatrix},$$

$$B = \begin{pmatrix} 6 & 1 & 6 & 1 & \cdots \\ 4 & 3 & 4 & 3 & \cdots \\ 3 & 4 & 3 & 4 & \cdots \end{pmatrix},$$

and

$$c = \begin{pmatrix} A \\ A \\ \vdots \\ A \\ B \end{pmatrix}.$$

Clearly, c is a 4-hued 6-coloring of G, so we have $\chi_4(G) \leq 6$, hence $\chi_4(G) = 6$.

(vi) When $n \equiv 4(mod\ 5)$, i. e., $n = 5k + 4\ for\ k \geq 1$, we can obtain that $\chi_4(G) \geq 6$ similarly. We consider the coloring c of G as following:

$$A = \begin{pmatrix} 1 & 5 & 1 & 5 & \cdots \\ 2 & 6 & 2 & 6 & \cdots \\ 3 & 1 & 3 & 4 & \cdots \\ 4 & 3 & 4 & 3 & \cdots \\ 5 & 2 & 5 & 2 & \cdots \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 4 & 1 & 4 & \cdots \\ 6 & 5 & 6 & 5 & \cdots \\ 3 & 2 & 3 & 2 & \cdots \\ 4 & 1 & 4 & 1 & \cdots \end{pmatrix},$$

and

$$c = \begin{pmatrix} A \\ A \\ \vdots \\ A \\ B \end{pmatrix}.$$

It is not difficult to verify that c is a 4-hued 6-coloring of G, so we have $\chi_4(G) \leq 6$. Hence $\chi_4(G) = 6$ in this case.

Subcase 3.3 For $r = 5$, when $n = 5$, it is clear that $d(u_i, v_1) = 5$, then $|c(N(u_1, v_j))| = 5$ by the definition. Let $c(u_1, v_1) = 1$, $c(u_2, v_1) = 2$, $c(u_3, v_1) = 3$, $c(u_4, v_1) = 4$ and $c(u_5, v_1) = 5$. Then $c(u_1, v_2) = 6$, $c(u_2, v_2) = 7$, $c(u_3, v_2) = 8$, $c(u_4, v_2) = 9$, and $c(u_5, v_2) = 10$ by the definition of r-hued coloring, so we obtain that $\chi_5(G) \geq 10$.

In the upper coloring c, for $j \geq 3$, let $c(u_i, v_j) = c(u_i, v_1)$ when j is odd, and $c(u_i, v_j) = c(u_i, v_2)$ when j is even. It is not difficult to verify that c is a 5-hued 10-coloring of G. Hence $\chi_5(G) = 10$.

Subcase 3.3.1 If $6|n$, then the coloring c of the first column is unique, and $\chi_5(G) \geq 6$ clearly. We can find the coloring c as following:

$$\begin{pmatrix} 1 & 4 & 1 & 4 & \cdots \\ 2 & 5 & 2 & 5 & \cdots \\ 3 & 6 & 3 & 6 & \cdots \\ 4 & 1 & 4 & 1 & \cdots \\ 5 & 2 & 5 & 2 & \cdots \\ 6 & 3 & 6 & 3 & \cdots \\ 1 & 4 & 1 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

It is not difficult to verify that c is a 5-hued 6-coloring of G, so we have $\chi_5(G) = 6$.

Subcase 3.3.2 Suppose that $6 \nmid n$.

(i) If $7|n$, then it is not difficult to verify that $\chi_5(G) \geq 7$. We can find a 5-hued 7-coloring c as following:

$$\begin{pmatrix} 1 & 4 & 1 & 4 & \cdots \\ 2 & 5 & 2 & 5 & \cdots \\ 3 & 6 & 3 & 6 & \cdots \\ 4 & 7 & 4 & 7 & \cdots \\ 5 & 1 & 5 & 1 & \cdots \\ 6 & 2 & 6 & 2 & \cdots \\ 7 & 3 & 7 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Thus we have $\chi_5(G) \leq 7$, hence $\chi_5(G) = 7$.

(ii) For $n = 13$, if we use five colors to color the first column, WLOG, assume that $c(u_i, v_1) \equiv i(mod\ 5)$ for $1 \leq i \leq 10$, then $c(u_{11}, v_1) = 6$, $c(u_{12}, v_1) = 7, c(u_{13}, v_1) = 8$ by the definition of r-hued coloring, so we need at least eight colors.

If we use six colors to color the first column, WLOG, assume that $c(u_i, v_1) \equiv i(mod\ 6)$ for $1 \leq i \leq 12$, then we have $c(u_{13}, v_1) = 7$ by the definition of r-hued coloring.

We consider the coloring c of G as following:

$$\begin{pmatrix} 1 & 4 & 1 & 4 & \cdots \\ 2 & 5 & 2 & 5 & \cdots \\ 3 & 6 & 3 & 6 & \cdots \\ 4 & 1 & 4 & 1 & \cdots \\ 5 & 2 & 5 & 2 & \cdots \\ 6 & 3 & 6 & 3 & \cdots \\ 1 & 4 & 1 & 4 & \cdots \\ 2 & 5 & 2 & 5 & \cdots \\ 3 & 6 & 3 & 6 & \cdots \\ 4 & 7 & 4 & 7 & \cdots \\ 5 & 1 & 5 & 1 & \cdots \\ 6 & 2 & 6 & 2 & \cdots \\ 7 & 3 & 7 & 3 & \cdots \end{pmatrix}.$$

Clearly, it is a 5-hued 7-coloring of G, so we have $\chi_5(G) \leq 7$. Hence $\chi_5(G) = 7$.

