On Optimization Criteria of Fuzzy Linear Programming

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***Abstract*—In this paper, we consider the expansion of linear inequality and some related theorems in the fuzzy case, and propose an optimization criterion of fuzzy linear problems. Firstly, the fuzzy version of Tucker's lemma is carried out, and then based on the fuzzy Tucker's lemma, it is proved that Tucker-First existence theorm and Tucker-second existence theorem are established, thus the Motzkin's theorem is deduced. Finally, Farka's theorem is deduced to construct the optimization criterion of fuzzy linear programming by fuzzy Farka's theorem.**

***Index Terms*—Fuzzy numbers, Linear optimization, Duality method.**

I. Introduction

AFTER leading the concept of fuzzy set in [28], many applications of fuzzy sets have been developed. One of them is fuzzy optimization, it explains any imprecision in the optimization problems. The fuzzy optimization problems were introduced by Bellman and Zadeh in [1]. They think that a fuzzy decision can be considered as a fuzzy target and a constraint problem at the intersection. Then, many articles about fuzzy optimization were published. In this direction, we are here to mention that some of the recent work have been carried out , [15] used unified approach to investigate Fuzzy mathematical programming. The large scale fuzzy and possibilistic optimization problems have been studied by Lodwick and Bachman in [8]. Lodwick et al. in [9] and [10] have studied distinctions and relation-ships between fuzzy and possibilistic, individually.Barkley and Abdallah [2] have applied the Monte Carlo methods in fuzzy theory. [3] basically introduced the main models and methods of fuzzy linear programming. Several authors put forward the optimality conditions for the fuzzy optimization problem (see [22] , [23] and [12]).

By using the fuzzy set theory in [7] [19], the fuzzy optimization problem has been widely studied since 70s. The first method to solve the fuzzy linear programming problem in the method of [27], [4]. In [27], the main focus is the fuzzy goal and the application of the strategic theory to the fuzzy decision problem. However, the focus is on the concepts of mathematical programming and using the level set, and some classical results are generalized to include fuzzy constraints and objective functions. In [4], the fuzzy set theory is applied to the fuzzy linear programming problem, which proves how the fuzzy linear programming problem is solved without increasing the computation amount.

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At present, there has been a great development in both the optimization field and the fuzzy field. In [24], we have learned about the new method to solve the linear optimization problem, a new linear relaxation technique is proposed, and the problem EP is reduced to a sequence of linear programming problems by using the new linear relaxation technique. In [14], the author studied edge detection method by fuzzy C-means. In [20], the author solved the problem of low prediction accuracy in utilizing the current filter methods for adaptive neural fuzzy inference system (ANFIS) parameters learning. In [25] [13], optimization of convex and generalized convex fuzzy mappings are derived, and the author studied the fuzzy differential equations in the quotient space of fuzzy numbers. In [21], the author establishes a Stackelberg game for manufacturers in a fuzzy decision-making environment.

In this paper, we first introduce and comment on the necessary knowledge of triangular fuzzy numbers and a partial order relation, which provides a basis for judging the order relations in the future. Then, the theorems of the alternative in the linear problem is proved in the fuzzy domain. Through fuzzy linear theorems of the alternative, we get the optimization criteria for linear problems in the fuzzy case, thus the optimization of fuzzy linear problem can be solved.

II. Preliminaries

The following notations, definitions and results will be needed in the sequel.

We denote $\mathcal{K}_C$ as the family of all bounded closed intervals in $\mathbb{R}$ [5], that is,

$$\mathcal{K}_C = \{[a_L, a_R] | a_L, a_R \in \mathbb{R} \ and \ a_L \leq a_R\}.$$

A fuzzy set $\widetilde{x}$ of $\mathbb{R}$ is characterized by a membership function $\mu_{\widetilde{x}} : \mathbb{R} \to [0,1]$. For each such fuzzy set $\widetilde{x}$, we denote by $[\widetilde{x}]^\alpha = \{x \in \mathbb{R} : \mu_{\widetilde{x}}(x) \geq \alpha\}$ for any $\alpha \in (0,1]$, its α-level set. Define the set $[\widetilde{x}]^0$ by $[\widetilde{x}]^0 = \overline{\bigcup_{\alpha\in(0,1]} [\widetilde{x}]^\alpha}$, where $\overline{A}$ denotes the closure of a crisp set A. A fuzzy number $\widetilde{x}$ is a fuzzy set with non-empty bounded closed level sets $[\widetilde{x}]^\alpha = [\widetilde{x}_L(\alpha), \widetilde{x}_R(\alpha)]$ for all $\alpha \in [0,1]$, where $[\widetilde{x}_L(\alpha), \widetilde{x}_R(\alpha)]$ denotes a closed interval with the left end point $\widetilde{x}_L(\alpha)$ and the right end point $\widetilde{x}_R(\alpha)$ [6]. We denote the class of fuzzy numbers by $\mathscr{F}$.

Definition 1: [4] Let $\tau : \mathscr{F} \to \mathbb{R}$ be a function defined by

$$\tau(\widetilde{x}) = \int_0^1 \alpha[\widetilde{x}_L(\alpha) + \widetilde{x}_R(\alpha)]d\alpha.$$

for all $\widetilde{x} \in \mathscr{F}$. Then τ is called ranking value function.

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Definition 2: [4] Suppose that $\widetilde{x}$ and $\widetilde{y}$ are two fuzzy numbers. Then $\widetilde{x}$ precedes $\widetilde{y}$ ($\widetilde{x} \preceq \widetilde{y}$) if

$$\int_0^1 \alpha[\widetilde{x}_L(\alpha) + \widetilde{x}_R(\alpha)]d\alpha \leq \int_0^1 \alpha[\widetilde{y}_L(\alpha) + \widetilde{y}_R(\alpha)]d\alpha.$$

Note that the order relation $\preceq$ is reflexive and transitive. Moreover, any two elements of $\mathscr{F}$ are comparable under the ordering $\preceq$. For more details see [16], [17].

