Higher Order Derivatives Sampling of Random Signals Related to the Fractional Fourier Transform

Rui-Meng Jing, Bing-Zhao Li

Abstract—**Multirate or multichannel sampling related theory and methods are some of the hottest research topics in modern signal processing community. Among them, the sampling associated with the signal and its derivatives is often encountered in various real applications. In this paper, we investigate the sampling theory related to the higher order derivatives of random signals with the fractional Fourier transform. We first obtain the uniform sampling theorem associated with the higher order derivatives of random signals, and then we generalize this results associated with the periodic nonuniform sampling model for random signals. The corresponding sampling rate will be reduced by a factor of n so that the workload will be greatly reduced. Finally, the simulations are performed to verify the proposed theorem.**

Index Terms—**higher order derivative sampling, fractional Fourier domain, periodic nonuniform sampling, power spectral density, random signal, mean square error.**

I. Introduction

SHANNON sampling is the classical uniform sampling theorem, which states that, for a complete reconstruction of an original bandlimited signal, the sampling rate must be at least twice the maximum frequency presented in the signal [1], [2]. Its theory is the milestone both in terms of achievement and conciseness. However, in most practical applications, such as in the field of synthetic aperture Radar (SAR), astronomies and geophysics, the Shannon sampling conditions are not satisfied. And therefore the research of the sampling theory becomes one of the hottest research topics in modern signal processing and applied mathematical community. There have been published many references about the generalizations of the classical Shannon sampling with certain situations [3], [4], [5], [6], [7], [8], [9]. Among them, how to reduce the sampling rate of a signal by multi-channel sampling or derivative sampling is one of the challenging problems in the sampling field.

The early works on sampling with derivative values for deterministic signals in Fourier domain are proposed by Fogel [4], Jagerman and Fogel [5], and Linden and Ahramon [6]. In these works, Shannon sampling theorem is extended to the uniform sampling [9] and periodic nonuniform sampling model [7], [8] of a deterministic bandlimited signal $f(t)$ and their derivative $f^{(l)(knT)}(l = 0, 1, \ldots, n-1)$ and $f^{(l)(n(t_p+kNT))}(l = 0, 1, \ldots, n-1; p = 1, 2, \ldots, N)$ in the Fourier domain. However, all the sampling theories mentioned above are related to the Fourier transform. It is shown in recent research results that many natural signals are better represented by alternative bases other than the Fourier basis. For example, the fractional Fourier transform basis, linear canonical transform basis, wavelet transform basis and the sparse sampling [10], [11], [12], [13], [14], [15], [16].

As one of the generalizations of the classical Fourier transform, the fractional Fourier transform has been applied in signal processing [17], [18], [19], [20], [21], [22]. The study of the sampling theorems associated with the fractional Fourier transform has developed in recent years [23], [24], [25], [26]. The sampling expansion and spectral properties for a uniformly sampled signal which is bandlimited in the fractional Fourier domain have been derived from different ways.

The spectral analysis and reconstruction of periodic nonuniformly sampled are presented in [27]. A more general sampling theorem is considered in [28], where the multichannel sampling theorem in the fractional Fourier domain is also studied. In literature [29], for the original bandlimited random signals in fractional Fourier domain, it can be reconstructed from its uniform samples and multi-channel samples in the mean square error (MSE). However, for the best of our knowledge, there are no papers have been published related to higher order uniform or periodic nonuniform derivative sampling theorems for random signals in the fractional Fourier domain to reduce the sampling rate. Therefore, it is worthwhile and interesting to investigate the sampling theorem associated with the higher order derivatives of random signals in fractional Fourier transform domain.

In this paper, we investigate the uniform and periodic nonuniform derivatives sampling problems of random signals based on the fractional Fourier transform. The perfect reconstruction formulas for a random signal from the uniform and periodic nonuniform sampling points of its higher derivatives are obtained. And the sampling rate can be reduced n times as the Nyquist sampling rate. One comparison and a quantitative analysis are provided to support our conclusion. The paper is organized as follows. The preliminaries are summarized in section 2. In section 3, the uniform and periodic nonuniform derivative sampling theorems for bandlimited random signals in the fractional Fourier domain are derived. In section 4, the simulation results are presented to show the accuracy and usefulness of derived results. Section 5 concludes this paper.

