

# The Nonlinear HSS-like Iterative Method for Absolute Value Equations

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**Abstract**—The Picard-HSS iterative method is proposed to solve the absolute value equation (AVE). To further improve its performance, a nonlinear HSS-like iterative method is proposed. Compared to that the Picard-HSS method is an inner-outer double-layer iterative scheme, the proposed nonlinear HSS-like iteration is only a monolayer iterative method and the iteration vector could be updated timely. Some numerical experiments are used to demonstrate that the nonlinear HSS-like method is feasible, robust and effective.

**Index Terms**—absolute value equation, nonlinear HSS-like iteration, fixed point iteration, positive definite.

## I. INTRODUCTION

THE solution of the absolute value equation (AVE) with the following form is considered:

$$Ax - |x| = b. \quad (1)$$

Here,  $A \in \mathbb{R}^{n \times n}$ ,  $x, b \in \mathbb{R}^n$  and  $|x|$  denotes the component-wise absolute value of vector  $x$ , i.e.,

$$|x| = (|x_1|, |x_2|, \dots, |x_n|)^T.$$

The AVE (1) is a special case of the generalized absolute value equation (GAVE) of the type

$$Ax - B|x| = b, \quad (2)$$

where  $A, B \in \mathbb{R}^{m \times n}$  and  $x, b \in \mathbb{R}^m$ . The GAVE (2) was introduced in [1] and investigated in a more general context in [2]–[4]. Recently, these problems have been investigated in the literature [4]–[8].

The AVE (1) arises in linear programs, quadratic programs, bimatrix games and other problems, which can all be reduced to a linear complementarity problem (LCP) [9]–[11], and the LCP is equivalent to the AVE (1). This implies that AVE is NP-hard in its general form [4], [6], [7]. Beside, if  $B = 0$ , then the generalized AVE (2) reduces to the system of linear equations  $Ax = b$ , which have many applications in scientific computation [7], [12], [13].

The main research of the AVE includes two aspects: one is the theoretical analysis, which focuses on the theorem of alternatives, various equivalent reformulations, and the existence of solutions; see [1], [2], [5], [14]. And the other is how to solve the AVE. We mainly pay attention to the latter.

In the last decade, based on the fact that the LCP is equivalent to the AVE and the special structure of AVE,

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a large variety of methods for solving AVE (1) can be found in the literature; See [3], [7], [8], [15]. The finite succession of linear programs (SLP) is established in [6], [15], which arise from a reformulation of the AVE as the minimization of a piecewise-linear concave function on a polyhedral set and solving the latter by successive linearization. Mangasarian [16] and Caccetta [14] present the semi-smooth Newton method and the smoothing Newton method to solve the AVE, respectively. In 2015, Haghani [17] propose an improved Newton method with two-step form, called Traub's method, whose effectiveness is better than that of Mangasarian in [16]. To utilize the semi-smooth property, the generalized Newton method [18], the modified generalized Newton method [19], the improved generalized Newton method [20] and the inexact semi-smooth Newton algorithm [21] are further put forward for solving the AVE. These methods are all globally convergent under certain conditions.

Recently, the Picard-HSS iterative method is proposed to solve AVE by Salkuyeh in [22], which is originally designed to solve weakly nonlinear systems [23] and its generalizations are also paid attention [24], [25]. The sufficient conditions to guarantee the convergence and some numerical experiments are given to show the effectiveness of the method. However, the numbers of the inner HSS iteration steps are often problem-dependent and difficult to be determined in actual computations. Moreover, the iteration vector can not be updated timely. In this paper, we present the nonlinear HSS-like iterative method to overcome the defect of the mentioned above method in [22].

The rest of this paper is organized as follows. In Section II, the HSS and Picard-HSS iteration methods are reviewed. In Section III, the nonlinear HSS-like iterative method for solving AVE (1) is described. Numerical experiments are presented in Section IV, to further shown the feasibility and effectiveness of the nonlinear HSS-like method. Finally, some conclusions are draw in Section V.

## II. THE HSS AND PICARD-HSS ITERATION METHODS

IN this section, the HSS iterative method for solving the non-Hermitian linear systems and the Picard-HSS iterative method for solving the AVE (1) are reviewed.