Lemma 1: [4] For $\widetilde{x}, \widetilde{y} \in \mathscr{F}$, then $\widetilde{x} \preceq \widetilde{y}$ if and only if $\tau(\widetilde{x}) \leq \tau(\widetilde{y})$.

Lemma 2: [11] Let $\widetilde{a}_i$ be fuzzy numbers and $x_i \geq 0 (i = 1, 2, \cdots, n)$ be an real numbers. Then, $\sum_{i=1}^{n} \widetilde{a}_i x_i$ is a fuzzy numbers.

Definition 3: [22] Let $\widetilde{x}$ be fuzzy numbers, If the membership function $u_{\widetilde{x}}(x)$ of the fuzzy number $\widetilde{x}$ is denoted by

$$u_{\widetilde{x}}(x) = \begin{cases} 0, & x < b, x > c, \\ \frac{x-b}{a-b}, & b \leq x \leq a, \\ \frac{c-x}{c-a}, & a < x \leq c. \end{cases}$$

Then, $\widetilde{x}$ is called a triangular fuzzy number. Furthermore, the triangular fuzzy number $\widetilde{x}$ is presented by $\widetilde{x} = (b, a, c)$, the $\alpha-$ level set of $\widetilde{a}$ is the

$$\widetilde{a}_\alpha = [(1-\alpha)b + \alpha a, (1-\alpha)c + \alpha a].$$

Definition 4: [18] According to Zadeh's extension principle, we can define:

$$(\widetilde{a} + \widetilde{b})_\alpha = [\widetilde{a}^L_\alpha + \widetilde{b}^L_\alpha, \widetilde{a}^U_\alpha + \widetilde{b}^U_\alpha],$$

$$(\lambda \cdot \widetilde{a})_\alpha = [\lambda \widetilde{a}^L_\alpha, \lambda \widetilde{a}^U_\alpha], \quad \lambda \geq 0,$$

$$(\lambda \cdot \widetilde{a})_\alpha = [\lambda \widetilde{a}^U_\alpha, \lambda \widetilde{a}^L_\alpha], \quad \lambda < 0.$$

Definition 5: [24] Let $\widetilde{a} = (a^L, a, a^U)$ be a triangular fuzzy numbers, we assume that $a^L, a, a^U \geq 0$.The square and multiplicative inverse of $\widetilde{a}$ using its α-level sets are define as

$$(\widetilde{a}^2_\alpha) = [((1-\alpha)a^L + \alpha a)^2, ((1-\alpha)a^U + \alpha a)^2],$$

$$(1/\widetilde{a}_\alpha) = [1/((1-\alpha)a^L + \alpha a), 1/((1-\alpha)a^U + \alpha a)].$$

Definition 6: Suppose we have a $p \times q$ matrix A, that is $A = (a_{ij})_{p \times q}$, if a_{ij} is a triangular fuzzy number. Then we can say

$$\widetilde{A} = \begin{bmatrix} \widetilde{a}_{11} & \widetilde{a}_{12} & \dots & \widetilde{a}_{1q} \\ \widetilde{a}_{21} & \widetilde{a}_{22} & \dots & \widetilde{a}_{2q} \\ \dots & \dots & \dots & \dots \\ \widetilde{a}_{p1} & \widetilde{a}_{p2} & \dots & \widetilde{a}_{pq} \end{bmatrix}$$

is a triangular fuzzy matrix.

III. Theorems of the Alternative of Fuzzy Linear Programming

Lemma 3: For any given $p \times n$ matrix $\widetilde{A}$, let elements of $\widetilde{A}$ are triangular fuzzy numbers. The systems
(1) $\widetilde{A}x \underline{\underline{\succ}} \widetilde{0}$;
(2) $\widetilde{A}'x = \widetilde{0}, y \underline{\underline{\geq}} 0$;
possess solutions x and y satisfying

$$\widetilde{A}_1 x + y_1 \succ \widetilde{0}.$$

Proof. We prove by using mathematical induction on p. For $p = 1$, if $\widetilde{A}_1 = \widetilde{0}$, take $y_1 = 1, x = 0$; if $\widetilde{A}_1 \succ \widetilde{0}$, take $y_1 = 0, x = 1$; if $\widetilde{A}_1 \prec \widetilde{0}$, take $y_1 = 0, x = -1$;
Suppose

$$\widetilde{\widetilde{A}} = \begin{bmatrix} \widetilde{A} \\ \widetilde{A}_{p+1} \end{bmatrix} = \begin{bmatrix} \widetilde{A}_1 \\ \dots \\ \widetilde{A}_p \\ \widetilde{A}_{p+1} \end{bmatrix}$$

and $\widetilde{A}$ satisfy the above Lemma, that is $n = p$. When $n = p + 1$, we have known the following conclusions,

$$\widetilde{\widetilde{A}}x \underline{\underline{\succ}} \widetilde{0}, \widetilde{A}'y = \widetilde{0}, y \geq 0, \widetilde{A}_1 x + y_1 \succ \widetilde{0}.$$

(I) Let $\widetilde{A}_{p+1}x \underline{\underline{\succ}} \widetilde{0}$, $\bar{y} = (y, 0)$. Hence

$$\widetilde{\widetilde{A}} \underline{\underline{\succ}} \widetilde{0}, \ \widetilde{\widetilde{A}}' y = \widetilde{0}, \ \overline{y} \underline{\underline{\succ}} 0, \ \widetilde{A}_1 x + y_1 \succ \widetilde{0}.$$

(II)Suppose $\widetilde{A}_{p+1}x \prec \widetilde{0}$. The following triangular fuzzy matrices are constructed

$$\widetilde{B} = \begin{bmatrix} \widetilde{B}_1 \\ \widetilde{B}_2 \\ \dots \\ \widetilde{B}_p \end{bmatrix} = \begin{bmatrix} \widetilde{A}_1 + \lambda_1 \widetilde{A}_{p+1} \\ \dots \\ \widetilde{A}_p + \lambda_p \widetilde{A}_{p+1} \end{bmatrix}.$$