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II. Preliminaries

A. Fractional Fourier Transform (FRFT)

As the generalization of the classical Fourier transform, the αth FRFT [30] of a signal $x(t)$, denoted as $X_\alpha(u)$, is defined by

$$X_\alpha(u) = \mathcal{F}^\alpha[x(t)](u) = \int_{-\infty}^{+\infty} x(t)K_\alpha(u,t)dt \quad (1)$$

where the kernel function $K_\alpha(u,t)$ is given as follows:

$$K_\alpha(u,t) = \begin{cases} A_\alpha e^{j(t^2+u^2)\cot\alpha/2 - jtu\csc\alpha}, & \alpha \neq k\pi \\ \delta(t-u), & \alpha = 2k\pi \\ \delta(t+u), & \alpha = (2k-1)\pi \end{cases} \quad (2)$$

and $A_\alpha = \sqrt{(1-j\cot\alpha)/2\pi}$. The inverse FRFT is expressed as follows:

$$x(t) = \mathcal{F}^{-\alpha}[X_\alpha(u)](t) = \int_{-\infty}^{+\infty} X_\alpha(u)K_\alpha^*(u,t)du \quad (3)$$

When $\alpha = \pi/2$, the FRFT reduces to the Fourier transform (FT). Superscript $*$ is the complex conjugation.

B. Power Spectral Density

In traditional Fourier transform (FT) domain, for a random signal $\{x(t), -\infty < t < +\infty\}$, we often use the autocorrelation function and the power spectral density to represent its characters. If its autocorrelation function

$$R_{xx}(t_1,t_2) = R_{xx}(t_2+\tau,t_2) = E[x(t_1)x^*(t_2)] \quad (4)$$

is independent of t_2 and only depends on their difference $\tau = t_1 - t_2$, where $E\{\bullet\}$ indicates the statistical expectation, then the random signal $x(t)$ is said to be stationary in the wide sense [31].

Motivated by the fact that the FRFT generalizes the FT in a rotational manner, the αth fractional autocorrelation function of $x(t)$ is defined as [31], [32], [33], [34]

$$\begin{aligned} R_{xx}^\alpha(\tau) &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{xx}(t_2+\tau,t_2)e^{jt_2\tau\cot\alpha}dt_2 \\ &= P_{xx}(\rho+\tau,\rho)e^{j\rho\tau\cot\alpha} \end{aligned} \quad (5)$$

where $P_{xx}(t+\tau,t)$ is the power spectral density of $x(t)$ and the equation (5) is valid for all ρ. Likewise, the αth fractional power spectral density is given in [31]

$$P_{xx}^\alpha(u) = \sqrt{\frac{1+j\cot\alpha}{2\pi}}F_\alpha[R_{xx}^\alpha(\tau)](u)e^{-ju^2\cot\alpha/2} \quad (6)$$

When $\alpha = \pi/2$, equation (6) becomes the Wiener-Khinchine theorem.

In additional, a random signal $x(t)$ is said to be bandlimited in the αth fractional Fourier domain if its fractional power spectral density satisfies

$$P_{xx}^\alpha(u) = 0, |u| > u_r \quad (7)$$

where u_r is called the bandwidth of the random signal $x(t)$ in fractional Fourier domain. When $\alpha = \pi/2$, the $x(t)$ is bandlimited in Fourier [35].

C. The periodic nonuniform sampling model

The periodic nonuniform sampling is a special case of nonuniform sampling, which is known as block sampling. In the periodic nonuniform sampling model, the sampling points are divided into several groups of points. The groups have a recurrent period NT, and each group has N nonuniform sampling points. Denoting the points in one period by $t_p + nNT, p = 1,2,\ldots,N; n \in (-\infty,+\infty)$. The model of this periodic nonuniform sampling distribution is depicted in Fig.1, and that the version is redrawn based on Jenq's idea [8].