Let  $A \in \mathbb{R}^{n \times n}$  be a non-Hermitian positive definite matrix,  $B \in \mathbb{R}^{n \times n}$  be a zero matrix, the GAVE (2) reduced to the non-Hermitian system of linear equations

$$Ax = b. \quad (3)$$

Because any square matrix  $A$  possesses a Hermitian and skew-Hermitian splitting (HSS)

$$A = H + S, \quad H = \frac{1}{2}(A + A^H) \quad \text{and} \quad S = \frac{1}{2}(A - A^H), \quad (4)$$

the following HSS iterative method is first introduced by Bai, Golub and Ng in [26] for solving the non-Hermitian positive definite system of linear equations (3).

**Algorithm 1.** (The HSS iterative method.)

Given an initial guess  $x^{(0)} \in \mathbb{R}^n$ , compute  $x^{(k)}$  for  $k = 0, 1, 2, \dots$  using the following iterative scheme until  $\{x^{(k)}\}_{k=0}^{\infty}$  converges,

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b, \end{cases}$$

where  $\alpha$  is a positive constant and  $I$  is the identity matrix.

When the Hermitian part  $H = \frac{1}{2}(A + A^H)$  of the matrix  $A$  is positive definite, Bai et al. proved that the spectral radius of the HSS iteration matrix is less than 1 for any positive parameters  $\alpha$ , i.e., the HSS iterative method is unconditionally convergent; see [26].

For the convenience of the subsequent discussion, the AVE (1) can be rewritten as its equivalent form:

$$Ax = f(x), \quad f(x) = |x| + b.$$

Recalling that the linear term  $Ax$  and the nonlinear term  $f(x) = |x| + b$  are well separated and the Picard iterative method is a fixed-point iteration, the Picard iteration

$$Ax^{(k+1)} = f(x^{(k)}), \quad k = 0, 1, \dots,$$

can be used to solve the AVE (1). When the matrix  $A \in \mathbb{R}^{n \times n}$  is large sparse and positive definite, the next iteration  $x^{(k+1)}$  may be inexactly computed by HSS iteration. This naturally lead to the following iterative method proposed in [22] for solving the AVE (1).

**Algorithm 2.** (The Picard-HSS iterative method)

Let  $A \in \mathbb{R}^{n \times n}$  be a sparse and positive definite matrix,  $H = \frac{1}{2}(A + A^H)$  and  $S = \frac{1}{2}(A - A^H)$  be its Hermitian and skew-Hermitian parts respectively. Given an initial guess  $x^{(0)} \in \mathbb{R}^n$  and a sequence  $\{\ell_k\}_{k=0}^{\infty}$  of positive integers, compute  $x^{(k+1)}$  for  $k = 0, 1, 2, \dots$  using the following iterative scheme until  $\{x^{(k)}\}$  satisfies the stopping criterion:

(1) Set  $x^{(k,0)} := x^{(k)}$ ;

(2) For  $\ell = 0, 1, \dots, \ell_k - 1$ , solve the following linear systems to obtain  $x^{(k,\ell+1)}$ :

$$\begin{cases} (\alpha I + H)x^{(k,\ell+\frac{1}{2})} = (\alpha I - S)x^{(k,\ell)} + |x^{(k)}| + b, \\ (\alpha I + S)x^{(k,\ell+1)} = (\alpha I - H)x^{(k,\ell+\frac{1}{2})} + |x^{(k)}| + b, \end{cases}$$

where  $\alpha$  is a given positive constant and  $I$  is the identity matrix;

(3) Set  $x^{(k+1)} := x^{(k,\ell_k)}$ .

The advantage of the Picard-HSS iterative method is obvious. Firstly, the two linear sub-systems in all inner HSS iterations have the same shifted Hermitian coefficient matrix  $\alpha I + H$  and shifted skew-Hermitian coefficient matrix  $\alpha I + S$ , which are constant with respect to the iteration index  $k$ . Secondly, as the coefficient matrix  $\alpha I + H$  and  $\alpha I + S$  are Hermitian and skew-Hermitian respectively, the first sub-system can be solved exactly by making use of the Cholesky factorization and the second one by the LU factorization. The lastly, these two sub-systems can be solve approximately by the conjugate gradient method and a Krylov subspace method like GMRES, respectively; see [22], [23].