Let

$$\lambda_j = \frac{\tau(\widetilde{A}_j x)}{\tau(-\widetilde{A}_{p+1}x)} \underline{\underline{\geq}} 0, \quad j = 1, 2, \cdots, p.$$

Thus,

$$\widetilde{B}_j x = \widetilde{A}_j x + \lambda_j \widetilde{A}_{p+1} x,$$

because we can get $\tau(\widetilde{A}_j x + \lambda_j \widetilde{A}_{p+1} x) = 0$, then

$$\widetilde{B}_j x = \widetilde{0}.$$

Thus,

$$\widetilde{B}x = \widetilde{0}.$$

Suppose that v, u satisfy lemma 1, that is,

$$\widetilde{B}v \underline{\underline{\succ}} \widetilde{0}, \widetilde{B}'u = \widetilde{0}, u \geq 0,$$

and we can get $\widetilde{B}_1 v + u_1 \succ \widetilde{0}$.
Let $\bar{u} = (u, \Sigma_{j=1}^p \lambda_j u_j)^T$, so we can get $\bar{u} \underline{\underline{\geq}} 0$.
Hence

$$\widetilde{\widetilde{A}}' \bar{u} = \widetilde{A}' u + \widetilde{A}'_{p+1} \Sigma_{j=1}^p \lambda_j u_j = \widetilde{B}' u = \widetilde{0}.$$

Let $w = v - \frac{\tau(\widetilde{A}_{p+1}v)}{\tau(\widetilde{A}_{p+1}x)} x$, we can get

$$\widetilde{A}_{p+1} w = \widetilde{A}_{p+1} v - \widetilde{A}_{p+1} v$$

hence $\widetilde{A}_{p+1}\omega = 0$, because of

$$\widetilde{B}\omega = \widetilde{B}v - \frac{\tau(\widetilde{A}_{p+1}v)}{\tau(\widetilde{A}_{p+1}x)} \widetilde{B}x = \widetilde{B}v \underline{\underline{\succ}} \widetilde{0}.$$

And we know

$$\widetilde{B}\omega = \begin{bmatrix} \widetilde{B}_1 \\ \widetilde{B}_2 \\ \dots \\ \widetilde{B}_p \end{bmatrix} \omega = \begin{bmatrix} \widetilde{A}_1 + \lambda_1 \widetilde{A}_{p+1} \\ \dots \\ \widetilde{A}_p + \lambda_p \widetilde{A}_{p+1} \end{bmatrix} \omega \underline{\underline{\succ}} \widetilde{0},$$

hence $\widetilde{A}\omega \succeq \widetilde{0}$.

We can get

$$\widetilde{\widetilde{A}}'\omega = \begin{bmatrix} \widetilde{A}\omega \\ \cdots \\ \widetilde{A}_{p+1}\omega \end{bmatrix} \succeq \widetilde{0}$$

because of

$$\widetilde{B}_1\omega + u_1 = \widetilde{B}_1(v - \frac{\tau(\widetilde{A}_{p+1}v)}{\tau(\widetilde{A}_{p+1}x)}x) + u_1 = \widetilde{B}_1'v + u_1 \succ \widetilde{0}.$$

From the above we can know $\lambda_1\widetilde{A}_{p+1}\omega = 0$. Hence

$$\widetilde{A}_1\omega+\lambda_1\widetilde{A}_{p+1}\omega+u_1 = (\widetilde{A}_1+\lambda_1\widetilde{A}_{p+1})\omega+u_1 = \widetilde{B}_1\omega+u_1 \succ \widetilde{0},$$

that is,

$$\tau(\widetilde{A}_1\omega + \lambda_1\widetilde{A}_{p+1}\omega + u_1) > 0.$$

Since $\tau(\lambda_1\widetilde{A}_{p+1}\omega) = 0$ and $\tau(\widetilde{A}_1\omega + u_1) > 0$, we have $\widetilde{A}_1\omega + u_1 \succ \widetilde{0}$. To sum up, the conclusion is established.

Theorem 1: For any given $p\times n$ matrix $\widetilde{A}$, let elements of $\widetilde{A}$ are triangular fuzzy numbers. That is, $\widetilde{A} = (A^L, A, A^U)$, $[\widetilde{A}]^\alpha = [(1-\alpha)A^L + \alpha A, (1-\alpha)A^U + \alpha A]$. The systems
(1) $\widetilde{A}x \succeq \widetilde{0}$;
(2) $\widetilde{A}'y = \widetilde{0},\ y \geqq 0$;
possess solutions x and y satisfying

$$\widetilde{A}x + y \succ \widetilde{0}.$$

Proof. There exist $x^i \in R^n, y^i \in R^p, i = 1, 2, \cdots, p$. By Lemma 1, we have

$$\begin{cases} \widetilde{A}x^i \succeq \widetilde{0}; \\ \widetilde{A}'y^i = \widetilde{0}, y^i \geqq 0; \\ \widetilde{A}_i x^i + y^i_i \succ \widetilde{0}; \end{cases}$$

we define $x = \Sigma_{i=1}^p x^i, y = \Sigma_{i=1}^p y^i$.
Hence

$$\widetilde{A}x = \Sigma_{i=1}^p \widetilde{A}x^i, \widetilde{A}'y = \Sigma_{i=1}^p \widetilde{A}'y^i,$$

because of $\tau(\widetilde{A}x^i) \geq 0, \tau(\widetilde{A}'y^i) = 0$.