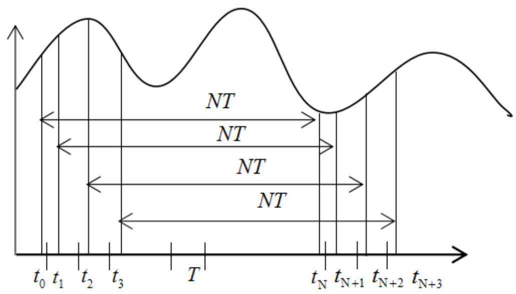

Fig. 1. Periodic nonuniform sampling model.

III. Main Results

Based on the latest results associated with the FRFT, we investigate and obtain the higher order derivatives sampling theorems of random signals in FRFT domain in this section. We discuss the uniform sampling case firstly and then investigate the case of periodic nonuniform sampling.

A. Uniform Sampling Theorem

We investigate the higher order derivatives sampling problems associated with the FRFT and the main result can be represented in Theorem 1.

Theorem 1: Let a random signal $x(t)$ be bandlimited in the αth FRFT domain with the bandwidth u_r, which has $n-1$ order continuous derivative. If $x_c(t) = x(t)e^{j(t^2/2)\cot\alpha}$ is stationary in the wise sense, then $x(t)$ can be reconstructed as

$$\begin{aligned} x(t) = l.i.m.e^{-\frac{j}{2}t^2\cot\alpha}\sum_{k=-\infty}^{+\infty}\sum_{l=0}^{n-1}(x(t)e^{\frac{j}{2}t^2\cot\alpha})^{(l)}|_{t=knT} \\ \cdot s_l(t-knT) \end{aligned} \quad (8)$$

in which,

$$s_l(t) = \sum_{r=l}^{n-1} a_{rl}\overline{s_r}(t), l = 0,1,\cdots,n-1, \quad (9)$$

$$\overline{s_r}(t) = \frac{1}{r!}t^r sinc^n\left(\frac{t}{nT}\right), \quad (10)$$

and the coefficients a_{rl} are the solutions of

$$s_l^{(l')}(0) = \sum_{r=l}^{n-1}\overline{s_r}^{(l')}(0)a_{rl} = \delta_{ll'} \quad (11)$$

$$l' = l,\cdots,n-1; l = 0,\cdots,n-1,$$

And $l.i.m.$ stands for limit in the mean square or convergence in probability as well, i.e.,

$$\lim_{K\to\infty} E\{|x(t)-\sum_{k=-K}^{K}\sum_{l=0}^{n-1} e^{-\frac{j}{2}t^2\cot\alpha}(x(t)e^{\frac{j}{2}t^2\cot\alpha})^{(l)}|_{t=knT} \cdot s_l(t-knT)|^2\}=0 \tag{12}$$

Proof: Let $x_c(t)$ denotes the chirped form of the signal $x(t)$, i.e., $x_c(t)=x(t)e^{j(t^2/2)\cot\alpha}$. Since the random signal $x(t)$ is bandlimited in the αth FRFT domain with the bandwidth u_r, by (6) and (7), we can derive

$$\begin{aligned} P_{xx}^{\alpha}(\tau) &= \int_{-u_r}^{u_r}[P_{xx}^{\alpha}(u)e^{\frac{ju^2\cot\alpha}{2}}]e^{\frac{-j}{2}(u^2+\tau^2)\cot\alpha+ju\tau\csc\alpha}du \\ &= \int_{-u_r}^{u_r} P_{xx}^{\alpha}(u)e^{-\frac{j}{2}\cot\alpha+ju\tau\csc\alpha}du \end{aligned} \tag{13}$$

Since $x_c(t)$ is wise-sense stationary, we can obtain its autocorrelation function from

$$E[x(t_1)e^{j\frac{t_1^2}{2})\cot\alpha}x^*(t_2)e^{-j\frac{t_2^2}{2}\cot\alpha}]=e^{j((t_2+\tau)^2-t_2^2/2)\cot\alpha} \cdot E[x(t_2+\tau),x^*(t_2)]=e^{\frac{j}{2}\tau^2\cot\alpha+jt_2\tau\cot\alpha}R_{xx}(t_2+\tau,t_2) \tag{14}$$

and equation (5). The autocorrelation function of $x_c(t)$ can be written as

$$R_{x_cx_c}(\tau)=e^{\frac{j}{2}\tau^2\cot\alpha}R_{xx}(t_2+\tau,t_2)=e^{\frac{j}{2}\tau^2\cot\alpha}R_{xx}^{\alpha}(\tau). \tag{15}$$