(ii) For $n = 19$, if we use five colors to color the first column, then $c(u_i, v_1) \equiv i(mod\ 5)$ for $1 \leq i \leq 15$, and $c(u_{16}, v_1) = 6$, $c(u_{17}, v_1) = 7$, $c(u_{18}, v_1) = 8$, $c(u_{19}, v_1) = 9$ by the definition of r-hued coloring. If we use six colors to color the first column, then $c(u_i, v_1) \equiv i(mod\ 6)$ for $1 \leq i \leq 18$, and $c(u_{19}, v_1) = 7$ by the definition of r-hued coloring. Hence $\chi_5(G) \geq 7$.

We consider a 5-hued 7-coloring c of G as following:

$$\begin{pmatrix} 1 & 4 & 1 & 4 & \cdots \\ 2 & 5 & 2 & 5 & \cdots \\ 3 & 6 & 3 & 6 & \cdots \\ 4 & 1 & 4 & 1 & \cdots \\ 5 & 2 & 5 & 2 & \cdots \\ 6 & 3 & 6 & 3 & \cdots \\ 1 & 4 & 1 & 4 & \cdots \\ 2 & 5 & 2 & 5 & \cdots \\ 3 & 6 & 3 & 6 & \cdots \\ 4 & 1 & 4 & 1 & \cdots \\ 5 & 2 & 5 & 2 & \cdots \\ 6 & 3 & 6 & 3 & \cdots \\ 1 & 4 & 1 & 4 & \cdots \\ 2 & 5 & 2 & 5 & \cdots \\ 3 & 6 & 3 & 6 & \cdots \\ 4 & 7 & 4 & 7 & \cdots \\ 5 & 1 & 5 & 1 & \cdots \\ 6 & 2 & 6 & 2 & \cdots \\ 7 & 3 & 7 & 3 & \cdots \end{pmatrix}.$$

Then we have $\chi_5(G) \leq 7$, hence $\chi_5(G) = 7$.

Similar as in (i) and (ii), if $n \in \{20, 25, 26, 27\}$, then we can obtain that $\chi_5(G) = 7$.

(iii) Suppose that $n \geq 30$ but $6 \nmid n$. It is not difficult to verify that there exist nonnegative integers x and y such that $n = 6x + 7y$ from [11].

Because $d(u_i, v_1) = 5$, we use at least five colors in the first column. When $n \equiv 0(mod\ 5)$, i. e., $n = 5k$ for $k \geq 6$, WLOG, assume that the coloring c of the first column is $c(u_i, v_1) \equiv i(mod\ 5)$ for $1 \leq i \leq 5k$, then $c(u_1, v_2) = 6$, $c(u_2, v_2) = 7$, $c(u_3, v_2) = 8$, $c(u_4, v_2) = 6$, and $c(u_5, v_2) = 7$ by the definition of r-hued coloring, but $|c(N(u_3, v_2))| \leq 4$, a contradiction. So we need at least nine colors in this coloring.

When $n \equiv 1(mod\ 5)$, i. e., $n = 5k+1$ for $k \geq 6$, if we use six colors in the first column, WLOG, assume that the coloring c satisfies $c(u_i, v_1) \equiv i(mod\ 5)$ for $1 \leq i \leq 5k$, then we have $c(u_{5k+1}, v_1) = 6$, $c(u_1, v_2) = 4$, $c(u_2, v_2) = 5$, $c(u_3, v_2) = 6$, $c(u_4, v_2) = 7$, $c(u_5, v_2) = 8$, $c(u_6, v_2) = 6$, and $c(u_7, v_2) = 9$ by the definition of r-hued coloring, so we need at least nine colors in this coloring.

Similarly, we obtain the same result in the case of using six colors in the first column.

Moreover, we can define the coloring c with seven colors on the first column as $c(u_i, v_1) \equiv i(mod\ 6)$ for $1 \leq i \leq 6x$, $c(u_i, v_1) \equiv i(mod\ 7)$ for $6x+1 \leq i \leq 7y$, and then $c(u_i, v_2) \equiv (c(u_i, v_1) + 3)(mod\ 6)$ for $1 \leq i \leq 6x$, $c(u_i, v_2) \equiv (c(u_i, v_1) + 3)(mod\ 7)$ for $6x + 1 \leq i \leq 7y$, $c(u_i, v_j) = c(u_i, v_1)$ when j is odd, and $c(u_i, v_j) = c(u_i, v_2)$ when j is even. Clearly, c is a 5-hued 7-coloring of G and optimal, so $\chi_5(G) = 7$.

Similarly, we can obtain the same result in the case of $n \equiv i(mod\ 5)$ for $i \in [2, 4]$.

Subcase 3.3.3 Suppose that $n \in \{8, 15, 22, 23, 29\}$.

(i) For $n = 8$, it is not difficult to verify that the coloring c on the first column as $c(u_i, v_1) \equiv i(mod\ 8)$ is optimal, so $\chi_5(G) \geq 8$. We consider a 8-coloring of G as following:

$$\begin{pmatrix} 1 & 4 & 1 & 4 & \cdots \\ 2 & 5 & 2 & 5 & \cdots \\ 3 & 6 & 3 & 6 & \cdots \\ 4 & 7 & 4 & 7 & \cdots \\ 5 & 8 & 5 & 8 & \cdots \\ 6 & 1 & 6 & 1 & \cdots \\ 7 & 2 & 7 & 2 & \cdots \\ 8 & 3 & 8 & 3 & \cdots \end{pmatrix}.$$

Clearly, it is a 5-hued 8-coloring of G, so we have $\chi_5(G) \leq 8$. Hence $\chi_5(G) = 8$.

(ii) For $n = 15$, if we use five colors to color the first column, WLOG, assume that $c(u_i, v_1) \equiv i(mod\ 5)$ for $1 \leq i \leq 15$, then, by the definition of r-hued coloring, we have $c(u_1, v_2) = 6$, $c(u_2, v_2) = 7$, $c(u_3, v_2) = 8$, $c(u_4, v_2) = 6$, and $c(u_5, v_2) = 9$, so we need at least nine colors in this coloring.