$$\Sigma_{i=1}^p \tau(\widetilde{A}x^i) = \tau(\Sigma_{i=1}^p \widetilde{A}x^i) \geq 0,$$

$$\Sigma_{i=1}^p \tau(\widetilde{A}'y^i) = \tau(\Sigma_{i=1}^p \widetilde{A}'y^i) = 0,$$

then

$$\widetilde{A}x \succeq \widetilde{0};$$

$$\widetilde{A}'y = \widetilde{0};$$

and $y = \Sigma_{i=1}^p y^i \geqq 0$. Then when $i = 1, 2, ..., p$,

$$\begin{aligned} \widetilde{A}_i x + y_i &= \widetilde{A}_i \Sigma_{i=1}^p x^i + \Sigma_{k=1}^p y^k_i \\ &= \widetilde{A}_i x^i + y^i_i + \Sigma_{k=1, k\neq i}^p (\widetilde{A}_i x^k + y^k_i) \end{aligned}$$

and we already know

$$\widetilde{A}_i x^i + y^i_i \succ 0, \widetilde{A}_i x^i \succeq \widetilde{0}$$

so

$$\tau(\widetilde{A}_k x^i) \geq 0$$

that is

$$\widetilde{A}_k x^i \succeq \widetilde{0},\ k = 1, 2, \cdots, p.$$

Since $y^i \geqq 0$, then $y^i_k \geq 0,\ k = 1, 2, \cdots, p$.
So

$$\Sigma_{k=1,}^p (\widetilde{A}_i x^k + y^k_i) \succeq \widetilde{0}$$

hence, we can get

$$\widetilde{A}_i x + y_i \succ \widetilde{0}$$

then

$$\widetilde{A}x + y \succ \widetilde{0}.$$

Theorem 2: For any given $p_1 \times n$ matrix $\widetilde{A}$,and $p_2 \times n$ matrix $\widetilde{B}$,$\widetilde{A}$ is non empty. Then the systems
(1) $\widetilde{A}x \succeq \widetilde{0}$, $\widetilde{B}x = \widetilde{0}$;
(2) $\widetilde{A}'y_1 + \widetilde{B}'y_2 = \widetilde{0},\ y \geqq 0$;
possess solutions $x\epsilon R^n, y_1\epsilon R^{p_1}, y_2\epsilon R^{p_2}$ satisfying

$$\widetilde{A}x + y_1 \succ \widetilde{0}.$$

Proof. By Theorem 1, we can construct the following form

$$\begin{bmatrix} \widetilde{A} \\ \widetilde{B} \\ -\widetilde{B} \end{bmatrix} x \succeq \widetilde{0}, [\widetilde{A'}, \widetilde{B'}, -\widetilde{B'}] \begin{bmatrix} y_1 \\ z_1 \\ z_2 \end{bmatrix} = 0, \begin{bmatrix} y_1 \\ z_1 \\ z_2 \end{bmatrix} \geqq 0.$$

The conclusion obtained by Theorem 1

$$\widetilde{A}x + y_1 \succ \widetilde{0},$$

$$\widetilde{B}x + z_1 \succ \widetilde{0},$$

$$-\widetilde{B}x + z_2 \succ \widetilde{0}.$$

Also because

$$\begin{bmatrix} \widetilde{A} \\ \widetilde{B} \\ -\widetilde{B} \end{bmatrix} \succeq \widetilde{0},$$

then

$$\widetilde{A}x \succeq \widetilde{0}, \quad , \quad \widetilde{B}x \succeq \widetilde{0}, \quad , \quad -\widetilde{B}x \succeq \widetilde{0}.$$

And $\tau(\widetilde{B}x) = 0$, $\widetilde{B}x = \widetilde{0}$, because

$$[\widetilde{A'} \quad , \quad \widetilde{B'} \quad , \quad -\widetilde{B'}] \begin{bmatrix} y_1 \\ z_1 \\ z_2 \end{bmatrix} = \widetilde{A'}y_1 + \widetilde{B'}z_1 - \widetilde{B'}z_2 = \widetilde{0},$$

which implies $y_2 = z_1 - z_2$.Then

$$\begin{aligned} \widetilde{A'}y_1 + \widetilde{B'}z_1 - \widetilde{B'}z_2 &= \widetilde{A'}y_1 + \widetilde{B'}(y_2 + z_2) - \widetilde{B'}z_2 \\ &= \widetilde{A'}y_1 + \widetilde{B'}y_2 + \widetilde{B'}z_2 - \widetilde{B'}z_2 = \widetilde{0}, \end{aligned}$$

which implies

$$\tau(\widetilde{A'}y_1 + \widetilde{B'}(y_2 + z_2) - \widetilde{B'}z_2) = 0,$$

and $\tau(\widetilde{B'}z_2 + \widetilde{B'}z_2) = 0$. Then $\tau(\widetilde{A'}y_1 + \widetilde{B'}y_2) = 0$ and $\widetilde{A'}y_1 + \widetilde{B'}y_2 = \widetilde{0}$.

Corollary 1: For any given $p_1 \times n$ matrix $\widetilde{A}$, $p_2 \times n$ matrix $\widetilde{B}$,$p_3 \times n$ matrix $\widetilde{C}$ and $p_4 \times n$ matrix $\widetilde{D}$,$\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ is non empty. If satisfying the following conditions
(1) $\widetilde{A}x \succeq \widetilde{0}$, $\widetilde{B}x \succeq \widetilde{0}$, $\widetilde{C}x \succeq \widetilde{0}$, $\widetilde{D}x = \widetilde{0}$;
(2) $\widetilde{A}'y_1 + \widetilde{B}'y_2 + \widetilde{\widetilde{C}}'y_3 + \widetilde{D}'y_4 = \widetilde{0},\ y_i \geqq 0\ i = 1, 2, 3$;
then $x\epsilon R^n$, $y_1\epsilon R^{p1}$, $y_2\epsilon R^{p2}$, $y_3\epsilon R^{p3}$, $y_4\epsilon R^{p4}$ are the solution of conditions (1) and (2), then they are satisfied

$$\widetilde{A}x + y_1 \succ \widetilde{0},$$

$$\widetilde{B}x + y_2 \succ \widetilde{0},$$

$$\widetilde{C}x + y_3 \succ \widetilde{0}.$$

Theorem 3: For any given triangular fuzzy matrixes $\widetilde{A}$, $\widetilde{C}$ and $\widetilde{D}$. $\widetilde{A}$ is non empty. Either
(I) $\widetilde{A}x \succ \widetilde{0}$, $\widetilde{C}x \succeq \widetilde{0}$ and $\widetilde{D}x = \widetilde{0}$ has a solution x
or
(II) $\widetilde{A}'y_1 + \widetilde{C}'y_3 + \widetilde{D}'y_4 = \widetilde{0}, y_1 \geq 0, y_3 \geq 0$ has a solution y_1, y_3, y_4
but never both.