Combining equations (13) and (15), we have

$$R_{x_cx_c}(\tau)=e^{\frac{j}{2}\tau^2\cot\alpha}R_{xx}^{\alpha}(\tau)=\int_{-u_r}^{u_r}P_{xx}^{\alpha}e^{ju\tau\csc\alpha}du \tag{16}$$

We can see that $x_c(t)$ is conventionally bandlimited with $u_r\csc\alpha$. Therefore, let the estimate be $\widehat{x}_c(t)$, where

$$\widehat{x}_c(t)=\sum_{k=-\infty}^{+\infty}\sum_{l=0}^{n-1}x_c^{(l)}(knT)s_l(t-knT) \tag{17}$$

Then, we have

$$E\{[x_c(t)-\widehat{x}_c(t)]x_c^{(l)*}(mnT)\}=E\{x_c(t)x_c^{(l)*}(mnT)\}- \sum_{k=-\infty}^{+\infty}\sum_{l=0}^{n-1}E\{x_c^{(l)}(knT)x_c^{(l)*}(mnT)\}s_l(t-knT) \tag{18}$$

Let $R_{x_cx_c}^{c(l)}(t)$ express the autocorrelation function of $x_c^{(l)}(t)$ and $p(t)=E\{x_c(t)x_c^{(l)*}(mnT)\}$. Noted that the random signal $x_c(t)$ is bandlimited in the FT domain. Due to $x_c(t)$ is stationary, so the FT auto-correlation function of $x_c(t)$ can be expressed as $R_{x_cx_c}(\tau)=E\{x_c^*(t)x_c(t+\tau)\}$, and $R_{x_cx_c}(\tau)$ is only a function of the variable τ. Then, we have

$$R_{x_cx_c}^{(l)}(\tau)=E\{x_c^*(t)x_c^{(l)}(t+\tau)\}, l=0,1,\ldots,n-1 \tag{19}$$

and the equation is valid for all t.

Applying the higher order derivative uniform sampling to the deterministic function $p(t)$ yields

$$\begin{aligned} p(t) &= E\{x_c(t)x_c^{(l)*}(mnT)\} \\ &= \sum_{k=-\infty}^{+\infty}\sum_{l=0}^{n-1}p^{(l)}(knT)s_l(t-knT) \\ &= \sum_{k=-\infty}^{+\infty}\sum_{l=0}^{n-1}E\{x_c^{(l)}(t)|_{t=knT}x_c^{(l)*}(mnT)\}s_l(t-knT) \\ &= \sum_{k=-\infty}^{+\infty}\sum_{l=0}^{n-1}R_{x_cx_c}^{c(l)}(knT-mnT)s_l(t-knT) \end{aligned} \tag{20}$$

Substituting equation (20) into equation (18), we deduce

$$\begin{aligned} &E\{[x_c(t)-\widehat{x}_c(t)]x_c^{(l)*}(mnT)\} \\ &=\sum_{k=-\infty}^{+\infty}\sum_{l=0}^{n-1}R_{x_cx_c}^{c(l)}(knT-mnT)s_l(t-knT)- \\ &\sum_{k=-\infty}^{+\infty}\sum_{l=0}^{n-1}R_{x_cx_c}^{c(l)}(knT-mnT)s_l(t-knT)=0 \end{aligned} \tag{21}$$

This means that, for very m, $[x_c(t)-\widehat{x}_c(t)]$ is orthogonal to $x_c^{(l)}(mnT)$. Since $\widehat{x}_c(t)$ is a linear summation of $x_c^{(l)}(mnT)$, $[x_c(t)-\widehat{x}_c(t)]$ is also orthogonal to $\widehat{x}_c(t)$, i.e.,

$$E\{[x_c(t)-\widehat{x}_c(t)]\widehat{x}_c^*(t)\}=0 \tag{22}$$

On the other hand, we can obtain from equation (18)

$$\begin{aligned} &E\{[x_c(t)-\widehat{x}_c(t)]x_c^*(t)\} \\ &= R_{x_cx_c}(t,t)-\sum_{k=-\infty}^{+\infty}\sum_{l=0}^{n-1}E\{x_c^{(l)}(knT)x_c^*(t)\}s_l(t-knT) \\ &= R_{x_cx_c}(0)-\sum_{k=-\infty}^{+\infty}\sum_{l=0}^{n-1}R_{xx}^{(l)}(knT-t)s_l(t-knT) \end{aligned} \tag{23}$$