III. THE NONLINEAR HSS-LIKE ITERATIVE METHOD

IN the Picard-HSS iteration, the numbers  $\ell_k, k = 0, 1, 2, \dots$  of the inner HSS iteration steps are often problem-dependent and difficult to be determined in actual computations [23]. Moreover, the iteration vector can not be updated timely. Thus, to avoid these defect and still preserve the advantages of the Picard-HSS iterative method, based on the HSS (4) and the nonlinear fixed-point equations

$$(\alpha I + H)x = (\alpha I - S)x + |x| + b,$$

and

$$(\alpha I + S)x = (\alpha I - H)x + |x| + b,$$

the following nonlinear HSS-like iterative method is proposed to solve the AVE (1).

**Algorithm 3.** (The nonlinear HSS-like iterative method.)

Let  $A \in \mathbb{R}^{n \times n}$  be a sparse and positive definite matrix,  $H = \frac{1}{2}(A + A^H)$  and  $S = \frac{1}{2}(A - A^H)$  be its Hermitian and skew-Hermitian parts, respectively. Given an initial guess  $x^{(0)} \in \mathbb{R}^n$ , compute  $x^{(k+1)}$  for  $k = 0, 1, 2, \dots$  using the following iterative scheme until  $\{x^{(k)}\}$  satisfies the stopping criterion:

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + |x^{(k)}| + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + |x^{(k+\frac{1}{2})}| + b, \end{cases}$$

where  $\alpha$  is a given positive constant and  $I$  is the identity matrix.

It is obvious that both  $x$  and  $|x|$  in the second step are updated in the nonlinear HSS-like iteration, but only  $x$  is updated in the Picard-HSS iteration. Furthermore, the nonlinear HSS-like iteration is a monolayer iterative scheme, and the Picard-HSS is an inner-outer double-layer iterative scheme.

To obtain a one-step form of the nonlinear HSS-like iteration, the following symbols are introduced

$$\begin{aligned} U(x) &= (\alpha I + H)^{-1}((\alpha I - S)x + |x| + b), \\ V(x) &= (\alpha I + S)^{-1}((\alpha I - H)x + |x| + b), \end{aligned}$$

and

$$\psi(x) := V \circ U(x) = V(U(x)).$$

Then the nonlinear HSS-like iterative scheme can be equivalently expressed as

$$x^{(k+1)} = \psi(x^{(k)}).$$

The Ostrowski theorem, i.e., Theorem 10.1.3 in [27], gives a local convergence theory about a one-step stationary nonlinear iteration. Based on this, Bai et al. established the local convergence theory for the nonlinear HSS-like iterative method in [23]. However, this convergence theory has a strict requirement that  $f(x) = |x| + b$  must be  $F$ -differentiable at a point  $x^* \in \mathbb{D}$  such that  $Ax^* - |x^*| = b$ . Obviously, the absolute value function  $|x|$  is non-differentiable. Thus, the convergence analysis of the nonlinear HSS-like iterative method for solving weakly nonlinear linear systems is unsuitable for solving AVE, and need further discuss.

At the end of this section, we remark that the nonlinear HSS-like iterative method can be alternatively reformulated into residual-updating form as follows.

**Algorithm 4.** (The residual-updating variant of the nonlinear HSS-like method) Given an initial guess  $x^{(0)} \in \mathbb{D} \subset \mathbb{R}^n$ , compute  $x^{(k+1)}$  for  $k = 0, 1, 2, \dots$  using the following iterative procedure until  $\{x^{(k)}\}$  satisfies the stopping criterion:

- (1) Set  $r^{(k)} := |x^{(k)}| + b - Ax^{(k)}$ ,
- (2) Solve  $(\alpha I + H)v = r^{(k)}$ ,
- (3) Set  $x^{(k+\frac{1}{2})} = x^{(k)} + v$ ,  $r^{(k)} := |x^{(k+\frac{1}{2})}| + b - Ax^{(k+\frac{1}{2})}$ ,
- (4) Solve  $(\alpha I + S)v = r^{(k)}$ ,
- (5) Set  $x^{(k+1)} = x^{(k+\frac{1}{2})} + v$ ,

where  $\alpha$  is a given positive constant and  $I$  is the identity matrix.