Proof.($I \Rightarrow \overline{II}$)If both I and II hold, then we would have x, y_1, y_3, y_4 such that

$$\widetilde{A}x \succ \widetilde{0}, \widetilde{C}x \succeq \widetilde{0}, \widetilde{D}x = \widetilde{0},$$

$$\widetilde{A'}y_1 + \widetilde{C'}y_3 + \widetilde{D'}y_4 = \widetilde{0}, y_1 \geq 0, y_3 \geq 0,$$

so $\tau(x\widetilde{A'}y_1) > 0$, then $x\widetilde{A'}y_1 \succ \widetilde{0}$.
The same can be obtained: $x\widetilde{C'}y_3 \succeq \widetilde{0}, x\widetilde{D'}y_4 = \widetilde{0}$.
Then $x\widetilde{A'}y_1 + x\widetilde{C'}y_3 + x\widetilde{D'}y_4 \succ \widetilde{0}$.
But now we have a contradiction to the first equality of II.
Hence, I and II cannot hold simultaneously. Thus, $I \Rightarrow \overline{II}$.
($\overline{I} \Rightarrow II$)Suppose $\overline{I}$ is as follows:

$$\widetilde{A}x \succeq \widetilde{0}, \widetilde{C}x \succeq \widetilde{0}, \widetilde{D}x = \widetilde{0}$$

which is

$$\widetilde{A}x \succeq \widetilde{0}, \widetilde{C}x \succeq \widetilde{0}, \widetilde{D}x = \widetilde{0}$$

$$\widetilde{A'}y_1 + \widetilde{C'}y_3 + \widetilde{D'}y_4 = \widetilde{0}, y_1 \geq 0, y_3 \geq 0$$

By corollary 1 we can get: $y_1 \geq 0$, which implies $\overline{I} \Rightarrow II$.

Theorem 4: For any given $p \times n$ triangular fuzzy matrix $\widetilde{A}$, and each given triangular fuzzy vector b. Either
(I) $\widetilde{A}x \preceq \widetilde{0}$, $\widetilde{b'}x \succ \widetilde{0}$ has a solution $x \epsilon R^n$
or
(II) $\widetilde{A}'y = \widetilde{b}, y \geq 0$ has a solution $y \epsilon R^p$
but never both.

Proof. $\widetilde{A}x \preceq \widetilde{0}$, which shows $\tau(\widetilde{A}x) \leq 0$, then

$$-\tau(\widetilde{A}x) = \tau(-\widetilde{A}x) \geq 0, -\widetilde{A}x \geq \succ \widetilde{0}$$

Condition (I) can be converted to: $\widetilde{b}' \succ \widetilde{0}$, $(-\widetilde{A})x \succeq \widetilde{0}$.
From the above, condition (I) is equivalent to condition (I') in theorem 3.
Suppose that the condition (II') in theorem 3 is:

$$\widetilde{b}\eta + (-\widetilde{A'})y_3 = 0, \eta \geq 0, y_3 \geq 0, \eta \epsilon R, y_3 \ \epsilon R^p$$

Let $y = \frac{y_3}{\eta}$, then condition (II') changed to $\widetilde{A'}y = b$. which $(II') \Leftrightarrow (II)$.
From theorem 3 we can get the conclusion above.

IV. Optimality criteria for fuzzy linear programme

Through the above theorem 4, we can define the optimization criterion of fuzzy linear problem :
Find an $\overline{x}$, if it exists such that

$$-\widetilde{b'}\overline{x} = min(-\widetilde{b'}x), \overline{x} \epsilon X = \{x \ \epsilon R^n, \widetilde{A}x \preceq \widetilde{c}\}, b \ \epsilon \ F^n, c \ \epsilon \ F^n$$

and $\widetilde{A}$ is a given $m \times n$ triangular fuzzy matrix.
Let $\overline{x}$ be a solution of the linear programming problem, then there exists a $\overline{u} \ \epsilon \ R^m$ such that $(\overline{x}, \overline{u})$ satissfy

$$\begin{cases} \widetilde{A}\overline{x} \preceq \widetilde{c}, \widetilde{A'}\overline{u} = \widetilde{b} \\ \overline{u} \geq 0, \widetilde{b'}x = \widetilde{c'}u \end{cases}$$

Proof.(Necessity)Define the set of indicators P, Q and M. $P \bigcup Q = \{1, 2, ..., m\}$

$$P \ = \ \{i \mid \ \widetilde{A_i}\overline{x} = \widetilde{c_i}\}, Q = \{i \mid \widetilde{A_i}\overline{x} \prec \widetilde{c_i}\}$$

(1) If $P = \emptyset$, we assert that $\widetilde{A}\overline{x} - \widetilde{c} \prec -\delta\widetilde{e}$ and $\delta > 0, \widetilde{e}$ is a m dimensional triangular fuzzy vector, $-\delta\widetilde{e} < \varepsilon$.
For any $x \epsilon R^n$, We can find a real number $\alpha > 0$. then we have

$$\widetilde{A}(\overline{x} \ + \ \alpha x) - \widetilde{c} = \widetilde{A}\overline{x} + \widetilde{A}\alpha x - \widetilde{c} \prec -\delta\widetilde{e} \ + \ \widetilde{A}\alpha x \preceq \widetilde{0}$$

From the above: $\overline{x} + \alpha x \epsilon X$, then we can get

$$-\widetilde{b'}\overline{x} = min(-\widetilde{b'}x)$$

That is

$$-\widetilde{b'}\overline{x} \preceq - \widetilde{b'}(\overline{x} + \alpha x)$$

$$\tau(-\widetilde{b'}\overline{x}) \leq \tau(-\widetilde{b'}(\overline{x}+\alpha x)), \tau(\widetilde{b'}\overline{x}) \ \geq \ \tau(\widetilde{b'}(\overline{x}+\alpha x)), \alpha > 0.$$