Similarly, applying the higher order derivative uniform sampling to the deterministic function$R_{x_cx_c}(\tau-\tau_0)$ yields

$$R_{x_cx_c}(\tau-\tau_0)=\sum_{k=-\infty}^{+\infty}\sum_{l=0}^{n-1}R_{x_cx_c}^{(l)}(knT-\tau_0)s_l(t-knT) \tag{24}$$

Choosing the variables $\tau=\tau_0=t$ in equation (24), we obtain

$$R_{xx}(0)=\sum_{k=-\infty}^{+\infty}\sum_{l=0}^{n-1}R_{xx}^{(l)}(knT-t)s_l(t-knT) \tag{25}$$

Substituting equation (25) into equation (23) yields

$$E\{[x(t)-\widehat{x}(t)]x^*(t)\}=0 \tag{26}$$

Therefore, combining equations (19) and (22), we can obtain

$$\begin{aligned} E[|x(t)-\widehat{x}(t)|^2] &= E\{[x(t)-\widehat{x}(t)][x^*(t)-\widehat{x}^*(t)] \\ &= E\{[x(t)-\widehat{x}(t)]x^*(t)\}-E\{[x(t)-\widehat{x}(t)]\widehat{x}^*(t)\}=0 \end{aligned} \tag{27}$$

So we have proven the following equation,

$$x_c(t)=l.i.m.\sum_{k=-\infty}^{+\infty}\sum_{l=0}^{n-1}x_c^{(l)}(knT)s_l(t-knT) \tag{28}$$

then,

$$x(t) = l.i.m.e^{-\frac{j}{2}t^2 \cot\alpha} \sum_{k=-\infty}^{+\infty} \sum_{l=0}^{n-1} (x(t)e^{\frac{j}{2}t^2 \cot\alpha})^{(l)}|_{t=knT} \cdot s_l(t - knT) \quad (29)$$

Conclusion comes from sorting out. ■

B. Periodic Nonuniform Sampling

In the periodic nonuniform sampling model, the groups have a recurrent period NT, and each group has N nonuniform sampling points. Denote the points in one period by $t_p + nNT, p = 1, 2, \ldots, N; n \in (-\infty, +\infty)$.

Theorem 2: Let a random signal $x(t)$ satisfy the conditions of Theorem 1, then $x(t)$ can be reconstructed from the periodic nonuniform sampling model by the following reconstruction formula,

$$x(t) = l.i.m.e^{-\frac{j}{2}t^2 \cot\alpha} \sum_{k=-\infty}^{+\infty} \sum_{l=0}^{n-1} (x(t)e^{\frac{j}{2}t^2 \cot\alpha})^{(l)}|_{t=n(t_p+kNT)} \overline{s_{lp}}[t - n(t_p + kNT)] \quad (30)$$

in which,

$$\overline{s_{lp}}(t) = \sum_{r=l}^{n-1} a_{rp} \overline{\overline{s_{rp}}}(t), l = 0, \cdots, n-1, p = 1, \cdots, N \quad (31)$$

$$\overline{\overline{s_{rp}}}(t) = \frac{t^r}{r!} \left[\frac{\prod_{q=1}^{N} \sin \pi((t + n(t_p - t_q))/nNT)}{\prod_{q=1, q\neq p}^{N} \sin \pi(n(t_p - t_q)/nNT)} \frac{1}{\pi(\frac{t}{nNT})} \right]^n \quad (32)$$

and the coefficients a_{rp} are solutions of

$$\overline{s_{lp}}(0) = \sum_{r=l}^{n-1} a_{rp} \overline{\overline{s_{rp}}}(0) = \delta_{ll'}, l' = l, \cdots, n-1; p = 1, \cdots, N. \quad (33)$$