#### IV. NUMERICAL EXPERIMENTS

**I**N this section, the numerical properties of the Picard, Picard-HSS and nonlinear HSS-like methods are examined and compared experimentally by a suit of test problems.

All the tests are performed in MATLAB R2013a on Intel(R) Core(TM) i5-3470 CPU 3.20 GHz and 8.00 GB of RAM, with machine precision  $10^{-16}$ , and terminated when the current residual satisfies

$$\frac{\|Ax^{(k)} - |x^{(k)}| - b\|_2}{\|b\|_2} \leq 10^{-5},$$

where  $x^{(k)}$  is the computed solution by each of the methods at iteration  $k$ , and a maximum number of the iterations 500 is used.

In addition, the stopping criterion for the inner iterations of the Picard-HSS method is set to be

$$\frac{\|b^{(k)} - As^{(k, \ell_k)}\|_2}{\|b^{(k)}\|_2} \leq \eta_k,$$

where  $b^{(k)} = |x^{(k)}| + b - Ax^{(k)}$ ,  $s^{(k, \ell_k)} = x^{(k, \ell_k)} - x^{(k, \ell_k - 1)}$ ,  $\ell_k$  is the number of the inner iteration steps and  $\eta_k$  is the prescribed tolerance for controlling the accuracy of the inner iterations at the  $k$ -th outer iteration. If  $\eta_k$  is fixed for all  $k$ , then it is simply denoted by  $\eta$ . Here, we take  $\eta = 0.1$ .

The first subsystem with the Hermitian positive definite coefficient matrix  $(\alpha I + H)$  in (3) is solved by the Cholesky factorization, and the second subsystem with the skew-Hermitian coefficient matrix  $(\alpha I + S)$  in (3) is solved by the LU factorization.

The optimal parameters employed in the Picard-HSS and nonlinear HSS-like iteration methods have been obtained experimentally. In fact, the experimentally found optimal parameters are the ones resulting in the least numbers of iterations and CPU times [22]. As mentioned in [23] the computation of the optimal parameter is often problem-dependent and generally difficult to be determined.

We consider the two-dimensional convection-diffusion equation

$$\begin{cases} -(u_{xx} + u_{yy}) + q(u_x + u_y) + pu = f(x, y), & (x, y) \in \Omega, \\ u(x, y) = 0, & (x, y) \in \partial\Omega, \end{cases}$$

where  $\Omega = (0, 1) \times (0, 1)$ ,  $\partial\Omega$  is its boundary,  $q$  is a positive constant used to measure the magnitude of the diffusive term and  $p$  is a real number.

Let  $h = 1/(m + 1)$  and  $Re = (qh)/2$  denote the equidistant step size and the mesh Reynolds number, respectively. We use the five-point finite difference scheme to the diffusive term and the central difference scheme to the convective term. Then we get a system of linear equations  $Ax = d$ , where  $A$  is a matrix of order  $n = m^2$  of the form

$$A = T_x \otimes I_m + I_m \otimes T_y + pI_n, \tag{5}$$

with

$$T_x = \text{tridiag}(t_2, t_1, t_3)_{m \times m} \text{ and } T_y = \text{tridiag}(t_2, 0, t_3)_{m \times m},$$

where  $t_1 = 4$ ,  $t_2 = -1 - Re$ ,  $t_3 = -1 + Re$ ,  $I_m$  and  $I_n$  are the identity matrices of order  $m$  and  $n$  respectively,  $\otimes$  means the Kronecker product.

In our numerical experiments, the matrix  $A$  in AVE (1) is defined by (5) with different values of  $q$  ( $q = 0, 1, 10, 100$  and  $1000$ ) and  $p$  ( $p = 0$  and  $0.5$ ). It is easy to find that the matrix  $A$  is in general non-symmetric positive definite for any nonnegative number  $q$  [22]. The zero vector is used as the initial guess, and the right-hand side vector  $b$  of AVE (1) is taken in such a way that the vector  $x = (x_1, x_2, \dots, x_n)^T$  with  $x_k = (-1)^k i$  ( $k = 1, 2, \dots, n$ ) is the exact solution, where  $i$  denotes the imaginary unit.

