Adjacent Vertex Distinguishing Proper Edge Colorings of Bicyclic Graphs*

Xiang'en Chen† Shunyi Liu‡

Abstract—An adjacent vertex distinguishing proper edge coloring of a graph $G$ is a proper edge coloring of $G$ such that no pair of adjacent vertices meets the same set of colors. Let $\chi_a'(G)$ be the minimum number of colors required to give $G$ an adjacent vertex distinguishing proper edge coloring. In this paper, we show that $\chi_a'(G) \leq \Delta(G) + 1$ for bicyclic graphs $G$, where $\Delta(G)$ is the maximum degree of $G$.

Keywords: Adjacent vertex distinguishing proper edge coloring; Adjacent vertex distinguishing proper edge chromatic number; Bicyclic graph

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A proper edge coloring of $G$ is a mapping $\varphi : E(G) \to \{1, 2, \ldots, k\}$ such that no two adjacent edges meet the same color. Denote by $C_\varphi(v) = \{\varphi(uw) | uw \in E(G)\}$ the color set of the vertex $v$. We say that a proper edge coloring $\varphi$ of $G$ is adjacent vertex distinguishing, or an avd-coloring, if $C_\varphi(u) \neq C_\varphi(v)$ for any pair of adjacent vertices $u$ and $v$. It is obvious that an avd-coloring exists provided that $G$ contains no isolated edge. A $k$-avd-coloring of $G$ is an avd-coloring of $G$ using at most $k$ colors. Let $\chi_a'(G)$ be the minimum number of colors in an avd-coloring of $G$. We use $d_2(u)$ to denote the degree of the vertex $u$ of $G$, and $\Delta(G)$ denotes the maximum degree of $G$. Clearly, $\chi_a'(G) \geq \Delta(G)$, and $\chi_a'(G) \geq \Delta(G) + 1$ if there exist two adjacent vertices $u$ and $v$ with $d_2(u) = d_2(v) = \Delta(G)$.

The adjacent vertex distinguishing proper edge coloring was first introduced by Zhang et al., and the following conjecture was proposed [17].

Conjecture 1. (AVDPEC Conjecture) If $G$ is a simple connected graph on at least 3 vertices and $G \neq C_5$ (a 5-cycle), then $\Delta(G) \leq \chi_a'(G) \leq \Delta(G) + 2$.

In [2], Balister et al. proved that Conjecture 1 holds for bipartite graphs and for graphs with $\Delta(G) \leq 3$. Edwards et al. [6] showed that $\chi_a'(G) \leq \Delta(G) + 1$ if $G$ is a planar bipartite graph with $\Delta(G) \geq 12$. Hornák et al. [12] showed that $\chi_a'(G) \leq \Delta(G) + 2$ for all planar graphs $G$ with $\Delta(G) \geq 12$. Akbari et al. [1] obtained $\chi_a'(G) \leq 3\Delta(G)$ for all graphs $G$ without isolated edges. This bound was recently improved to $3\Delta(G) - 1$ by Zhu et al. [19]. The best general result is due to Hatami [10] who bounded (by a probabilistic method) $\chi_a'(G)$ from above by $\Delta(G) + 300$ provided that $\Delta(G) \geq 10^{10}$. For more on the avd-colorings of graphs, see [3–5, 7–9, 11, 13–16, 18].

A bicyclic graph is a connected graph in which the number of edges equals the number of vertices plus one. In this paper, we investigate the avd-coloring of bicyclic graphs and show that $\chi_a'(G) \leq \Delta(G) + 1$ for bicyclic graphs $G$. This implies that Conjecture 1 holds for all bicyclic graphs.

The rest of the paper is organized as follows. In Section 2, we obtain $\chi_a'(G)$ for bicyclic graphs $G$ without pendant vertex. This plays an important role in Section 3 where we obtain the exact value of $\chi_a'(G)$ for bicyclic graphs $G$ with at least one pendant vertex. In Section 4, we give the conclusion of this paper.

2 Bicyclic graphs without pendant vertex

In this section, we obtain the exact value of $\chi_a'(G)$ for bicyclic graphs $G$ without pendant vertex.

It is easy to see that if $G$ is a bicyclic graph without pendant vertex, then $G$ must be some $H_i$ for $1 \leq i \leq 5$ (see Figure 1).

The following lemma is obvious.

Lemma 1. Let $P$ be a path of $G$ whose internal vertices are all of degree 2 in $G$. If $\varphi$ is a 3-avd-coloring of $G$, then the colors of any three consecutive edges of $P$ are pairwise distinct.

