Differential Quadrature Method for Fractional Logistic Differential Equation

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***Abstract*—In this paper, we suppose and analyze a differential quadrature method for the numerical solution of fractional Logistic differential equation. The fractional derivative is described in the Caputo sense. Explicit expressions of weighting coefficients for approximation of fractional derivatives are derived and are utilized to reduce the Logistic differential equation to system of algebraic equations. The convergence order of the proposed method is investigated in the infinity norm. Numerical examples are presented to verify the efficiency and accuracy of the proposed method. The results reveal that the method is accurate and easy to implement.**

***Index Terms*—Logistic differential equation, Differential quadrature method, Caputo derivative, Convergence analysis, Chebyshev polynomial.**

I. Introduction

We consider the fractional Logistic differential equation of the form

$$D^{\alpha}y(t) = \rho y(t)(1 - y(t)), \quad t \in [0, 1], \quad \rho > 0, \tag{1}$$

with the initial condition

$$y(0) = y_0, \quad y_0 > 0. \tag{2}$$

In this paper, $D^{\alpha}y(t), 0 < \alpha \leq 1$ denotes the Caputo derivative of order α. For $\alpha = 1$ equation (1) is the standard Logistic equation

$$y'(t) = \rho y(t)(1 - y(t)), \quad y(0) = y_0.$$

The exact solution to the problem is

$$y(t) = \frac{y_0}{(1 - y_0)e^{-\rho t} + y_0}.$$

The existence and uniqueness of the problem (1) are introduced in [1]. During the past decades, the problem of fractional differential equations have been used to model physical and engineering processes that are found to be best described by fractional differential equations. Consequently, the field of the fractional differential equations has attracted interest of researcher in several areas including physics, chemistry, engineering [2], [3]. Fractional Logistic differential equation has been used to construct many mathematical models in various ?elds, such as population growth model [4], electroanalytical chemistry [5], and signal process [6]. Most of fractional differential equations do not have exact analytical solutions, hence considerable heed has been focused on the approximate and numerical solutions of these equations. Recently, several numerical methods to solve the fractional differential equations have been given such as wavelet method [7], [8], [9], Legendre polynomials operational matrix method [10], [11], [12], homotopy perturbation method [13], Adomian decomposition method [14], Adams-Bashforth-Moulton method [15], [16], ant colony algorithm [17], Laplace transform method [18] and other methods [19], [20]. In particular, fractional Logistic differential equations have been solved by using the variational method, finite difference method [21], embedding technique [22], [4], Laguerre collocation method [23]. The main aim of the presented paper is concerned with an extension of the previous work on fractional differential equations and derive some general approximate formulae of differential quadrature method and then we applied these formulae to obtain the numerical solution of fractional Logistic differential equation. Also, we presented study of the convergence analysis of the proposed method.

The differential quadrature method was introduced by Richard Bellman and his associates in the early of 1970s, following the ideas of integral quadrature[24]. The basic idea of the differential quadrature method is that any derivative at a mesh point can be approximated by a weighted linear sum of all the functional values along a mesh line. The key procedure in the differential quadrature method is the determination of weighting coefficients. Later, Shu and Richards [25] obtained explicit formulations to compute weighting coefficients using Lagrange interpolation polynomials as a set of basis. Fung [26] introduced a modified differential quadrature method to incorporate initial conditions. He also discussed at length the stability of various grid pattern in the differential quadrature method. In addition, Chan and Striz [27] derived different weighting coefficients form previous studies and applied them to forth order differential equations. They defined the weighting coefficient as Lagrange and Chebyshev basis functions.

For simplicity, the one-dimensional problem is chosen to demonstrate the differential quadrature method. If a function $u(x)$ is sufficiently smooth on the domain $a \leq x \leq b$, the first and the second order derivatives of the function $u(x)$ with respect to x at a grid point x_i are approximated by a linear sum of all the functional values in the domain, that is,

$$u'(x_i) = \sum_{j=0}^{N} d_{ij}u(x_j), i = 0, 1, 2, \cdots, N. \tag{3}$$

where d_{ij} represent the weighting coefficients of the first order derivative approximations. Then we can write (3)

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in the matrix form

$$U^{(1)} = D^{(1)}U \tag{4}$$

where

$$U^{(1)} = (u'(x_0), u'(x_1), \cdots, u'(x_N))^T$$

and

$$U = (u(x_0), u(x_1), \cdots, u(x_N))^T$$

Obviously, the key procedure in differential quadrature method is to determine the weighting coefficients matrix $D^{(1)}$.