If we use six colors to color the first column, assume that $c(u_i, v_1) \equiv i(mod\ 6)$ for $1 \leq i \leq 12$, then we have $c(u_{13}, v_1) = 7$, $c(u_{14}, v_1) = 8$, and $c(u_{15}, v_1) = 9$, so we need at least nine colors in this coloring.

If we use seven colors to color the first column, assume that $c(u_i, v_1) \equiv i(mod\ 7)$ for $1 \leq i \leq 14$, then $c(u_{15}, v_1) = 8$, so we need at least eight colors in this coloring, too.

We consider the coloring c of G as following:

$$\begin{pmatrix} 1 & 4 & 1 & 4 & \cdots \\ 2 & 5 & 2 & 5 & \cdots \\ 3 & 6 & 3 & 6 & \cdots \\ 4 & 7 & 4 & 7 & \cdots \\ 5 & 1 & 5 & 1 & \cdots \\ 6 & 2 & 6 & 2 & \cdots \\ 7 & 3 & 7 & 3 & \cdots \\ 1 & 4 & 1 & 4 & \cdots \\ 2 & 5 & 2 & 5 & \cdots \\ 3 & 6 & 3 & 6 & \cdots \\ 4 & 7 & 4 & 7 & \cdots \\ 5 & 8 & 5 & 8 & \cdots \\ 6 & 1 & 6 & 1 & \cdots \\ 7 & 2 & 7 & 2 & \cdots \\ 8 & 3 & 8 & 3 & \cdots \end{pmatrix}.$$

Clearly, it is a 5-hued 8-coloring of G, so we have $\chi_5(G) \leq 8$. Hence $\chi_5(G) = 8$.

Similar as (i) and (ii), we can obtain that $\chi_5(G) = 8$ in the case of $n \in \{22, 23, 29\}$.

Subcase 3.3.4 $n \in \{9, 10, 11, 17\}$.

(i) For $n = 9$, it is not difficult to verify that the coloring c with $c(u_i, v_1) \equiv 1(mod\ 9)$ is optimal, so we have $\chi_5(G) \geq 9$. We consider a 9-coloring of G as following:

$$\begin{pmatrix} 1 & 4 & 1 & 4 & \cdots \\ 2 & 5 & 2 & 5 & \cdots \\ 3 & 6 & 3 & 6 & \cdots \\ 4 & 7 & 4 & 7 & \cdots \\ 5 & 8 & 5 & 8 & \cdots \\ 6 & 9 & 6 & 9 & \cdots \\ 7 & 1 & 7 & 1 & \cdots \\ 8 & 2 & 8 & 2 & \cdots \\ 9 & 3 & 9 & 3 & \cdots \end{pmatrix}.$$

Clearly, it is a 5-hued 9-coloring of G, so we have $\chi_5(G) \leq 9$. Hence $\chi_5(G) = 9$.

(ii) For $n = 17$, if we use five colors to color the first column, WLOG, assume that $c(u_i, v_1) \equiv i(mod\ 5)$ for $1 \leq i \leq 15$. Then, by the definition of r-hued coloring, we have $c(u_{16}, v_1) = 6$ and $c(u_{17}, v_1) = 7$, so we have $c(u_1, v_2) = 4$, $c(u_2, v_2) = 5$, $c(u_3, v_2) = 6$, $c(u_4, v_2) = 7$, $c(u_5, v_2) = 8$, $c(u_6, v_2) = 6$, and $c(u_7, v_2) = 9$, thus we need at least nine colors in this coloring.

If we use six colors to color the first column, WLOG, assume that $c(u_i, v_1) \equiv i(mod\ 6)$ for $1 \leq i \leq 17$. Then $c(u_{13}, v_2) = 4$, and thus $c(u_{14}, v_2) = 5$, $c(u_{15}, v_2) = 6$, $c(u_{16}, v_2) = 7$, $c(u_{17}, v_2) = 8$, $c(u_1, v_2) = 6$, and $c(u_2, v_2) = 9$.

If we use seven colors to color the first column, WLOG, assume that $c(u_i, v_1) \equiv i(mod\ 7)$ for $1 \leq i \leq 14$. Then $c(u_{15}, v_1) = 3$, $c(u_{16}, v_1) = 4$, $c(u_{17}, v_1) = 5$ and $c(u_{17}, v_2) = 6$, so we have $c(u_{16}, v_2) = 2$, $c(u_{15}, v_2) = 1$, $c(u_{14}, v_2) = 8$, $c(u_{13}, v_2) = 2$, and $c(u_{12}, v_2) = 9$.

If we use eight colors to color the first column, WLOG, assume that $c(u_i, v_1) \equiv i(mod\ 8)$ for $1 \leq i \leq 16$, then we have $c(u_{17}, v_1) = 9$.

Therefore, we need at least nine colors in every case.

We define the coloring c of G as follows:

$$\begin{pmatrix} 1 & 4 & 1 & 4 & \cdots \\ 2 & 5 & 2 & 5 & \cdots \\ 3 & 6 & 3 & 6 & \cdots \\ 4 & 7 & 4 & 7 & \cdots \\ 5 & 8 & 5 & 8 & \cdots \\ 6 & 1 & 6 & 1 & \cdots \\ 7 & 2 & 7 & 2 & \cdots \\ 8 & 3 & 8 & 3 & \cdots \\ 1 & 4 & 1 & 4 & \cdots \\ 2 & 5 & 2 & 5 & \cdots \\ 3 & 6 & 3 & 6 & \cdots \\ 4 & 7 & 4 & 7 & \cdots \\ 5 & 8 & 5 & 8 & \cdots \\ 6 & 9 & 6 & 9 & \cdots \\ 7 & 1 & 7 & 1 & \cdots \\ 8 & 2 & 8 & 2 & \cdots \\ 9 & 3 & 9 & 3 & \cdots \end{pmatrix}.$$

Clearly, c is a 5-hued 9-coloring of G, so we have $\chi_5(G) \leq 9$. Hence $\chi_5(G) = 9$.