TABLE I: The optimal parameters values  $\alpha$  ( $p=0$ ).

Optimal parameters		$m=10$	$m=20$	$m=40$	$m=80$
$q=0$	HSS-like	1.3	1.0	1.0	1.0
	Picard-HSS	1.1	0.5	0.2	0.1
$q=1$	HSS-like	1.4	1.0	1.0	1.0
	Picard-HSS	1.1	0.6	0.3	0.2
$q=10$	HSS-like	1.7	1.1	1.0	1.0
	Picard-HSS	1.6	0.8	0.4	0.2
$q=100$	HSS-like	2.5	2.7	1.7	1.2
	Picard-HSS	2.4	2.7	1.8	0.9

TABLE II: The optimal parameters values  $\alpha$  ( $p=0.5$ ).

Optimal parameters		$m=10$	$m=20$	$m=40$	$m=80$
$q=0$	HSS-like	2.4	2.2	2.1	2.0
	Picard-HSS	2.2	2.0	1.8	1.8
$q=1$	HSS-like	2.4	2.2	2.1	2.0
	Picard-HSS	2.3	2.0	1.8	1.8
$q=10$	HSS-like	2.6	2.3	2.2	2.1
	Picard-HSS	2.4	2.3	2.0	1.9
$q=100$	HSS-like	3.4	2.9	2.3	2.3
	Picard-HSS	3.5	3.0	2.3	2.1

The numerical results of the Picard, Picard-HSS and nonlinear HSS-like iterations are list in Tables III and IV, and the experimentally optimal parameters used in the Picard-HSS and nonlinear HSS-like iterations are those given in Tables I and II. We give the elapsed CPU time in seconds for the convergence (denoted as CPU), the norm of absolute residual vectors (denoted as RES), and the number of outer, inner and total iteration steps (outer and inner iterations only for Picard-HSS) for the convergence (denoted as  $IT_{out}$ ,  $IT_{int}$  and  $IT$ , respectively). The number of outer iteration steps for Picard-HSS and the number of iteration steps for Picard and HSS-like iteration methods larger than 500 are simply listed by the symbol "-".

From these two tables, we see that both the HSS-like and Picard-HSS methods can successfully produced approximate solution to the AVE for all of the problem-scales  $n = m^2$  and the convective measurements  $q$ , while the Picard iteration converges only for some special cases. Here, it is necessary to mention that the shifted matrices  $\alpha I + H$  and  $\alpha I + S$  are usually more well-conditioned than the matrix  $A$  [22].

For the convergent cases, the number of iteration steps for the Picard and HSS-like methods and the number of inner iteration steps for the Picard-HSS method are increase rapidly with the increasing of problem-scale, while the number of outer iteration steps is fixed. The CPU time also increases rapidly with the increasing of the problem-scale for all iteration methods.

When the convective measurements  $q$  become large, for all iterative method, both the number of iteration steps (except outer iteration for Picard-HSS) and the amount of CPU times decrease slightly.

Clearly, the iteration steps of the nonlinear HSS-like method are more robust than those of the Picard-HSS, and the iteration steps of the Picard method are more than 500, then the nonlinear HSS-like method performs much better than the Picard-HSS in terms of iteration step; In terms of CPU time, the situation is almost the same, i.e., the nonlinear HSS-like iterative method is the

most time-efficient in the convergent cases. Therefore, the nonlinear HSS-like method are the winners for solving this test problem when the convective measurements  $q$  is not large.

V. CONCLUSIONS

**I**N this paper, the nonlinear HSS-like iterative method is proposed to solve the absolute value equation (AVE), which is based on two aspects: the first is the separable property of the linear term  $Ax$  and nonlinear term  $|x| + b$ , and the second is the Hermitian and skew-Hermitian splitting of the involved matrix  $A$ . Compared to that the Picard-HSS iterative scheme is an inner-outer double-layer iterative scheme, the new nonlinear HSS-like iteration is a monolayer iterative method and the iteration vector could be updated timely. Numerical experiments have shown that the nonlinear HSS-like method is feasible, robust and efficient nonlinear solver.