II. Preliminaries and notations

In this section, we present some notations, definitions, and preliminary facts that will be used further in this paper.

A. Basic definitions of fractional calculus

There are various definitions of fractional integration and derivatives. The widely used definition of a fractional integration is the Riemann-Liouville definition and of a fractional derivative is the Caputo definition.

Definition 2.1: A real function $f(t), t > 0$, is said to be in the space $C_\mu, \mu \in R$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0, \infty)$, and it is said to be in space C_μ^n if and only if $f^{(n)} \in C_\mu, n \in N$.

Definition 2.2: The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as

$$\begin{aligned} J^\alpha f(t) =& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \alpha > 0 \\ J^0 f(t) =& f(t). \end{aligned}$$

We note that

$$J^\alpha(D^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!}. \tag{5}$$

Definition 2.3: The fractional derivative D^α of $f(t)$ in the Caputo's sense is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d(\tau), \tag{6}$$

for $n-1 < \alpha \leq n, n \in N, t > 0, f(t) \in C_{-1}^n$.
For the Caputo derivative we have

$$D^\alpha x^\beta = \begin{cases} 0, & \text{for } \beta \in N_0 \text{ and } \beta < \lceil \alpha \rceil; \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, & \text{for } \beta \in N_0 \text{ and } \beta \geq \lceil \alpha \rceil \\ & \text{or } \beta \notin N_0 \text{ and } \beta > \lfloor \alpha \rfloor. \end{cases} \tag{7}$$

We use the ceiling function $\lceil \alpha \rceil$ to denote the smallest integer greater than or equal to α, and the floor function $\lfloor \alpha \rfloor$ to denote the largest integer less than or equal to α. Also $N_0 = \{0, 1, 2, \cdots\}$.

B. Chebyshev polynomials and their properties

The well known Chebyshev polynomial of the first kind $T_n(x)$ is the polynomial of degree n defined for $x \in [-1, 1]$ by

$$T_n(x) = \cos(n \arccos(x)), \quad n = 0, 1, \cdots.$$

Also they have the following properties:

- Three-term recurrence relation:

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x),$$

  with $T_0(x) = 1$ and $T_1(x) = x$.
- The expression of $T_n(x)$ in terms of x is given by [28]

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} c_k^{(n)} x^{n-2k} \tag{8}$$

  where

$$c_k^{(n)} = (-1)^k 2^{n-2k-1} \frac{n}{n-k} \binom{n-k}{k}, \ (2k < n),$$

  and

$$c_k^{(2k)} = (-1)^k, \ (k \geq 0).$$

- Discrete orthogonality relation.
  With the extrema of $T_n(x)$ as nodes: Let $n > 0$, $r, s \leq n$, and $x_i = -\cos(i\pi/n)$, $i = 0, 1, \cdots, n$. Then

$$\sum_{i=0}^{n}{}'' T_r(x_i) T_s(x_i) = K_r \delta_{rs}, \tag{9}$$

  where $K_0 = K_n = n$ and $K_r = n/2$ when $1 \leq r \leq n-1$. The double prime indicates that the term suffixes $i = 0$ and $i = n$ are to be halved.

This discrete orthogonality property leads us to a very efficient interpolation formula. For later use, we write the interpolation polynomial $I_N u(x)$, interpolating $u(x)$ in the points $x_i = -\cos(i\pi/N)$, $i = 0, 1, \cdots, N$, as a sum of Chebyshev polynomials in the form

$$I_N u(x) = \sum_{k=0}^{N}{}'' c_k T_k(x). \tag{10}$$

The coefficient c_k in (10) are given by the explicit formula[29]

$$c_k = \frac{2}{N} \sum_{i=0}^{n}{}'' u(x_i) T_k(x_i), \ i = 0, 1, \cdots, N. \tag{11}$$

III. Weighting coefficients for Chebyshev-based differential quadrature

A continuous and bounded function $u(x)$ can be approximated by first kind Chebyshev polynomials in the interval $[-1, 1]$ by the formula

$$u(x) \approx u_N(x) = \sum_{k=0}^{N}{}'' c_k T_k(x) = T(x)^T C, \tag{12}$$

where C and $T(x)$ are $(N+1) \times 1$ vectors given by

$$C = (c_0/2, c_1, \cdots, c_{N-1}, c_N/2)^T, \tag{13}$$

and

$$T(x) = (T_0(x), T_1(x), \cdots, T_{N-1}(x), T_N(x))^T. \tag{14}$$

In order to obtain the weighting coefficients, the points $x_i = -\cos(i\pi/N), i = 0, 1, \cdots, N$ and (10) were applied. In abbreviated form, $u_N(x)$ can expressed as