(iii) For $n = 11$, similar as (i) and (ii), it is not difficult to show that $\chi_5(G) \geq 9$. We consider the coloring c of G as following:

$$\begin{pmatrix} 1 & 7 & 1 & 7 & \cdots \\ 2 & 8 & 2 & 8 & \cdots \\ 3 & 9 & 3 & 9 & \cdots \\ 4 & 1 & 4 & 1 & \cdots \\ 5 & 2 & 5 & 2 & \cdots \\ 6 & 3 & 6 & 3 & \cdots \\ 7 & 4 & 7 & 4 & \cdots \\ 8 & 5 & 8 & 5 & \cdots \\ 9 & 6 & 9 & 6 & \cdots \\ 4 & 2 & 4 & 2 & \cdots \\ 5 & 3 & 5 & 3 & \cdots \end{pmatrix}.$$

Clearly, c is a 5-hued 9-coloring, so we have $\chi_5(G) \leq 9$, hence $\chi_5(G) = 9$.

(iv) For $n = 10$, it is not difficult to prove $\chi_5(G) \geq 9$ similar as (i) and (ii).

We consider a 9-coloring of G as following:

$$\begin{pmatrix} 1 & 6 & 2 & 7 & 1 & 6 & \cdots \\ 2 & 7 & 3 & 8 & 2 & 7 & \cdots \\ 3 & 8 & 4 & 9 & 3 & 8 & \cdots \\ 4 & 6 & 5 & 7 & 4 & 6 & \cdots \\ 5 & 9 & 1 & 6 & 5 & 9 & \cdots \\ 1 & 7 & 2 & 8 & 1 & 7 & \cdots \\ 2 & 6 & 3 & 7 & 2 & 6 & \cdots \\ 3 & 8 & 4 & 9 & 3 & 8 & \cdots \\ 4 & 7 & 5 & 8 & 4 & 7 & \cdots \\ 5 & 9 & 1 & 6 & 5 & 9 & \cdots \end{pmatrix}.$$

Clearly, if m is odd, then it is a 5-hued 9-coloring, so we have $\chi_5(G) \leq 9$, hence we obtain that $\chi_5(G) = 9$ in this case.

Suppose that m is even. If we use nine colors to color G, then there is some vertex in any adjacent two columns whose color number of neighbors must be 4 no matter what the coloring is. Therefore, the color number of the neighbors of this vertex in the first column or the last column is 4 which contradicts the 5-hued nature. Hence $\chi_5(G) \geq 10$. Consider a 10-coloring c of G with $c(u_i, v_j) \equiv (2(i-1)+j)(mod\ 10)$ as following:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 1 & \cdots \\ 5 & 6 & 7 & 8 & 9 & 10 & 1 & 2 & 3 & \cdots \\ 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 & \cdots \\ 9 & 10 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 1 & \cdots \\ 5 & 6 & 7 & 8 & 9 & 10 & 1 & 2 & 3 & \cdots \\ 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 & \cdots \\ 9 & 10 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \end{pmatrix}.$$

Clearly, $|c(N(u_i, v_j))| = min\{5, d(u_1, v_j)\}$, so c is a 5-hued 10-coloring of G, hence we have $\chi_5(G) = 10$ for even m.

Subcase 3.4 Suppose that $r \geq 6$. If $n \equiv 0(mod 7)$, then $\chi_r(C_m^2) \geq 7$ by Observation 1.1. Define the coloring c of G as following:

$$\begin{pmatrix} 1 & 4 & 7 & 3 & 6 & 2 & 5 & 1 & 4 & 7 & \cdots \\ 2 & 5 & 1 & 4 & 7 & 3 & 6 & 2 & 5 & 1 & \cdots \\ 3 & 6 & 2 & 5 & 1 & 4 & 7 & 3 & 6 & 2 & \cdots \\ 4 & 7 & 3 & 6 & 2 & 5 & 1 & 4 & 7 & 3 & \cdots \\ 5 & 1 & 4 & 7 & 3 & 6 & 2 & 5 & 1 & 4 & \cdots \\ 6 & 2 & 5 & 1 & 4 & 7 & 3 & 6 & 2 & 5 & \cdots \\ 7 & 3 & 6 & 2 & 5 & 1 & 4 & 7 & 3 & 6 & \cdots \\ 1 & 4 & 7 & 3 & 6 & 2 & 5 & 1 & 4 & 7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Clearly, c is a r-hued 7-coloring of G, so we have $\chi_r(G) \leq 7$. Hence $\chi_r(G) = 7$. Suppose that $n \equiv i(mod 7)$ for $i \in [1, 6]$ in the following.

Subcase 3.4.1 Suppose that $n \in \{15, 22, 23, 29, 30, 31\}$, or $8 \mid n$, or $n \geq 36$.