In what follows, we say that two vertices $u$ and $v$ are distinguished from each other in a given coloring if the set of colors incident to $u$ is not equal to the set of colors

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incident to \(v\). We also say that the coloring distinguishes \(u\) and \(v\) in this case, or that \(u\) and \(v\) are distinguishable.

From Lemma 1 we can immediately obtain the following result.

**Lemma 2.** If a graph \(G\) has a cycle \(C\) of length \(r\) so that there exists exactly one vertex of \(C\) whose degree is greater than 2 in \(G\), where \(r \equiv 1 \pmod{3}\), then \(\chi_c^r(G) \geq 4\).

Let \(P = u_1u_2\cdots u_r\) be a path of \(G\). We say that “\(P\) is cyclically colored by colors 1, 2 and 3” if the colors assigned to \(u_1u_2, u_2u_3\) and \(u_1u_4\) are 1, 2 and 3 respectively, and \(u_1u_5, u_5u_6\) and \(u_6u_7\) are colored by 1, 2 and 3 respectively, and the remaining edges are colored in a similar manner until the last one \(u_{r−1}u_r\) is colored. We may similarly give a definition of “\(s\) distinct edges \(e_1, e_2, \cdots, e_s\) are cyclically colored by colors 1, 2 and 3”. We use \(l(P)\) to denote the length of \(P\).

**Lemma 3.** Let \(C\) be a cycle of \(G\) of length \(r\), where \(r \equiv 1 \pmod{3}\). If \(C\) has exactly two vertices of degree 3 in \(G\) such that these two 3-vertices are not adjacent in \(G\) and their respective adjacent vertices not belonging to \(C\) are also not adjacent in \(G\), and the rest of \(r − 2\) vertices of \(C\) are all of degree 2 in \(G\), then the edges incident to the vertices of \(C\) can be properly colored using 3 colors such that any two consecutive vertices of \(C\) are distinguished from each other.

**Proof.** Suppose that \(C = x_1x_2\cdots x_rx_1\), where \(d_G(x_j) = 3, 3 \leq j \leq r − 1\); and \(d_G(x_i) = 2, i \neq 1, j\). Let \(e_i\) and \(e_j\) be the edges incident to \(x_i\) and \(x_j\), respectively, where \(e_i\) and \(e_j\) are not the edges of \(C\). Let \(P_1\) and \(P_2\) be the two paths connecting \(x_i\) and \(x_j\) in \(C\), respectively. We cyclically color \(e_1, x_1x_2, x_2x_3, \cdots, x_{j−1}x_j, e_j, x_jx_{j+1}, x_{j+1}x_{j+2}, \cdots, x_{r−1}x_r, x_rx_1\) by colors 1, 2 and 3. It is easy to verify that the resulting coloring satisfies the conditions of the lemma.

We call the coloring method used in the proof of Lemma 3 the \(\xi\)-coloring of \(C \cup \{e_1, e_j\}\). Let \(\varphi\) be a \(\xi\)-coloring of \(C \cup \{e_1, e_j\}\). It is obvious that \(\varphi\) is a partial avd-coloring of \(G\). Clearly, we can obtain a \(\xi\)-coloring such that the color of \(e_1\) is 2 or 3 by permuting the order of colors.

**Proposition 1.**

\[
\chi_c^r(H_1) = \begin{cases} 
4, & \text{if there are exactly two numbers of } r, s \text{ and } t \text{ both congruent to 1 modulo 3;} \\
3, & \text{otherwise.}
\end{cases}
\]

**Proof.** Set \(P_1 = xu_1u_2\cdots u_{r−1}y\), \(P_2 = xv_1v_2\cdots v_{s−1}y\), and \(P_3 = xw_1w_2\cdots w_{t−1}y\). Clearly, \(\chi_c^r(H_1) \geq 3\). By the symmetry of \(P_1, P_2\) and \(P_3\), we only describe 10 cases in which we can find a suitable corresponding edge coloring (see Table 1).