$$u_N(x) = T(x) \cdot P \cdot U, \tag{15}$$

where

$$P = \begin{pmatrix} \frac{1}{2N}T_0(x_0) & \frac{2}{2N}T_0(x_1) & \cdots & \frac{1}{2N}T_0(x_N) \\ \frac{1}{N}T_1(x_0) & \frac{2}{N}T_1(x_1) & \cdots & \frac{1}{N}T_1(x_N) \\ \frac{1}{N}T_2(x_0) & \frac{2}{N}T_2(x_1) & \cdots & \frac{1}{N}T_2(x_N) \\ \vdots & \cdots & \ddots & \vdots \\ \frac{1}{2N}T_N(x_0) & \frac{2}{2N}T_N(x_1) & \cdots & \frac{1}{2N}T_N(x_N) \end{pmatrix},$$

$$U = \begin{pmatrix} u(x_0), u(x_1), u(x_2), \cdots, u(x_N) \end{pmatrix}^T.$$

The derivative $u'_N(x)$ is as

$$u'_N(x) = T'(x) \cdot P \cdot U. \tag{16}$$

We know that

$$T'(x) = T(x) \cdot 2M, \tag{17}$$

in which M is the $(N+1) \times (N+1)$ operational matrix of derivative given by

$$M = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \cdots & m_1 \\ 0 & 0 & 2 & 0 & 4 & 0 & \cdots & m_2 \\ 0 & 0 & 0 & 3 & 0 & 5 & \cdots & m_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(N+1)\times(N+1)},$$

so that m_1, m_2 and m_3 are respectively $N/2, 0, N$ for odd N and $0, N, 0$ for even N. Then, we substitute equation (17) into (16) to get

$$u'_N(x) = T(x) \cdot 2M \cdot P \cdot U. \tag{18}$$

Therefore the $u'_N(x)$ can be expressed in the form, as follows:

$$U^{(1)} = Q \cdot 2M \cdot P \cdot U,$$

where

$$Q = \begin{pmatrix} T_0(x_0) & T_1(x_0) & T_2(x_0) & \cdots & T_N(x_0) \\ T_0(x_1) & T_1(x_1) & T_2(x_1) & \cdots & T_N(x_1) \\ T_0(x_2) & T_1(x_2) & T_2(x_2) & \cdots & T_N(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_0(x_N) & T_1(x_N) & T_2(x_N) & \cdots & T_N(x_N) \end{pmatrix}.$$

So, we can get the the weighting coefficients matrix

$$D^{(1)} = Q \cdot 2M \cdot P. \tag{19}$$

Furthermore, the weighting coefficient of the n-order derivative can be completely determined from those of the first derivative

$$D^{(n)} = D^{(1)}D^{(1)} \cdots D^{(1)} = P \cdot 2^n M^n \cdot Q.$$

IV. Calculation of weighting coefficients of fractional order derivatives

In order to apply the Chebyshev polynomials in the interval $[0, 1]$, we used the shifted Chebyshev polynomials $T^*_n(x)$ which defined in terms of the Chebyshev polynomials $T_n(x)$ by the following relation

$$T^*_n(x) = T_n(2x - 1) \tag{20}$$

A function $u(x) \in [0, 1]$ is approximate by means of the shifted Chebyshev polynomials. Similarly, the function $u(x)$ is approximated as

$$u_N(x) = T^*(x)P^*U, \tag{21}$$

where

$$T^* = (T^*_0(x), T^*_1(x), \cdots, T^*_{N-1}(x), T^*_N(x)),$$

$$P^* = \begin{pmatrix} \frac{1}{2N}T^*_0(x_0) & \frac{2}{2N}T^*_0(x_1) & \cdots & \frac{1}{2N}T^*_0(x_N) \\ \frac{1}{N}T^*_1(x_0) & \frac{2}{N}T^*_1(x_1) & \cdots & \frac{1}{N}T^*_1(x_N) \\ \vdots & \cdots & \ddots & \vdots \\ \frac{1}{2N}T^*_N(x_0) & \frac{2}{2N}T^*_N(x_1) & \cdots & \frac{1}{2N}T^*_N(x_N) \end{pmatrix}$$

$$U = \ (u(x_0), u(x_1), u(x_2), \cdots, u(x_N))^T,$$

where $x_i = \frac{1}{2}\left[1 - \cos(i\pi/N)\right], \ i = 0, 1, 2, \cdots, N$. According to the definition of Caputo fractional derivative, we can write