(i) If $8 \mid n$, then the coloring c of the first column is $c(u_i, v_1) \equiv i(mod\ 8)$ which is a optimal coloring, so we have $\chi_r(G) \geq 8$. Define the coloring c of G as following:

$$\begin{pmatrix} 1 & 4 & 7 & 2 & 5 & 8 & 3 & 6 & 1 & 4 & 7 & \cdots \\ 2 & 5 & 8 & 3 & 6 & 1 & 4 & 7 & 2 & 5 & 8 & \cdots \\ 3 & 6 & 1 & 4 & 7 & 2 & 5 & 8 & 3 & 6 & 1 & \cdots \\ 4 & 7 & 2 & 5 & 8 & 3 & 6 & 1 & 4 & 7 & 2 & \cdots \\ 5 & 8 & 3 & 6 & 1 & 4 & 7 & 2 & 5 & 8 & 3 & \cdots \\ 6 & 1 & 4 & 7 & 2 & 5 & 8 & 3 & 6 & 1 & 4 & \cdots \\ 7 & 2 & 5 & 8 & 3 & 6 & 1 & 4 & 7 & 2 & 5 & \cdots \\ 8 & 3 & 6 & 1 & 4 & 7 & 2 & 5 & 8 & 3 & 6 & \cdots \\ 1 & 4 & 7 & 2 & 5 & 8 & 3 & 6 & 1 & 4 & 7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Clearly, c is a 6-hued 8-coloring of G, so we have $\chi_r(G) \leq 8$. Hence $\chi_r(G) = 8$.

(ii) $n = 15$. If we use five colors to color the first column, WLOG, assume that $c(u_i, v_1) \equiv i(mod\ 5)$ for $1 \leq i \leq 15$. Then, by the definition of r-hued coloring, we have $c(u_1, v_2) = 6$, $c(u_2, v_2) = 7$, $c(u_3, v_2) = 8$, $c(u_4, v_2) = 9$, and $c(u_5, v_2) = 10$, so we need at least ten colors in this coloring.

If we use six colors to color the first column, WLOG, assume that $c(u_i, v_1) \equiv i(mod\ 6)$ for $1 \leq i \leq 12$. Then $c(u_1, v_{13}) = 7$, $c(u_1, v_{14}) = 8$, and $c(u_1, v_{15}) = 9$, so we need at least nine colors in this coloring.

If we use seven colors to color the first column, WLOG, assume that $c(u_i, v_1) \equiv i(mod\ 7)$ for $1 \leq i \leq 14$. Then $c(u_1, v_{15}) = 8$, so we need at least eight colors in this coloring.

Therefore, we need at least eight colors in every case.

We can find a coloring c of G as following:

$$\begin{pmatrix} 1 & 4 & 7 & 3 & 6 & 1 & 4 & \cdots \\ 2 & 5 & 8 & 4 & 7 & 2 & 5 & \cdots \\ 3 & 6 & 1 & 5 & 8 & 3 & 6 & \cdots \\ 4 & 7 & 2 & 6 & 1 & 4 & 7 & \cdots \\ 5 & 8 & 3 & 7 & 2 & 5 & 8 & \cdots \\ 6 & 1 & 4 & 8 & 3 & 6 & 1 & \cdots \\ 7 & 2 & 5 & 1 & 4 & 7 & 2 & \cdots \\ 8 & 3 & 6 & 2 & 5 & 8 & 3 & \cdots \\ 1 & 4 & 7 & 3 & 6 & 1 & 4 & \cdots \\ 2 & 5 & 1 & 4 & 7 & 2 & 5 & \cdots \\ 3 & 6 & 2 & 5 & 1 & 3 & 6 & \cdots \\ 4 & 7 & 3 & 6 & 2 & 4 & 7 & \cdots \\ 5 & 1 & 4 & 7 & 3 & 5 & 1 & \cdots \\ 6 & 2 & 5 & 1 & 4 & 6 & 2 & \cdots \\ 7 & 3 & 6 & 2 & 5 & 7 & 3 & \cdots \end{pmatrix}.$$

Clearly, c is a 6-hued 8-coloring of G, so we have $\chi_r(G) \leq 8$. Hence $\chi_r(G) = 8$.

(iii) If $n \in \{22, 23, 29, 30, 31\}$, or $n \geq 36$, then we obtain $\chi_r(G) \geq 8$ as (i) similarly. It is not difficult to verify that there exist nonnegative integers x and y such that $n = 7x + 8y$ from [11]. We consider the coloring c of G with $c(u_i, v_1) \equiv i(mod\ 7)$, $c(u_i, v_j) \equiv (c(u_i, v_{j-1}) + 3)(mod\ 7)$ for $j \geq 2$ and $1 \leq i \leq 7x$ as following:

$$\begin{pmatrix} 1 & 4 & 7 & 3 & 6 & 1 & 4 & \cdots \\ 2 & 5 & 1 & 4 & 7 & 2 & 5 & \cdots \\ 3 & 6 & 2 & 5 & 1 & 3 & 6 & \cdots \\ 4 & 7 & 3 & 6 & 2 & 4 & 7 & \cdots \\ 5 & 1 & 4 & 7 & 3 & 5 & 1 & \cdots \\ 6 & 2 & 5 & 1 & 4 & 6 & 2 & \cdots \\ 7 & 3 & 6 & 2 & 5 & 7 & 3 & \cdots \end{pmatrix},$$

$c(u_i, v_1) \equiv i(mod\ 8)$, and $c(u_i, v_j) \equiv (c(u_i, v_{j-1}) + 3)(mod\ 8)$ for $j \geq 2$ and $7x + 1 \leq i \leq n$ as following:

$$\begin{pmatrix} 1 & 4 & 7 & 3 & 6 & 1 & 4 & \cdots \\ 2 & 5 & 8 & 4 & 7 & 2 & 5 & \cdots \\ 3 & 6 & 1 & 5 & 8 & 3 & 6 & \cdots \\ 4 & 7 & 2 & 6 & 1 & 4 & 7 & \cdots \\ 5 & 8 & 3 & 7 & 2 & 5 & 8 & \cdots \\ 6 & 1 & 4 & 8 & 3 & 6 & 1 & \cdots \\ 7 & 2 & 5 & 1 & 4 & 7 & 2 & \cdots \\ 8 & 3 & 6 & 2 & 5 & 8 & 3 & \cdots \end{pmatrix}.$$

Clearly, c is a r-hued 8-coloring of G, so we have $\chi_r(G) \leq 8$. Hence $\chi_r(G) = 8$.