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<tr>
<th>Conditions</th>
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<td>(r \equiv s \equiv t \equiv 0 \pmod{3})</td>
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<td>(312)</td>
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<tr>
<td>(r \equiv 1, s \equiv t \equiv 0 \pmod{3})</td>
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<tr>
<td>(r \equiv 2, s \equiv t \equiv 0 \pmod{3})</td>
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<tr>
<td>(r \equiv s \equiv 1, t \equiv 0 \pmod{3})</td>
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It remains to show that there exists no 3-avd-coloring when \(r \equiv s \equiv 1, t \equiv 0 \pmod{3}\) or \(r \equiv 2, s \equiv t \equiv 1 \pmod{3}\). We consider the latter case only. Suppose that \(\varphi\) is a 3-avd-coloring of \(H_1\) when \(r \equiv 2, s \equiv t \equiv 1 \pmod{3}\). Clearly the colors of \(xu_1, xu_1\) and \(xw_1\) are pairwise distinct. Without loss of generality, we assume that \(\varphi(xu_1) = 1, \varphi(xu_1) = 2\) and \(\varphi(xw_1) = 3\). It follows from Lemma 1 and \(s \equiv t \equiv 1 \pmod{3}\) that \(\varphi(v_{s−1}y) = 2\) and \(\varphi(v_{s−1}y) = 3\). Clearly, the coloring of \(P_1\) must be \((123)\overline{123}\) or \((123)\overline{123}\), which results in that \(\varphi(u_{r−1}y) = \varphi(v_{s−1}y)\) or \(\varphi(u_{r−1}y) = \varphi(v_{s−1}y)\), a contradiction.

**Proposition 2.** \(\chi_c^r(H_i) = 4, i = 2, 3, 4\).

**Proof.** Since \(H_i\) has a 4-vertex or two adjacent 3-vertices, we have \(\chi_c^r(H_i) \geq 4, i = 2, 3, 4\). It remains to prove that \(H_i\) has a 4-avd-coloring, \(i = 2, 3, 4\). For \(H_2\), we assign colors 4, 2 and 3 to \(xy, u_{r−1}y\) and \(v_{s−1}y\), respectively. Then we cyclically color \(xu_1, u_1u_2, \cdots, u_{r−2}u_{r−1}\) by colors 1,
the color of $x^v_b$ by colors 3, 2 and 1. Clearly, the resulting coloring is a 4-avd-coloring of $G$.

For $H_4$, we assign colors 2, 4 and 4 to $xy$, $u_{r-1}y$ and $v_{r-1}y$, respectively. Then the path $xuv_1v_2\cdots v_{s-1}$ is cyclically colored by colors 1, 2 and 3. We assign colors 1 and 4 to $xu$, $v_1u_2\cdots u_{s-1}$ and we cyclically color of $H_5$ is cyclically colored by colors 3, 2 and 1.

It is easy to see that the resulting coloring is a 4-avd-coloring in each case.

**Proposition 3.**

\[ \chi'_a(H_5) = \begin{cases} 4, & \text{if } r \equiv 1 \pmod{3} \text{ or } s \equiv 1 \pmod{3}; \\ 3, & \text{otherwise.} \end{cases} \]

**Proof.** When $r \equiv 1 \pmod{3}$ or $s \equiv 1 \pmod{3}$, it follows from Lemma 2 that $\chi'_a(H_5) \geq 4$. So it is sufficient to give $H_5$ a 4-avd-coloring. We cyclically color $xuv_1, u_1u_2, \ldots, u_{s-2}v_{s-1}$ by colors 1, 2 and 3; and we cyclically color $w_{s-2}v_{s-1}$ by colors 4, 3 and 2. We assign colors 1 and 4 to $v_1w_2\cdots w_{s-1}$ and $u_{r-1}x$, respectively. Finally, $xu$, $u_1u_2\cdots u_{s-1}$ are cyclically colored by colors 3, 2 and 1. Clearly, the resulting coloring is a 4-avd-coloring of $H_5$.

When $r \not\equiv 1 \pmod{3}$ and $s \not\equiv 1 \pmod{3}$, it is sufficient to give $H_5$ a 3-avd-coloring. We cyclically color $xuv_1, w_1u_2, \ldots, w_{r-2}u_{r-1}$ by colors 1, 2 and 3; and we cyclically color $w_{r-2}v_{r-1}$ by colors 4, 3 and 2. We assign colors 1 and 4 to $v_1w_2\cdots w_{r-1}$ and $u_{r-1}x$, respectively. Finally, $xu$, $u_1u_2\cdots u_{r-1}$ can be colored in a similar manner.

3 Bicyclic graphs with pendant vertices

In this section, we investigate the avd-coloring of bicyclic graphs with at least one pendant vertex.

Let $G$ be a bicyclic graph, and let $G_1$ be the graph obtained from $G$ by deleting all the pendant vertices of $G$ (if $G$ contains no pendant vertex, then $G_1 = G$). Similarly, $G_2$ is the graph obtained from $G_1$ by deleting all the pendant vertices of $G_1$ (if $G_1$ contains no pendant vertex, then $G_2 = G_1$). This process continues, and we finally obtain a graph $H$ such that $H$ has no pendant vertex. Denote $H$ by $H(G)$.