$$D^\alpha u_N(x) = D^\alpha T^*(x) \cdot P^* \cdot U, \tag{22}$$

where $\alpha > 0$. The Caputo fractional derivative of the vector $T^*(x)$ in (21) can be expressed as

$$D^\alpha T^*(x) = D^\alpha XN \tag{23}$$

where

$$N = \begin{pmatrix} 1 & -1 & 1 & \cdots & (-1)^N \\ 0 & 2 & -8 & \cdots & (-1)^{N-1}2N^2 \\ 0 & 0 & 8 & \cdots & (-1)^{N-2}\frac{2}{3}N^2(N^2-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2^{2N-1} \end{pmatrix},$$

and

$$X = (1, x, x^2, \cdots, x^N).$$

Using (7) we have

$$\begin{aligned} D^\alpha X = \Big(&0, \cdots, 0, c_{\lceil\alpha\rceil}x^{\lceil\alpha\rceil-\alpha}, \\ &c_{\lceil\alpha\rceil+1}x^{\lceil\alpha\rceil+1-\alpha}, \cdots, c_{N-\lceil\alpha\rceil}x^{N-\alpha}\Big), \end{aligned} \tag{24}$$

where

$$c_{\lceil\alpha\rceil} = \frac{\Gamma(\lceil\alpha\rceil+1)}{\Gamma(\lceil\alpha\rceil+1-\alpha)},$$

$$c_{\lceil\alpha\rceil+1} = \frac{\Gamma(\lceil\alpha\rceil+2)}{\Gamma(\lceil\alpha\rceil+2-\alpha)},$$

$$c_{\lceil\alpha\rceil+1} = \frac{\Gamma(\lceil\alpha\rceil+3)}{\Gamma(\lceil\alpha\rceil+3-\alpha)},$$

$$\vdots$$

$$c_{N-\lceil\alpha\rceil} = \frac{\Gamma(N+1)}{\Gamma(N+1-\alpha)}$$

Employing (22),(23) and (24) we get

$$U^{(\alpha)} = \Gamma \cdot N \cdot P^* \cdot U, \quad (25)$$

where

$$U^{(\alpha)} = \left(u^{(\alpha)}(x_0), u^{(\alpha)}(x_1), u^{(\alpha)}(x_2), \cdots, u^{(\alpha)}(x_N)\right)^T,$$

$$\Gamma = \begin{pmatrix} 0 & \cdots & 0 & c_{\lceil\alpha\rceil}x_0^{\lceil\alpha\rceil-\alpha} & c_{\lceil\alpha\rceil+1}x_0^{\lceil\alpha\rceil+1-\alpha} & c_{\lceil\alpha\rceil+2}x_0^{\lceil\alpha\rceil+2-\alpha} & \cdots & c_{N-\lceil\alpha\rceil}x_0^{N-\alpha} \\ 0 & \cdots & 0 & c_{\lceil\alpha\rceil}x_1^{\lceil\alpha\rceil-\alpha} & c_{\lceil\alpha\rceil+1}x_1^{\lceil\alpha\rceil+1-\alpha} & c_{\lceil\alpha\rceil+2}x_1^{\lceil\alpha\rceil+2-\alpha} & \cdots & c_{N-\lceil\alpha\rceil}x_1^{N-\alpha} \\ 0 & \cdots & 0 & c_{\lceil\alpha\rceil}x_2^{\lceil\alpha\rceil-\alpha} & c_{\lceil\alpha\rceil+1}x_2^{\lceil\alpha\rceil+1-\alpha} & c_{\lceil\alpha\rceil+2}x_2^{\lceil\alpha\rceil+2-\alpha} & \cdots & c_{N-\lceil\alpha\rceil}x_2^{N-\alpha} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & c_{\lceil\alpha\rceil}x_N^{\lceil\alpha\rceil-\alpha} & c_{\lceil\alpha\rceil+1}x_N^{\lceil\alpha\rceil+1-\alpha} & c_{\lceil\alpha\rceil+2}x_N^{\lceil\alpha\rceil+2-\alpha} & \cdots & c_{N-\lceil\alpha\rceil}x_N^{N-\alpha} \end{pmatrix}.$$