So we get that $\tau(\alpha\widetilde{b'}x) \leq 0$, $\tau(\widetilde{b'}x) \leq 0$.
That is $\widetilde{b'}x \preceq 0$, because of the arbitrariness of x, we can assume $x_i = \widetilde{b_i}, i = \{1, 2, ..., n\}$ then we can get $\widetilde{b'}x = \sum_{i=1}^{n} \widetilde{b_i}$ and $\tau(\widetilde{b_i^2}) \geq 0$. That is mean: $\widetilde{b'}x \succeq 0$.
By combining the above conclusion, we can get $\widetilde{b'}x = 0, \widetilde{b'} = \widetilde{0}$, so the equation is set up.
(2) If $P \neq \emptyset$, we assert that the system $\{\widetilde{A}_P x \preceq \widetilde{0}, \widetilde{b'}x \succ \widetilde{0}\}$ have solutions $x \epsilon R^n$, if $\alpha > 0$, αx is also a solution of inequality.
Then we can get

$$-\widetilde{b'}(\overline{x} \ + \ \alpha x) = -\widetilde{b'}\overline{x} \ - \ \alpha\widetilde{b'}x, \tau(\alpha\widetilde{b'}x) = \alpha\tau(\widetilde{b'}x) > 0.$$

and hence

$$\tau(-\alpha\widetilde{b'}x) < 0, \tau(-\widetilde{b'}\overline{x} - \alpha\widetilde{b'}x) = \tau(-\alpha\widetilde{b'}x) + \ \tau(-\widetilde{b'}\overline{x}) < \tau(-\widetilde{b'}\overline{x}).$$

which implies that

$$-(\widetilde{b'}\overline{x} \ + \ \alpha\widetilde{b'}x) \prec -\widetilde{b'}\overline{x},$$

so we can know $-\widetilde{b'}(\overline{x} \ + \ \alpha x) \prec -\widetilde{b'}\overline{x}, \alpha > 0$.
Because of $\widetilde{A}_P(\overline{x} + \alpha x) - \widetilde{c}_P = \widetilde{A}_P\overline{x} + \alpha\widetilde{A}_P x - \widetilde{c}_P$, and from the above, we can know:

$$\widetilde{A}_P\overline{x} = \widetilde{c}_P, \tau(\widetilde{A}_P\overline{x} - \widetilde{c}_P) = 0,$$

therefore, $\tau(\alpha\widetilde{A}_P x) \leq 0$.
Thus we can get the following conclusions:

$$\tau(\widetilde{A}_P(\overline{x} + \alpha x)) \leq 0, \widetilde{A}_P(\overline{x} + \alpha x) \preceq \widetilde{0}.$$

Because of (1), there exist $\alpha, \widetilde{A}_Q(\overline{x}+\alpha x)-c_Q \precsim \widetilde{0}$.
We deduce:$\overline{x}+\alpha x \epsilon X$, and $-\widetilde{b}(\overline{x}+\alpha x) \prec -\widetilde{b}\overline{x}$.
Contradiction with original hypothesis.
Then $\{\widetilde{A}_P x \precsim \widetilde{0}, \widetilde{b}'x \succ \widetilde{0}\}$ has no solution. It is necessary to get the following solution from the theorem 4.

$$\widetilde{A'}_P y = \widetilde{b}, y \geqq 0$$

Suppose $0 \epsilon R^q$, then we can get that
$\widetilde{A'}_P y + \widetilde{A'_P} * 0 = \widetilde{b}, y \geqq 0$, and $\widetilde{A'}\overline{u} = \widetilde{b}, \overline{u} \geqq 0$.
Hence

$$\widetilde{c'}_P y + \widetilde{c'}_Q y * 0 = \widetilde{c'}_P y = y'\widetilde{c}_P = y'\widetilde{A}_P\overline{x} = (\widetilde{A'}_P y)'\overline{x} = \widetilde{b'}\overline{x}.$$

Then from the above we know that

$$\overline{u} = (y, 0), \widetilde{A}\overline{x} \precsim \widetilde{c}$$

is clearly established.
To sum up, the inequalities are set up.
(Sufficiency) Through conditions (1), we know the following conclusions are established:

$$\widetilde{A}\overline{x} \precsim \widetilde{c}, \widetilde{A'}\overline{u} = \widetilde{b}, \overline{u} \geqq 0, \widetilde{b'}x = \widetilde{c'}u.$$

Because we know that

$$-\widetilde{b'}x - (-\widetilde{b'}\overline{x}) = -\widetilde{b'}x + \widetilde{b'}\overline{x} = -\overline{u'}\widetilde{A}x + \widetilde{c'}\overline{u},$$

and $\overline{u} \geqq 0$, we can get that
$\tau(-\overline{u'}\widetilde{A}x + \widetilde{c'}\overline{u}) = \tau(\widetilde{c'}\overline{u}) - \tau(\overline{u'}\widetilde{A}x) = \tau(\overline{u'}\widetilde{c}) - \tau(\overline{u'}\widetilde{A}x) = \overline{u'}\tau(\widetilde{c}) - \overline{u'}\tau(\widetilde{A}x)$.
Because of $\widetilde{A}\overline{x} \leqq \prec \widetilde{c}, \tau(\widetilde{A}\overline{x}) \leq \tau(\widetilde{c}), \tau(\widetilde{c'} - \overline{u}\widetilde{A}x) \geq 0$, and $\tau(-\widetilde{b'} - (-\widetilde{b'}\overline{x})) \geq 0$, we get

$$-\widetilde{b'}x + \widetilde{b'}\overline{x} \succeq 0,$$

in other words: $-\widetilde{b'}\overline{x} \precsim -\widetilde{b'}x$.