Subcase 3.4.2 Suppose that $n \in \{9, 17, 18, 25, 26, 27, 33, 34\}$.

(i) For $n = 9$, it is not difficult to verify that the coloring c of the first column with $c(u_i, v_1) \equiv i(mod\ 9)$ is an optimal coloring, so we have $\chi_r(G) \geq 9$. Define the coloring c of G as following:

$$\begin{pmatrix} 1 & 4 & 7 & 1 & 4 & \cdots \\ 2 & 5 & 8 & 2 & 5 & \cdots \\ 3 & 6 & 9 & 3 & 6 & \cdots \\ 4 & 7 & 1 & 4 & 7 & \cdots \\ 5 & 8 & 2 & 5 & 8 & \cdots \\ 6 & 9 & 3 & 6 & 9 & \cdots \\ 7 & 1 & 4 & 7 & 1 & \cdots \\ 8 & 2 & 5 & 8 & 2 & \cdots \\ 9 & 3 & 6 & 9 & 3 & \cdots \end{pmatrix}.$$

Clearly, it is a r-hued 9-coloring of G, so we have $\chi_r(G) \leq 9$. Hence we obtain that $\chi_r(G) = 9$.

(ii) For $n = 17$, if we use five colors to color the first column, WLOG, assume that $c(u_i, v_1) \equiv i(mod\ 5)$ for $1 \leq i \leq 15$. Then, by the definition of r-hued coloring, we have $c(u_3, v_2) = 6$, and $c(u_4, v_2) = 7$, $c(u_5, v_2) = 8$, $c(u_6, v_2) = 9$ and $c(u_7, v_2) = 10$.

If we use six colors to color the first column, WLOG, assume that $c(u_i, v_1) \equiv i(mod\ 6)$ for $1 \leq i \leq 17$. Then we have $c(u_1, v_2) = 6, c(u_2, v_2) = 7$, $c(u_3, v_2) = 8$, $c(u_{17}, v_2) = 9$, and $c(u_{16}, v_2) = 10$.

If we use seven colors to color the first column, WLOG, assume that $c(u_i, v_1) \equiv i(mod\ 7)$ for $1 \leq i \leq 14$. Then, by the definition of r-hued coloring, we obtain that $c(u_{15}, v_1) = 3$, $c(u_{16}, v_1) = 4$, $c(u_{17}, v_1) = 5$, $c(u_1, v_2) = 6$, $c(u_2, v_2) = 7$, $c(u_3, v_2) = 8$, $c(u_{17}, v_2) = 9$, and $c(u_{16}, v_2) = 10$.

If we use eight colors to color the first column, WLOG, assume that $c(u_i, v_1) \equiv i(mod\ 8)$ for $1 \leq i \leq 16$. Then we have $c(u_{17}, v_1) = 9$, so we need at least nine colors in this case.

We can find a coloring c of G as following:

$$\begin{pmatrix} 1 & 4 & 7 & 2 & 5 & 8 & 3 & 6 & 1 & 4 & 7 & \cdots \\ 2 & 5 & 8 & 3 & 6 & 1 & 4 & 7 & 2 & 5 & 8 & \cdots \\ 3 & 6 & 1 & 4 & 7 & 2 & 5 & 8 & 3 & 6 & 1 & \cdots \\ 4 & 7 & 2 & 5 & 8 & 3 & 6 & 1 & 4 & 7 & 2 & \cdots \\ 5 & 8 & 3 & 6 & 1 & 4 & 7 & 2 & 5 & 8 & 3 & \cdots \\ 6 & 1 & 4 & 7 & 2 & 5 & 8 & 3 & 6 & 1 & 4 & \cdots \\ 7 & 2 & 5 & 8 & 3 & 6 & 1 & 4 & 7 & 2 & 5 & \cdots \\ 8 & 3 & 6 & 1 & 4 & 7 & 2 & 5 & 8 & 3 & 6 & \cdots \\ 1 & 4 & 7 & 2 & 5 & 8 & 3 & 6 & 1 & 4 & 7 & \cdots \\ 2 & 5 & 8 & 3 & 6 & 9 & 4 & 7 & 2 & 5 & 8 & \cdots \\ 3 & 6 & 9 & 4 & 7 & 1 & 5 & 8 & 3 & 6 & 9 & \cdots \\ 4 & 7 & 1 & 5 & 8 & 2 & 6 & 9 & 4 & 7 & 1 & \cdots \\ 5 & 8 & 2 & 6 & 9 & 3 & 7 & 1 & 5 & 8 & 2 & \cdots \\ 6 & 9 & 3 & 7 & 1 & 4 & 8 & 2 & 6 & 9 & 3 & \cdots \\ 7 & 1 & 4 & 8 & 2 & 5 & 9 & 3 & 7 & 1 & 4 & \cdots \\ 8 & 2 & 5 & 9 & 3 & 6 & 1 & 4 & 8 & 2 & 5 & \cdots \\ 9 & 3 & 6 & 1 & 4 & 7 & 2 & 5 & 9 & 3 & 6 & \cdots \end{pmatrix}.$$

Clearly, c is a r-hued 9-coloring of G, so we have $\chi_r(G) \leq 9$. Hence $\chi_r(G) = 9$.