Then the weighting coefficient of the fractional derivative in matrix form:

$$D^{*(\alpha)} = \Gamma \cdot N \cdot P^*. \quad (26)$$

The weighting coefficients can be written collectively in matrix form as

$$D^{*(\alpha)} = \begin{pmatrix} d_{00}^{(\alpha)} & d_{01}^{(\alpha)} & \cdots & d_{0N}^{(\alpha)} \\ d_{10}^{(\alpha)} & d_{11}^{(\alpha)} & \cdots & d_{1N}^{(\alpha)} \\ \vdots & \vdots & \ddots & \vdots \\ d_{N0}^{(\alpha)} & d_{N1}^{(\alpha)} & \cdots & d_{NN}^{(\alpha)} \end{pmatrix}. \quad (27)$$

V. Applications to fractional differential equation

In order to show the fundamental importance of weighting coefficients of fractional order derivatives in the last section, we apply it for solving fractional Logistic differential equation. To solve the problem, we first consider incorporation of initial conditions. With the weighting coefficients $D^{*(\alpha)}, 0 < \alpha \le 1$ in (26),(27), initial condition is incorporated easily into the differential quadrature adopting the same strategy as [26]

$$y^{(\alpha)}(x_i) = \sum_{j=0}^{N} d_{ij}^{(\alpha)} y(x_j) = d_{i0}^{(\alpha)} y(0) + \sum_{j=1}^{N} d_{ij}^{(\alpha)} y(x_j)$$

The above equation can be rewritten in the matrix form as follow:

$$\begin{pmatrix} y^{(\alpha)}(x_1) \\ y^{(\alpha)}(x_2) \\ \vdots \\ y^{(\alpha)}(x_N) \end{pmatrix} = y(0) \begin{pmatrix} d_{10}^{(\alpha)} \\ d_{20}^{(\alpha)} \\ \vdots \\ d_{N0}^{(\alpha)} \end{pmatrix} + \begin{pmatrix} d_{11}^{(\alpha)} & d_{12}^{(\alpha)} & \cdots & d_{1N}^{(\alpha)} \\ d_{21}^{(\alpha)} & d_{22}^{(\alpha)} & \cdots & d_{2N}^{(\alpha)} \\ \vdots & \vdots & \ddots & \vdots \\ d_{N1}^{(\alpha)} & d_{N2}^{(\alpha)} & \cdots & d_{NN}^{(\alpha)} \end{pmatrix} \begin{pmatrix} y(x_1) \\ y(x_2) \\ \vdots \\ y(x_N) \end{pmatrix}. \quad (28)$$

TABLE I
Numerical results for various α using $N = 8$

t	$\alpha = 1/4$	$\alpha = 2/4$	$\alpha = 3/4$
0.1	0.3195860774	0.2853667858	0.2682051811
0.2	0.3282208872	0.3009352440	0.2815891319
0.3	0.3388679105	0.3137793200	0.2936890140
0.4	0.3492246185	0.3255048017	0.3052242536
0.5	0.3550030015	0.3356517345	0.3162870841
0.6	0.3593440986	0.3447525843	0.3269950903
0.7	0.3650080206	0.3534852539	0.3374840793
0.8	0.3702740202	0.3618243983	0.3478011827
0.9	0.3737782232	0.3695588313	0.3579396741
1.0	0.3778096931	0.3770071525	0.3679385593

TABLE II
Absolute errors for $\alpha = 1$ using $N = 4, 8, 12$

t	$N = 4$	$N = 8$	$N = 12$
0.1	2.2909×10^{-4}	4.8450×10^{-7}	3.8858×10^{-16}
0.2	2.2717×10^{-4}	8.5746×10^{-7}	3.4972×10^{-15}
0.3	1.7066×10^{-6}	1.2052×10^{-6}	1.6653×10^{-15}
0.4	4.5271×10^{-4}	1.4658×10^{-6}	1.9429×10^{-15}
0.5	1.1202×10^{-3}	1.7399×10^{-6}	1.2768×10^{-15}
0.6	1.9979×10^{-3}	2.0956×10^{-6}	2.2760×10^{-15}
0.7	3.0784×10^{-3}	2.4480×10^{-6}	3.8858×10^{-16}
0.8	4.3538×10^{-3}	2.7020×10^{-6}	5.5511×10^{-16}
0.9	5.8154×10^{-3}	2.9496×10^{-6}	6.6058×10^{-15}
1.0	7.4536×10^{-3}	3.1230×10^{-6}	3.3307×10^{-15}