V. Example

Example 1: The optimal solutions for the following fuzzy linear problems are solved:

$$Minimize - [(2,3,4),(3,4,5)] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$s.t. \begin{cases} (0,1,3)x_1 + (-1,1,3)x_2 \precsim (5,6,7) \\ (-1,1,3)x_1 + (1,2,3)x_2 \precsim (7,8,9) \\ x_1, x_2 \epsilon R, b \epsilon F^2, c \epsilon F^2 \end{cases}$$

We can get
$\widetilde{A} = \begin{bmatrix} (0,1,3) & (-1,1,3) \\ (-1,1,2) & (1,2,3) \end{bmatrix}, \widetilde{b} = \begin{bmatrix} (2,3,4) \\ (3,4,5) \end{bmatrix},$

$\widetilde{c} = \begin{bmatrix} (5,6,7) \\ (7,8,9) \end{bmatrix}.$

Thus the α-level is

$$\widetilde{A} = \begin{bmatrix} (\alpha, 3-2\alpha) & (2\alpha-1, 3-2\alpha) \\ (2\alpha-1, 2-\alpha) & (\alpha+1, 3-\alpha) \end{bmatrix},$$

$\widetilde{b} = \begin{bmatrix} (2+\alpha, 4-\alpha) \\ (3+\alpha, 5-\alpha) \end{bmatrix}, \widetilde{c} = \begin{bmatrix} (5+\alpha, 7-\alpha) \\ (7+\alpha, 9-\alpha) \end{bmatrix}.$
Through our linear problem optimization criteria, we can solve the problem.

Through $\widetilde{A'}\overline{u} = \widetilde{b}$, we can get:

$$(u_1, u_2) = (\frac{16}{9}, \frac{10}{9})$$

We put it in $\widetilde{b'}x = \widetilde{c'}u$, we can get: $3x_1 + 4x_2 = \frac{176}{9}$. and we have known inequalities to be established:$\widetilde{A}\overline{x} \precsim \widetilde{c}$.
Finally we get the optimal solution of linear inequality:

$$(x_1, x_2) = (\frac{8}{3}, \frac{26}{9}).$$

Example 2: The optimal solutions for the following fuzzy linear problems are solved:

$$Minimize - [(0,1,2), -(0,1,2)] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$s.t. \begin{cases} (0,1,2)x_1 - (0,1,2)x_2 \precsim (0,1,2) \\ (1,2,3)x_1 + (0,1,2)x_2 \precsim (0,1,2) \\ x_1, x_2 \epsilon R, b \epsilon F^2, c \epsilon F^2 \end{cases}$$

We can get
$\widetilde{A} = \begin{bmatrix} (0,1,2) & -(0,1,2) \\ (1,2,3) & (0,1,2) \end{bmatrix}, \widetilde{b} = \begin{bmatrix} (0,1,2) \\ -(0,1,2) \end{bmatrix},$

$\widetilde{c} = \begin{bmatrix} (0,1,2) \\ (0,1,2) \end{bmatrix}.$

Thus the α-level is

$$\widetilde{A} = \begin{bmatrix} (\alpha, 2-\alpha) & (\alpha-2, -\alpha) \\ (\alpha+1, 3-\alpha) & (\alpha, 2-\alpha) \end{bmatrix},$$

$\widetilde{b} = \begin{bmatrix} (\alpha, 2-\alpha) \\ (\alpha-2, -\alpha) \end{bmatrix}, \widetilde{c} = \begin{bmatrix} (\alpha, 2-\alpha) \\ (\alpha, 2-\alpha) \end{bmatrix}.$
Through our linear problem optimization criteria, we can solve the problem.
Through $\widetilde{A'}\overline{u} = \widetilde{b}$, we can get:

$$(u_1, u_2) = (1, 0)$$

We put it in $\widetilde{b'}x = \widetilde{c'}u$, we can get: $x_1 - x_2 = u_1 + u_2 = 1$. and we have known inequalities to be established:$\widetilde{A}\overline{x} \precsim \widetilde{c}$.
Finally we get the optimal solution of linear inequality:

$$(x_1, x_2) = (\frac{2}{3}, -\frac{1}{3}).$$

Example 3: The optimal solutions for the following fuzzy linear problems are solved:

$$Minimize - [(0,1,2), -(0,1,2)] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$s.t. \begin{cases} (0,1,2)x_1 + (0,1,2)x_2 \precsim (1,2,3) \\ (0,1,2)x_1 + (1,2,3)x_2 \precsim (1,2,3) \\ x_1, x_2 \epsilon R, b \epsilon F^2, c \epsilon F^2 \end{cases}$$

We can get
$\widetilde{A} = \begin{bmatrix} (0,1,2) & (0,1,2) \\ (0,1,2) & (1,2,3) \end{bmatrix}, \widetilde{b} = \begin{bmatrix} (0,1,2) \\ -(0,1,2) \end{bmatrix},$

$\widetilde{c} = \begin{bmatrix} (1,2,3) \\ (1,2,3) \end{bmatrix}.$

Thus the α-level is

$$\widetilde{A} = \begin{bmatrix} (\alpha, 2-\alpha) & (\alpha, 2-\alpha) \\ (\alpha, 2-\alpha) & (\alpha+1, 3-\alpha) \end{bmatrix},$$

$$\widetilde{b} = \begin{bmatrix} (\alpha, 2-\alpha) \\ (\alpha-2, -\alpha) \end{bmatrix}, \widetilde{c} = \begin{bmatrix} (\alpha+1, 3-\alpha) \\ (\alpha+1, 3-\alpha) \end{bmatrix}.$$

Through our linear problem optimization criteria, we can solve the problem.
Through $\widetilde{A'}\overline{u} = \widetilde{b}$, we can get:
/

$$(u_1, u_2) = (3, -2)$$

We put it in $\widetilde{b'}x = \widetilde{c'}u$, we can get: $x_1 - x_2 = 2u_1 + 2u_2 = 2$. and we have known inequalities to be established: $\widetilde{A}\overline{x} \preceq \widetilde{c}$. Finally we get the optimal solution of linear inequality:

$$(x_1, x_2) = (2, 0).$$

VI. Conclusion

In this paper, the dual type method for solving linear optimization problems is generalized to the problem of fuzzy linear optimization. We have proved the feasibility of the dual type method in the fuzzy linear problem and the establishment of the relative optimization theorem in the fuzzy field. And an appropriate example is given to prove the correctness of the proposed method.