Here $l.i.m.$ stands for limit in the mean square or convergence in probability as well, i.e.,

$$\lim_{K\to\infty} E\{|x(t) - e^{-\frac{j}{2}t^2 \cot\alpha} \sum_{k=-K}^{K} \sum_{p=1}^{N} \sum_{l=0}^{n-1} (x(t)e^{\frac{j}{2}t^2 \cot\alpha})^{(l)} |_{t=n(t_p+kNT)} \cdot \overline{s_{lp}}[t - n(t_p + kNT)]|^2\} = 0 \quad (34)$$

Proof: Similar to the proof of Theorem 1, and we omit it here. ■

IV. Simulation Results

To illustrate the importance and correctness of the results in FRFT domain, the uniform and periodic nonuniform derivative sampling are chosen as examples to perform the simulation in this section.

A. Uniform Sampling

In this simulation, the nominal sampling period is choosed as $T = 0.1$ and parameter $\alpha = \arcsin(2/\pi)$, $n = 4$. By applying the uniform reconstruction theorems in FRFT domain, the reconstruction signal $x(t)$ can be obtained. We assume that the original random signal $x(t)$ is shown in Fig.2 and its u_r=5 in Fig.3. That is to say, when $|u| > 5$, the fractional power spectral density $P_{xx}^{\alpha}(u)$ of $x(t)$ equals 0.

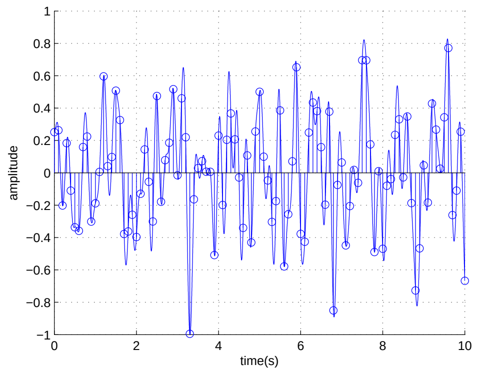

Fig. 2. Original random signal.

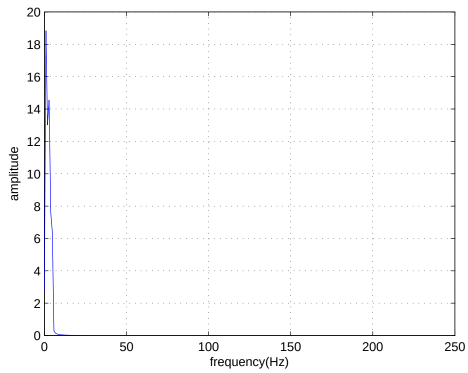

Fig. 3. u_r of the Original random signal.

We use theorem 1 to reconstruct the original random signal $x(t)$ and get the result show in Fig.4. At this time the sampling interval is $nT = 0.4$, which is 4 times the original sampling interval, that is, the sampling rate has been reduced by 4 times. In addition, if applied the classical Fourier higher derivative sampling theorem to signal $x(t)$, we can get the result show in Fig.5. We find that the reconstruction result is far from using theorem 1. Finally, we calculated the error of the two methods respectively and expressed in Fig.6.

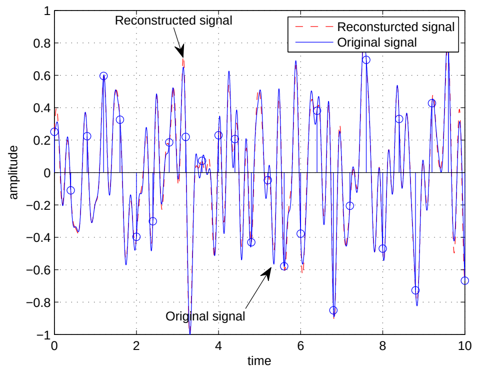

Fig. 4. Reconstructed signal and original signal.