In equation (28), the initial condition is naturally incorporated into the differential quadrature rule. By substituting the approximation (28) in (1) and by using the initial condition (2) we get a system of algebraic equation

$$\begin{pmatrix} y^{(\alpha)}(x_1) \\ y^{(\alpha)}(x_2) \\ \vdots \\ y^{(\alpha)}(x_N) \end{pmatrix} = \begin{pmatrix} \rho y(x_1) \\ \rho y(x_2) \\ \vdots \\ \rho y(x_N) \end{pmatrix} - \begin{pmatrix} \rho u^2(y_1) \\ \rho u^2(y_2) \\ \vdots \\ \rho u^2(y_N) \end{pmatrix} \quad (29)$$

Solving the system of algebraic equations, we can obtain the vector Y. Then using (21), we can get the output response

$$y_N(x) = T^*(x) \cdot P^* \cdot Y. \quad (30)$$

The numerical results for $\alpha = 1/4, 2/4, 3/4$ and $N = 8$ are shown in Table 1 and Figure 1. Also the absolute error for $\alpha = 1$ and $N = 4, 8, 12$ are shown in Table 2. The approximate solutions using the present method are high agreement with the exact solutions for $\alpha = 1$.

VI. Some useful Lemmas

In this section, we will provide some useful lemmas which play a significant role in the convergence analysis. We first introduce some notations that will be used. Let $I := (-1, 1)$ and $L^2_{\omega^{\alpha,\beta}}(I)$ be the space of measurable functions whose square is Lebesgue integrable in I relative to the weight function $\omega^{\alpha,\beta}(x)$. The inner produce and norm of $L^2_{\omega^{\alpha,\beta}}(I)$ are defined by

$$(u, v)_{\omega^{\alpha,\beta},I} = \int_{-1}^{1} u(x)v(x)\omega^{\alpha,\beta} dx, \quad \forall u, v \in L^2_{\omega^{\alpha,\beta}}(I).$$

$$\|u\|_{L^2_{\omega^{\alpha,\beta}},I} = (u, u)^{\frac{1}{2}}_{\omega^{\alpha,\beta},I}.$$

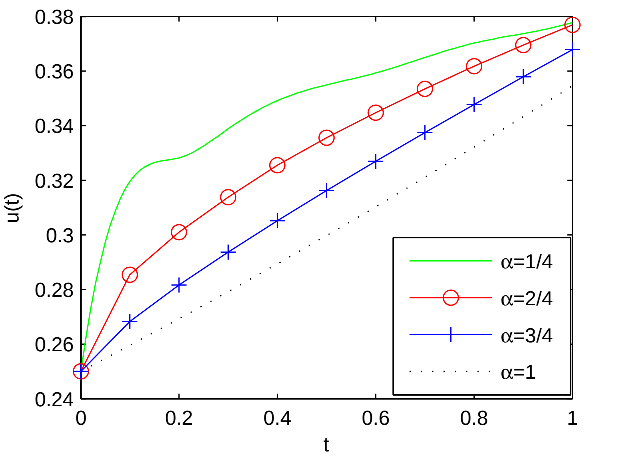

Fig. 1. Comparison of $u(t)$ for $N = 8$ and with $\alpha = 1/4, 2/4, 3/4, 1$.

and

$$\|u\|_{L^\infty I} = \sup_{-1\le x\le 1} |u(x)|.$$

For a non-negative integer m, define

$$H^m_{\omega^{\alpha,\beta},I} := \left\{v : \partial^k_x v \in L^2_{\omega^{\alpha,\beta}}(I), 0 \le k \le m\right\},$$

with the semi-norm and the norm as

$$|v|_{m,\omega^{\alpha,\beta}} = \|\partial^m_x v\|_{\omega^{\alpha,\beta}}, \quad \|v\|_{m,\omega^{\alpha,\beta}} = \left(\sum_{k=0}^{m} |v|^2_{k,\omega^{\alpha,\beta}}\right)^{\frac{1}{2}}.$$

$$|v|_{H^{m;N}_{\omega^{\alpha,\beta}}I} = \left(\sum_{k=min(m,N+1)}^{m} \|\partial^k_x v\|^2_{L^2_{\omega^{\alpha,\beta}}(I)}\right)^{\frac{1}{2}}.$$

Particularly, let

$$\omega(x) = \omega^{-\frac{1}{2},-\frac{1}{2}}(x)$$

be the Chebyshev weight function.