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TABLE III: Numerical results for test problems with different values of  $m$  and  $q$  ( $p = 0$ ,  $RES(\times 10^{-6})$ ).

Methods		$m=10$	$m=20$	$m=40$	$m=80$	
$q=0$	Picard	IT	-	-	-	
		CPU	-	-	-	
		RES	-	-	-	
	Picard-HSS	IT <sub>out</sub>	5	5	5	5
		IT <sub>int</sub>	7.2	13.8	33	62.6
		IT	36	69	165	313
		CPU	<b>0.0084</b>	0.0250	0.2310	2.0708
		RES	5.2907	7.1401	7.9627	9.1458
		HSS-like	IT	27	35	65
CPU	0.0375	<b>0.0146</b>	<b>0.1016</b>	<b>0.6085</b>		
RES	9.4084	8.7487	9.9395	9.9502		
$q=1$	Picard	IT	-	-	-	
		CPU	-	-	-	
		RES	-	-	-	
	Picard-HSS	IT <sub>out</sub>	5	5	5	5
		IT <sub>int</sub>	7.2	13.6	27	64.8
		IT	36	68	135	324
		CPU	0.0050	0.0317	0.2527	3.0404
		RES	6.3073	8.0703	7.7121	9.3360
		HSS-like	IT	28	38	65
CPU	<b>0.0044</b>	<b>0.0199</b>	<b>0.1343</b>	<b>0.8436</b>		
RES	8.7445	9.5272	9.9148	9.9588		
$q=10$	Picard	IT	-	-	-	
		CPU	-	-	-	
		RES	-	-	-	
	Picard-HSS	IT <sub>out</sub>	5	5	5	5
		IT <sub>int</sub>	3.8	7	13.2	25.4
		IT	19	35	66	127
		CPU	0.0031	<b>0.0174</b>	0.1285	1.2305
		RES	2.6888	4.0994	5.9529	7.1369
		HSS-like	IT	17	32	51
CPU	<b>0.0029</b>	0.0176	<b>0.1077</b>	<b>0.8857</b>		
RES	7.8979	7.2166	9.3825	9.8324		

TABLE IV: Numerical results for test problems with different values of  $m$  and  $q$  ( $p = 0.5$ ,  $RES(\times 10^{-6})$ ).

Methods		$m=10$	$m=20$	$m=40$	$m=80$	
$q=0$	Picard	IT	9	-	-	
		CPU	0.0010	-	-	
		RES	0.0016	-	-	
	Picard-HSS	IT <sub>out</sub>	5	5	5	5
		IT <sub>int</sub>	7	14.6	35	66.4
		IT	35	73	175	332
		CPU	0.0040	0.0261	0.2420	2.2039
		RES	5.4444	7.4483	8.1466	9.3423
		HSS-like	IT	29	38	36
CPU	<b>0.0037</b>	<b>0.0155</b>	<b>0.0590</b>	<b>0.2849</b>		
RES	7.7828	8.0756	9.6565	8.8724		
$q=1$	Picard	IT	9	-	-	
		CPU	0.0011	-	-	
		RES	0.0011	-	-	
	Picard-HSS	IT <sub>out</sub>	5	5	5	5
		IT <sub>int</sub>	7.8	14.4	28	42
		IT	39	72	140	210
		CPU	0.0052	0.0335	0.2612	1.9946
		RES	4.2330	5.3548	8.8367	8.5786
		HSS-like	IT	29	42	38
CPU	<b>0.0044</b>	<b>0.0218</b>	<b>0.0824</b>	<b>0.4113</b>		
RES	8.1442	8.5129	9.8553	8.2976		
$q=10$	Picard	IT	7	-	-	
		CPU	0.0009	-	-	
		RES	0.1525	-	-	
	Picard-HSS	IT <sub>out</sub>	5	5	5	5
		IT <sub>int</sub>	4	7	13.6	25
		IT	20	35	68	125
		CPU	0.0032	<b>0.0179</b>	0.1379	1.2244
		RES	1.5905	5.9853	5.1449	8.9996
		HSS-like	IT	18	34	45
CPU	<b>0.0030</b>	0.0183	<b>0.0960</b>	<b>0.4848</b>		
RES	8.1728	6.0961	8.8821	9.2731		

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