We use Monte Carlo method to circle the simulation 100 times, and get the mean value of 100 mean square errors of the original signal and reconstructed signal, as shown in Table.1. It can be seen from the table that the mean square error of fractional Fourier transform is 16.5102 (uniform sampling of 2001 points), but the mean square error with Fourier transform is 71.7477 (uniform sampling of 2001 points). We can see that using fractional Fourier transform

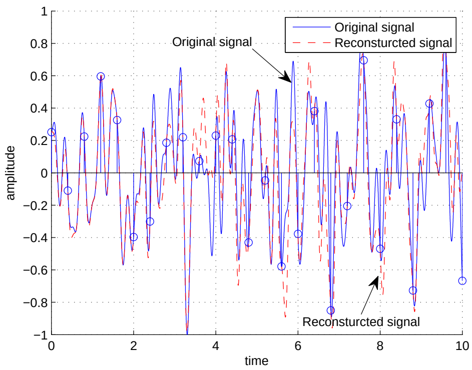

Fig. 5. Reconstructed signal and original signal.

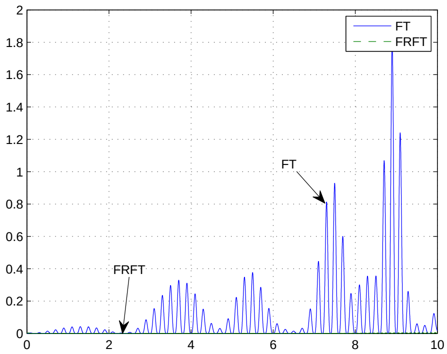

Fig. 6. MSE of theorem 1 and classical Fourier.

can not only make the sampling rate low, but also reduce the reconstruction error.

TABLE I
UNIFORM DERIVATIVE SAMPLING

Transformation Type	MSE
Higher Order FT (n=2)	71.7477
Higher Order FrFT (n=2)	16.5102

B. Periodic Uniform Sampling

In this simulation, the nominal sampling period is choosed as $T = 0.1$ and parameter $\alpha = \arcsin(2/\pi)$, $n = 2$. By applying the uniform reconstruction theorems in FRFT domain, the reconstruction signal $x(t)$ can be obtained. We assume that the original random signal $x(t)$ is shown in Fig.7 and its u_r=5 in Fig.8. That is to say, when $|u| > 5$, the fractional power spectral density $P_{xx}^{\alpha}(u)$ of $x(t)$ equals to 0.

We use theorem 2 to reconstruct the original random signal $x(t)$ and get the result show in Fig.9. We assume that $x(t)$ is a FRFT domain bandlimited signal of $\alpha = \arcsin(2/\pi)$. And we calculate the reconstruction formula of periodic nonuniform sampling when $n = 2$ and $N = 3$. So the original sampling points in time domain are $t_p + kNT = t_p + 3 \cdot 1 \cdot k; p = 1, 2, 3$ and sampling points of derivative are $2(t_p + kNT) = 2(t_p + 3 \cdot 1 \cdot k); p = 1, 2, 3$, that is to say, sampling rate of derivative is $\frac{1}{2}$ times the original sampling rate. And we assume that $x_1 = 0, x_2 = 0.105, x_3 = 0.195$. By bringing those values into equation (34), we can derive the reconstruction formula.

In addition, if applied the classical Fourier higher derivative sampling theorem to signal $x(t)$, we can get the result show in Fig.10. We find that the reconstruction result is far from using theorem 2. Finally, we calculate the errors of the two methods respectively and plot them in Fig.11.

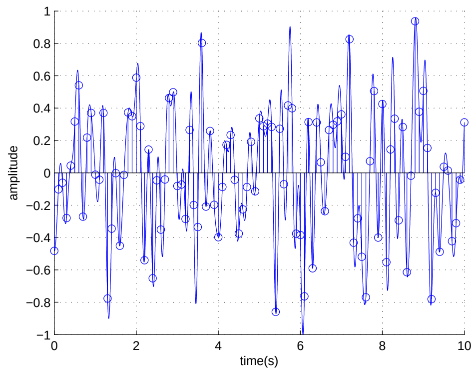

Fig. 7. Original random signal.