For a given positive integer N, we demote the points by $\{x_i\}_{i=0}^N$, which is the set of $N+1$ Gauss-Lobatto points, corresponding to the weight $\omega^{\alpha,\beta}(x)$. For P_N denote the space of all polynomials of degree not exceeding N. For all $v \in C[-1,1]$, we define the the Lagrange interpolating polynomial $I_N v \in P_N$, satisfying

$$I_N v(x_i) = v(x_i).$$

The Lagrange interpolating polynomial can be written in the form

$$I_N v(x) = \sum_{i=0}^{N} v(x_i) F_i(x), \quad 0 \le i \le N,$$

where $F_i(x)$ is the Lagrange interpolation basis function associated with $\{x_i\}_{i=0}^N$.

Lemma 6.1: ([30])Assume that $v \in H^m_\omega$ and denote $I_N v$ its interpolation polynomial associated with the Gauss-Lobatto points $\{x_i\}_{i=0}^N$, namely,

$$I_N v(x_i) = v(x_i).$$

Then the following estimates hold

$$\|v - I_N v\|_{L^\infty} \le CN^{\frac{1}{2}-m}|v|_{H^{m;N}_\omega}$$

VII. Convergence analysis

In this section, an error estimate of the applied method for the smooth solutions of fractional Logistic differential equation will be provided. For the sake of applying the theory of orthogonal polynomials we employ the variable transformations $t = (1+x)/2$, and let $u(x) = y((1+x)/2)$ to rewrite (1), (2) as follows

$$D^\alpha u(x) = \rho u(x)(1 - u(x)), \quad x > 0, \quad \rho > 0, \tag{31}$$

and

$$u(x) = \frac{1}{\Gamma(\alpha)}\left(\frac{T}{2}\right)^\alpha \int_{-1}^{x} (x-s)^{\alpha-1} D^\alpha u(s)ds + u(-1). \tag{32}$$

Theorem 7.1: Let $u(x)$ be the exact solution of the Logistic differential equation (31), which is assumed to be sufficiently smooth. Let the approximate solution $u_N(x)$ be obtained by using the differential quadrature method together with a polynomial interpolation. If $u(x) \in H^m_{\omega^c}(I)$, then for sufficiently large N the following error estimate holds

$$\|e(x)\|_{L^\infty} \le CN^{\frac{1}{2}-m}(|u|_{H^{m;N}_\omega} + |D^\alpha u|_{H^{m;N}_\omega} + |u^2|_{H^{m;N}_\omega}). \tag{33}$$

Proof: Firstly, equation (31) holds at the Gauss-Lobatto points $\{x_i\}_{i=0}^N$ on $[-1,1]$

$$D^\alpha u(x_i) = \rho u(x_i) - \rho u^2(x_i), \quad u(-1) = u_{-1}. \tag{34}$$

We use $u_i, 0 \le i \le N$ to approximate the function value $u(x_i), 0 \le i \le N$, and use

$$u_N(x) = \sum_{i=0}^{N} u_i F_i(x) \tag{35}$$

to approximate the function $u(x)$, namely, $u(x_i) \approx u_i, u(x) \approx u_N(x)$. Then, the numerical scheme (29) can be rewrite as

$$D^\alpha u_N(x_i) = \rho u_i - \rho u_i^2, \tag{36}$$

and

$$u_i = \frac{1}{\Gamma(\alpha)}\left(\frac{1}{2}\right)^\alpha \int_{-1}^{x_i} (x_i-s)^{\alpha-1} D^\alpha u_N(s)ds + u(-1). \tag{37}$$

Let $e(x)$ and $D^\alpha e(x)$ denote the error functions,

$$e(x) = u(t) - u_N(x), \quad D^\alpha e(x) = D^\alpha u(x) - D^\alpha u_N(x).$$

Subtracting (36) from (34) gives the error equations:

$$D^\alpha(u(x_i) - u_N(x_i)) = \rho(u(x_i) - u_i) + \rho(u^2(x_i) - u_i^2). \tag{38}$$

Multiply $F_i(x)$ on both sides of (37),(38) and summing up from $i = 0$ to $i = N$ yields

$$\begin{aligned} D^\alpha e(x) =& \rho e(x) + u(x) - I_N u(x) + D^\alpha u(x) \\ & - I_N D^\alpha u(x) + I_N\left(u^2(x) - u_N^x(t)\right). \end{aligned} \tag{39}$$

$$\begin{aligned} u_N(x) =& I_N\left(\frac{1}{\Gamma(\alpha)}\left(\frac{1}{2}\right)^\alpha \int_{-1}^{x} (x-s)^{\alpha-1} D^\alpha u(s)ds\right) \\ & - I_N\left(\frac{1}{\Gamma(\alpha)}\left(\frac{1}{2}\right)^\alpha \int_{-1}^{x} (x-s)^{\alpha-1} D^\alpha e(s)ds\right) \end{aligned} \tag{40}$$