(iii) If $n \in \{18, 25, 26, 27, 33, 34\}$, then there exist non-negative integers x and y such that $n = 8x + 9y$. Similar as (i) and (ii), we have $\chi_r(G) \geq 9$. We consider the coloring c of G with $c(u_i, v_1) \equiv i(mod\ 8)$ and $c(u_i, v_j) \equiv (c(u_i, v_{j-1}) + 3)(mod\ 8)$ for $j \geq 2$ and $1 \leq i \leq 8x$ as following:

$$\begin{pmatrix} 1 & 4 & 7 & 2 & 5 & 8 & 3 & 6 & 1 & 4 & 7 & \cdots \\ 2 & 5 & 8 & 3 & 6 & 1 & 4 & 7 & 2 & 5 & 8 & \cdots \\ 3 & 6 & 1 & 4 & 7 & 2 & 5 & 8 & 3 & 6 & 1 & \cdots \\ 4 & 7 & 2 & 5 & 8 & 3 & 6 & 1 & 4 & 7 & 2 & \cdots \\ 5 & 8 & 3 & 6 & 1 & 4 & 7 & 2 & 5 & 8 & 3 & \cdots \\ 6 & 1 & 4 & 7 & 2 & 5 & 8 & 3 & 6 & 1 & 4 & \cdots \\ 7 & 2 & 5 & 8 & 3 & 6 & 1 & 4 & 7 & 2 & 5 & \cdots \\ 8 & 3 & 6 & 1 & 4 & 7 & 2 & 5 & 8 & 3 & 6 & \cdots \end{pmatrix},$$

and $c(u_i, v_1) \equiv i(mod\ 9)$ and $c(u_i, v_j) \equiv (c(u_i, v_{j-1}) + 3)(mod\ 9)$ for $j \geq 2$ and $8x + 1 \leq i \leq n$ as following:

$$\begin{pmatrix} 1 & 4 & 7 & 2 & 5 & 8 & 3 & 6 & 1 & 4 & 7 & \cdots \\ 2 & 5 & 8 & 3 & 6 & 9 & 4 & 7 & 2 & 5 & 8 & \cdots \\ 3 & 6 & 9 & 4 & 7 & 1 & 5 & 8 & 3 & 6 & 9 & \cdots \\ 4 & 7 & 1 & 5 & 8 & 2 & 6 & 9 & 4 & 7 & 1 & \cdots \\ 5 & 8 & 2 & 6 & 9 & 3 & 7 & 1 & 5 & 8 & 2 & \cdots \\ 6 & 9 & 3 & 7 & 1 & 4 & 8 & 2 & 6 & 9 & 3 & \cdots \\ 7 & 1 & 4 & 8 & 2 & 5 & 9 & 3 & 7 & 1 & 4 & \cdots \\ 8 & 2 & 5 & 9 & 3 & 6 & 1 & 4 & 8 & 2 & 5 & \cdots \\ 9 & 3 & 6 & 1 & 4 & 7 & 2 & 5 & 9 & 3 & 6 & \cdots \end{pmatrix}.$$

Clearly, c is a r-hued 9-coloring of G, so we have $\chi_r(G) \leq 9$. Hence $\chi_r(G) = 9$.

Subcase 3.4.3 Suppose that $n \in \{5, 10, 11, 12, 13, 19, 20\}$.

(i) For $n = 5$, with the same reason in Subcase 3.3, we obtain that $\chi_r(G) \geq 10$. Let c be the coloring with $c(u_i, v_j) \equiv (2i + j)(mod\ 10)$. Then it is not difficult to verify that c is a r-hued 10-coloring of G, so we have $\chi_r(G) \leq 10$. Hence $\chi_r(G) = 10$.

(ii) For $n = 10$, if we use five colors to color the first column, WLOG, assume that $c(u_i, v_1) \equiv i(mod\ 5)$ for $1 \leq i \leq 10$. Then, by the definition of r-hued coloring, we have $c(u_1, v_2) = 6$, $c(u_2, v_2) = 7$, $c(u_3, v_2) = 8$, $c(u_4, v_2) = 9$, and $c(u_5, v_2) = 10$, so we need at least ten colors in this coloring.

If we use six colors to color the first column, WLOG, assume that $c(u_i, v_1) \equiv i(mod\ 6)$ for $1 \leq i \leq 6$. Then we have $c(u_7, v_1) = 2, c(u_8, v_1) = 3$, $c(u_9, v_1) = 4, c(u_{10}, v_1) = 5$, $c(u_1, v_2) = 6$, $c(u_2, v_2) = 7$, $c(u_3, v_2) = 8$, $c(u_{10}, v_2) = 9$, and $c(u_9, v_2) = 10$.

If we use seven colors to color the first column, WLOG, assume that $c(u_i, v_1) \equiv i(mod\ 7)$ for $1 \leq i \leq 7$. Then $c(u_8, v_1) = 3, c(u_9, v_1) = 4$, $c(u_{10}, v_1) = 5$, $c(u_1, v_2) = 6$, $c(u_2, v_2) = 7$, $c(u_3, v_2) = 8$, $c(u_4, v_2) = 1$, $c(u_5, v_2) = 2$, $c(u_6, v_2) = 9$, and $c(u_7, v_2) = 10$.

If we use eight colors to color the first column, WLOG, assume that $c(u_i, v_1) \equiv i(mod\ 8)$ for $1 \leq i \leq 8$. Then we have $c(u_9, v_1) = 4$ and $c(u_{10}, v_1) = 5$, and thus we obtain that $c(u_1, v_2) = 6$, $c(u_2, v_2) = 7$, $c(u_3, v_2) = 8$, $c(u_4, v_2) = 1$, $c(u_5, v_2) = 2$, $c(u_6, v_2) = 3$, $c(u_7, v_2) = 9$ and $c(u_8, v_2) = 10$.

If we use nine colors to color the first column, WLOG, assume that $c(u_i, v_1) \equiv i(mod\ 9)$ for $1 \le i \le 9$. Then we have $c(u_{10}, v_1) = 5$ and $c(u_1, v_2) \in \{4, 6, 7, 8\}$ by the definition of r-hued coloring. We can obtain that $\chi_r(G) \ge 10$ as upper paragraph similarly.