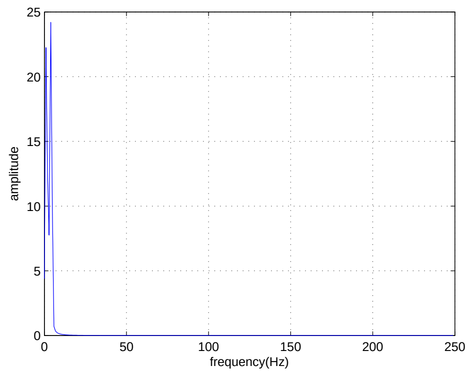

Fig. 8. u_r of the Original random signal.

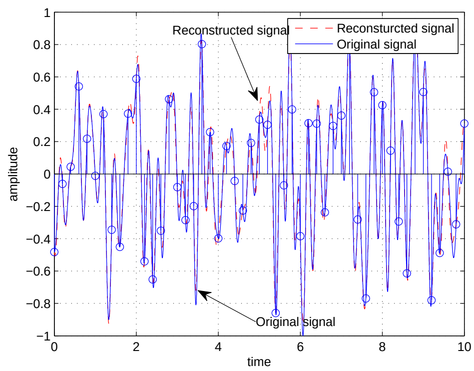

Fig. 9. Reconstructed signal and original signal.

We use Monte Carlo method to circle the simulation 100 times, and get the mean value of 100 mean square errors of the original signal and reconstructed signal, as shown in Table.2. It can be seen from the table that the mean square error of fractional Fourier transform is 14.7998, but the mean square error with Fourier transform is 91.4637. We can see that using fractional Fourier transform can not only make the sampling rate low, but also reduce the reconstruction error.

The derived theorems and the simulation results for random signals state that the random signal can be reconstructed from its uniform derivative samples or periodic nonuniform derivative samples in the FRFT domain perfectly. There are kinds of applications in signal processing, especially for the nonstationary random signals, since they are not bandlimited

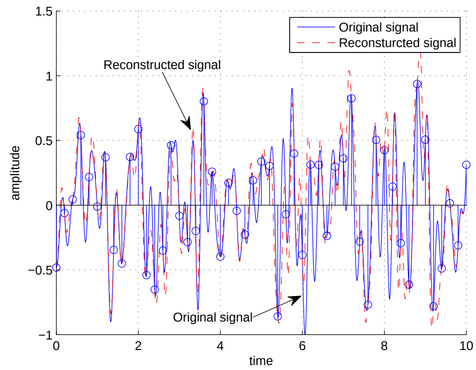

Fig. 10. Reconstructed signal and original signal.

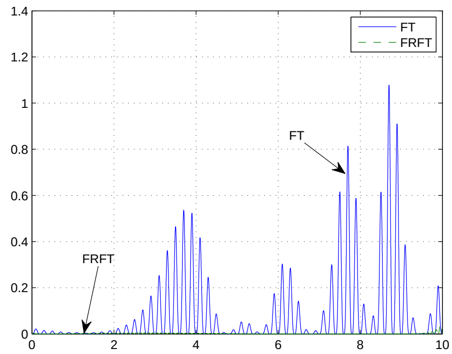

Fig. 11. MSE of theorem 2 and classical Fourier.

in the Fourier domain, whereas maybe bandlimited in the FRFT domain. Such as the periodic nonuniform sampling model as a simple and interesting nonuniform sampling model, which occurs in interleaved ADCs for the nonstationary signals in the FRFT domain. Hence, the current application of the periodic nonuniform sampled signal in the FRFT domain can be found in the interleaved A/D converters. In these cases, the derivative samples are obtained in these data acquisition systems. Based on the theorem 1 and theorem 2, we can reconstructed the original random nonstationary signal from its uniform or periodic nonuniform derivative samples and the sampling rate is far less Nyquist sampling rate.

V. Conclusion

In this paper, we have investigated the problem of higher order derivative sampling and reconstruction of random signals associated with FRFT domain. It is shown that for bandlimited random signals in the FRFT domain, the original signal can be reconstructed from its uniform and periodic nonuniform derivative sampling in the MSE sense. And sampling rate can be reduced by n times so that sampling can be easily realized in practical applications. The sampling error formula and analysis will be our future research directions.

TABLE II
Recurrent Nonuniform Derivative Sampling

Transformation Type	MSE
Higher Order FT (n=2)	91.4637
Higher Order FrFT (n=2)	14.7998

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