It follows from (5) that

$$u_N(x) = I_N u(x) - I_N\left(\frac{1}{\Gamma(\alpha)}(\frac{1}{2})^\alpha \int_{-1}^{x}(x-s)^{\alpha-1}D^\alpha e(s)ds\right). \quad (41)$$

Then we have

$$D^\alpha e(x) = \rho e(t) + J_1 + J_2 + J_3. \quad (42)$$

$$e(x) = \frac{1}{\Gamma(\alpha)}\left(\frac{1}{2}\right)^\alpha \int_{-1}^{x}(x-s)^{\alpha-1}D^\alpha e(s)ds + J_1 + J_3, \quad (43)$$

where

$$J_1 = u(x) - I_N u(x),$$
$$J_2 = D^\alpha u(x) - I_N D^\alpha u(x),$$
$$J_3 = II_N(u^2(x) - u_N^2(x)),$$

and

$$J_4 = \frac{1}{\Gamma(\alpha)}(\frac{1}{2})^\alpha\left[\int_{-1}^{x}(x-s)^{\alpha-1}D^\alpha e(s)ds - I_N(\int_{-1}^{x}(x-s)^{\alpha-1}D^\alpha e(s)ds)\right]$$

It follows from the results in [31] that

$$\|D^\alpha e(x)\|_{L^\infty} \le C\sum_{i=1}^{4}\|J_i\|_{L^\infty}, \quad (44)$$

$$\|e(x)\|_{L^\infty} \le C\sum_{i=1}^{4}\|J_i\|_{L^\infty}, \quad (45)$$

Applying Lemma (6.1) to J_1 and J_2, we have

$$\|J_1\|_{L^\infty} = \|u(t) - I_N u(t)\|_{L^\infty} \le CN^{\frac{1}{2}-m}|u|_{H_\omega^{m;N}}. \quad (46)$$

$$\|J_2\|_{L^\infty} = \|D^\alpha u(t) - I_N D^\alpha u(t)\|_{L^\infty} \le CN^{\frac{1}{2}-m}|D^\alpha u|_{H_{\omega^c}^{m;N}}. \quad (47)$$

We now estimate the third term $\|J_3\|_{L^\infty}$. By some simple calculation we can rewrite J_3 as

$$J_3 = u^2(x) - u_N^2(x) + I_N u^2(x) - u^2(x).$$

Therefore

$$\|J_3\|_{L^\infty} \le \|u^2(x) - u_N^2(x)\|_{L^\infty} + \|u^2(x) - I_N u^2(x)\|_{L^\infty}.$$

Since $u^2(x) - u_N^2(x) = 2u(x)e(x) - e(x)^2$, we have

$$\|u^2(x) - u_N^2(x)\|_{L^\infty} \le C\|u(x)e(x)\|_{L^\infty} + \|e(x)^2\|_{L^\infty}.$$

As the analysis in [32], applying Banach algebra theory we can obtain

$$\|u^2(x) - u_N^2(x)\|_{L^\infty} \le C\|u(x)\|_{L^\infty}\|e(x)\|_{L^\infty} + \|e(x)\|_{L^\infty}.$$

Due to Lemma 6.1, we have

$$\|u^2(x) - I_N u^2(x)\|_{L^\infty} \le CN^{-m}|u^2|_{H_{\omega^c}^{m;N}}.$$

Consequently,

$$\|J_3\|_{L^\infty} \le CN^{-m}|u^2|_{H_{\omega^c}^{m;N}}. \quad (48)$$

As for the bound of $\|J_4\|_{L^\infty}$, we use the same idea as ([31]).

$$\|J_4\|_{L^\infty} \le CN^{-\kappa}\log N\|D^\alpha e(t)\|_{L^\infty}, \quad (49)$$

where $0 < \kappa < 1/2$. Therefore, a combination of (46),(47) (48) and (49) yields the estimate (33). ■

VIII. Conclusion

A general formulation for the Chebyshev polynomial-based weighting coefficients matrix for approximation of fractional derivatives has been derived. The fractional derivatives are described in the Caputo sense. The matrix is used to approximate numerical solution of Logistic differential equations. Our numerical results are compared with exact solutions The solution obtained using the present method shows that this approach can solve the problem effectively.

The method of this article can be extended to the system of fractional differential equations, linear and nonlinear integro-differential equations of fractional order, but some modifications are required.

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