References

[1] R.E., Bellman, L.A., Zadeh, "Decision making in a fuzzy environment." Manag. Sci., vol. 17, no. 1, pp. 141-164, 1970.

[2] J.J., Buckley, A., Abdalla,"Monte Carlo methods in fuzzy queuing theory," Soft Comput., vol. 13, no. 1, pp. 1027-1033, 2009.

[3] J.M., Cadenas, J.L., Verdegay, "Towards a new strategy for solving fuzzy optimization problems," Fuzzy Optim. Decis. Mak., vol. 8, no. 1, pp. 231-244, 2009.

[4] Y. Chalco-Canoa, A. Rufián-Lizana, H. Román-Flores, R. Osuna-Gómez, "A note on generalized convexity for fuzzy mappings through a linear ordering," Fuzzy Sets Syst., vol. 231, no. 1, pp. 70-83, 2013.

[5] P.Diamond, P. Kloeden, "Metric spaces of fuzzy sets," Fuzzy Sets Syst., vol. 35, no. 1, pp. 241-249, 1990.

[6] D. Dubois, H. Prade, "Fuzzy sets and systems-theory and application," Academic Press, New York, vol. 144, no. 3, pp. 370-374, 1980.

[7] C. Kahraman, "Fuzzy theory and technology with applications," J. Intell. Fuzzy Syst., vol. 19, no. 4, pp. 231-233, 2008.

[8] W.A., Lodwick, K.A., Bachman, "Solving large-scale fuzzy and possibilistic optimization problems," Fuzzy Optim. Decis. Mak., vol. 4, no. 4, pp. 257-278, 2005.

[9] W.A., Lodwick, K.D., Jamison, "Theoretical and semantic distinctions of fuzzy, possibilistic," and mixed fuzzy/possibilistic optimization, Fuzzy Sets Syst., vol. 158, no. 17 pp. 1861-1872, 2007.

[10] W.A., Lodwick, "The relationship between interval, fuzzy and possibilistic optimization," Model. Decis. Artif. Intell., vol. 58, no. 61, pp. 55-59, 2009.

[11] L. Li, S.Y. Liu, J.K. Zhang, "On fuzzy generalized convex mappings and optimality conditions for fuzzy weakly univex mappings," Fuzzy Sets Syst., vol. 280, no. C, pp. 107-132, 2015.

[12] V.D. Pathak, U.M. Pirzada, "The optimality conditions for fuzzy optimization problem under the concept of generalized convexity," Adv. Appl. Math. Anal., vol. 1, no. 1, pp. 23-38, 2010.

[13] Dong Qiu, Chongxia Lu, and Chunlai Mu, "Convergence of Successive Approximations for Fuzzy Differential Equations in The Quotient Space of Fuzzy Numbers," IAENG International Journal of Applied Mathematics, vol. 46, no.4, pp. 512-517, 2016.

[14] YongSang Ryu, YoungSoo Park, JinSoo Kim, and SangHun Lee, "Image Edge Detection using Fuzzy C-means and Three Directions Image Shift Method," IAENG International Journal of Computer Science, vol. 45, no.1, pp1-6, 2018.

[15] I. Ramik, M. Vlach, "Fuzzy mathematical programming: a unified approach based on fuzzy relation,"Fuzzy Optim. Decis. Mak., vol. 1, no. 4, pp. 335-346, 2002.

[16] Y.R. Syau, E. Stanley Lee, "Fuzzy Weirstrass theorem and convex fuzzy mappings," Compu. Math. Appl, vol. 51, no. 12, pp. 1741-1750, 2006.

[17] Y.R. Syau, E. Stanley Lee, "Preinvexity and F1-convexity of fuzzy mappings through a linear ordering," Compu. Math. Appl., vol. 51, no. 3, pp. 405-418, 2006.

[18] S., Song, C., Wu, "Existence and uniqueness of solution to Cauchy problem of fuzzy differential equations." Fuzzy Sets Syst., vol. 110, no. 1, pp. 55-67,2000.

[19] H. Tanaka, T. Okuda, K. Asai, "On fuzzy mathematical programming," J. Cybernetics, vol. 3, no. 4, pp. 37-46, 1974.

[20] Hong Wang, and Tonghui Gao, "Training Algorithm of Adaptive Neural Fuzzy Inference System Based on Improved SRUKF," IAENG International Journal of Computer Science, vol. 44, no.4, pp396-403, 2017.

[21] Shuan Wang, "A Manufacturer Stackelberg Game in Price Competition Supply Chain under a Fuzzy Decision Environment," IAENG International Journal of Applied Mathematics, vol. 47, no.1, pp. 49-55, 2017.

[22] H.-C. Wu, "The Karush-Kuhn-Tucker optimality conditions for optimization problems with fuzzy-valued objective functions," Math. Methods Oper. Res., vol. 66, no. 2, pp. 203-224, 2007.

[23] H.-C. Wu, "The optimality conditions for optimization problems with fuzzy-valued objective functions," Optimization, vol. 57, no. 3, pp. 473-489, 2008.

[24] Yu Xiao, and L. I. Hua. "Improvement on Judgement Matrix Based on Triangle Fuzzy Number." Fuzzy Systems and Mathematics, vol. 1, no. 1, pp. 1-5, 2003.

[25] Yonghong Zhang, and Chunfeng Wang, "A New Branch and Reduce Approach for Solving Generalized Linear Fractional Programming," Engineering Letters, vol. 25, no.3, pp262-267, 2017.

[26] Wei Zhang, Dong Qiu, and Meilin Dong, "Optimizations of Convex and Generalized Convex Fuzzy Mappings in The Quotient Space of Fuzzy Numbers," IAENG International Journal of Applied Mathematics, vol. 47, no.4, pp.431-436, 2017.

[27] H.J. Zimmermann, "Description and optimization of fuzzy systems," Int. J. Gen. Syst., vol. 2, no. 4, pp. 209-215, 1976.

[28] L.A. Zadeh, "Fuzzy Sets," Inf. Control, vol. 8, no. 65, pp. 338-353, 1996.