We consider the coloring c with $c(u_i, v_j) \equiv (2i + j)(mod\ 10)$. It is not difficult to verify that c is r-hued 10-coloring of G, so we have $\chi_r(G) \le 10$. Hence $\chi_r(G) = 10$.

(iii) For $n \in \{11, 12, 13, 19, 20\}$, similar as (i) and (ii), we can obtain that $\chi_r(G) \ge 10$. If $n = 11$, then we consider the coloring c of G as following:

$$\begin{pmatrix} 1 & 4 & 7 & 10 & 2 & 5 & 8 & \cdots \\ 2 & 5 & 8 & 11 & 3 & 6 & 9 & \cdots \\ 3 & 6 & 9 & 1 & 4 & 7 & 10 & \cdots \\ 4 & 7 & 10 & 2 & 5 & 8 & 11 & \cdots \\ 5 & 8 & 11 & 3 & 6 & 9 & 1 & \cdots \\ 6 & 9 & 1 & 4 & 7 & 10 & 2 & \cdots \\ 7 & 10 & 2 & 5 & 8 & 11 & 3 & \cdots \\ 8 & 11 & 3 & 6 & 9 & 1 & 4 & \cdots \\ 9 & 1 & 4 & 7 & 10 & 2 & 5 & \cdots \\ 10 & 2 & 5 & 8 & 11 & 3 & 6 & \cdots \\ 11 & 3 & 6 & 9 & 1 & 4 & 7 & \cdots \end{pmatrix}.$$

It is not difficult to verify that c is a r-hued 11-coloring of G, so we have $\chi_r(G) \le 11$.

If $n = 12$, then we consider the coloring c of G as following:

$$\begin{pmatrix} 1 & 4 & 7 & 10 & 2 & 5 & 8 & \cdots \\ 2 & 5 & 8 & 11 & 3 & 6 & 9 & \cdots \\ 3 & 6 & 9 & 1 & 4 & 7 & 10 & \cdots \\ 4 & 7 & 10 & 2 & 5 & 8 & 11 & \cdots \\ 5 & 8 & 11 & 3 & 6 & 9 & 1 & \cdots \\ 6 & 9 & 1 & 4 & 7 & 10 & 2 & \cdots \\ 7 & 10 & 2 & 5 & 8 & 11 & 3 & \cdots \\ 8 & 11 & 3 & 6 & 9 & 1 & 4 & \cdots \\ 9 & 1 & 4 & 7 & 10 & 2 & 5 & \cdots \\ 10 & 2 & 5 & 8 & 11 & 3 & 6 & \cdots \\ 11 & 3 & 6 & 9 & 1 & 4 & 7 & \cdots \\ 6 & 9 & 1 & 4 & 7 & 10 & 2 & \cdots \end{pmatrix}.$$

Clearly, c is a r-hued 11-coloring of G, so we obtain that $\chi_r(G) \le 11$.

If $n = 13$, then we consider the coloring c of G as following:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 1 & 2 & \cdots \\ 3 & 4 & 5 & 6 & 3 & 4 & \cdots \\ 5 & 6 & 7 & 8 & 5 & 6 & \cdots \\ 7 & 8 & 9 & 10 & 7 & 8 & \cdots \\ 9 & 10 & 1 & 2 & 9 & 10 & \cdots \\ 1 & 2 & 3 & 4 & 1 & 2 & \cdots \\ 3 & 4 & 5 & 6 & 3 & 4 & \cdots \\ 5 & 6 & 7 & 8 & 5 & 6 & \cdots \\ 7 & 8 & 9 & 10 & 7 & 8 & \cdots \\ 9 & 10 & 1 & 2 & 9 & 10 & \cdots \\ 4 & 3 & 6 & 5 & 4 & 3 & \cdots \\ 6 & 5 & 8 & 7 & 6 & 5 & \cdots \\ 8 & 7 & 10 & 9 & 8 & 7 & \cdots \end{pmatrix}.$$

It is not difficult to verify that c is a r-hued 10-coloring of G, so we have $\chi_r(G) \le 10$. Hence $\chi_r(G) = 10$.

If $n = 19$, then we consider the coloring c of G as following:

$$\begin{pmatrix} 1 & 4 & 7 & 1 & 4 & \cdots \\ 2 & 5 & 8 & 2 & 5 & \cdots \\ 3 & 6 & 9 & 3 & 6 & \cdots \\ 4 & 7 & 1 & 4 & 7 & \cdots \\ 5 & 8 & 2 & 5 & 8 & \cdots \\ 6 & 9 & 3 & 6 & 9 & \cdots \\ 7 & 1 & 4 & 7 & 1 & \cdots \\ 8 & 2 & 5 & 8 & 2 & \cdots \\ 9 & 3 & 6 & 9 & 3 & \cdots \\ 1 & 4 & 7 & 1 & 4 & \cdots \\ 2 & 5 & 8 & 2 & 5 & \cdots \\ 3 & 6 & 9 & 3 & 6 & \cdots \\ 4 & 7 & 10 & 4 & 7 & \cdots \\ 5 & 8 & 1 & 5 & 8 & \cdots \\ 6 & 9 & 2 & 6 & 9 & \cdots \\ 7 & 10 & 3 & 7 & 10 & \cdots \\ 8 & 1 & 4 & 8 & 1 & \cdots \\ 9 & 2 & 5 & 9 & 2 & \cdots \\ 10 & 3 & 6 & 10 & 3 & \cdots \end{pmatrix}.$$

Clearly, it is a r-hued 10-coloring of G, so we have $\chi_r(G) \le 10$. Hence $\chi_r(G) = 10$.

If $n = 20$, then the coloring c with $c(u_i, v_j) \equiv (2i + j)(mod\ 10)$ is a r-hued 10-coloring of G, so we have $\chi_r(G) \le 10$. Hence $\chi_r(G) = 10$.

In a word, the proof of our result is completed. □

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