**Fact.** If $G$ is a bicyclic graph, then $H(G) \in \{H_1, H_2, H_3, H_4, H_5\}$ (see Figure 1).

We will classify all bicyclic graphs with at least one pendant vertex into three classes: $\alpha$-type, $\beta$-type and $\gamma$-type.

Let $G$ be a bicyclic graph containing a pendant vertex. We use $G_{\Delta}$ to denote the subgraph of $G$ induced by all the vertices of maximum degree of $G$.

We call $G$ an $\alpha$-type graph, if all the following conditions hold:

1. $\Delta(G) = 3$ and $G_{\Delta}$ is an empty graph (i.e. a graph without edges).
2. $G$ has a cycle $C$ of length $r$ ($r \equiv 1 \pmod{3}$) such that there exists exactly one vertex of $C$ whose degree is 3 in $G$.

We call $G$ a $\beta$-type graph, if all the following conditions hold:

1. $\Delta(G) = 3$ and $G_{\Delta}$ is an empty graph.
2. $H(G)$ is $H_1$, and there exist exactly two numbers of $r$, $s$ and $t$ which are both congruent to 1 modulo 3, and the other is congruent to 2 modulo 3.
3. There exists an internal vertex $z_0$ of one $(x, y)$-path $P$ in $H_1$ whose length is congruent to 2 modulo 3, and $d_P(z_0, x)$ and $d_P(z_0, y)$ are both congruent to 1 modulo 3, where $d_P(z_0, x)$ denotes the distance between $z_0$ and $x$ in $P$.
4. $d_G(v) = d_{H_1}(v)$ for $v \in V(H_1) \setminus z_0$.

If $G$ is neither $\alpha$-type nor $\beta$-type, then we call $G$ a $\gamma$-type graph.

**Theorem 1.** Let $G$ be a bicyclic graph on $n$ vertex. If $G$ is $\alpha$-type or $\beta$-type, then $\chi'_a(G) = 4$; if $G$ is $\gamma$-type, then

\[ \chi'_a(G) = \begin{cases} \Delta(G), & \text{if } G_{\Delta} \text{ is an empty graph;} \\ \Delta(G) + 1, & \text{otherwise.} \end{cases} \]

**Proof.** We divide the proof into three cases.

**Case 1.** $G$ is an $\alpha$-type graph.

It follows from Lemma 2 that $\chi'_a(G) \geq 4$. We prove by induction on the number of vertices of $G$ that there is a 4-avd-coloring of $G$.

When $n = 10$, $G$ must be the graph illustrated in Figure 2, and a 4-avd-coloring of $G$ is also presented.

![Figure 2: Basis step in Case 1.](https://via.placeholder.com/150)

Suppose that the theorem is true for $\alpha$-type graphs with fewer than $n$ vertices, and let $G$ be an $\alpha$-type graph with
When the neighbor of each pendant vertex of $G$ is not in $H$. Let $v_0$ be a pendant vertex of $G$ such that $d(v_0, H)$ is maximum, where $d(v, H) = \min\{d_G(v, u) | u \in V(H)\}$. Clearly, $d(v_0, H) \geq 2$. Let $w$ be the neighbor of $v_0$, and $u$ the exactly one neighbor of $w$ in $G$ which is not a pendant vertex. Set $G' = G - v_0$. It is easy to see that $G'$ is an $\alpha$-type graph with $n - 1$ vertices. By induction hypothesis, $G'$ has a 4-avd-coloring. There are at least 2 colors missing from the edges incident to $w$ (since $\Delta(G) = 3$ and $wv_0$ has not been colored). Hence we can assign one missing color to $wv_0$ such that $w$ and $u$ are distinguishable.

When the neighbor of each pendant vertex of $G$ is in $H$. Let $v$ be any pendant vertex of $G$, and $w$ the neighbor of $v$. Set $G' = G - v$. It is obvious that $G'$ is an $\alpha$-type graph with $n - 1$ vertices. By induction hypothesis, $G'$ has a 4-avd-coloring. We assign a color missing from the edges incident to $w$ to $wv$ such that the coloring is proper. Note that $G$ is an empty graph, it is easy to verify that the resulting coloring is a 4-avd-coloring of $G$.

**Case 2.** $G$ is a $\beta$-type graph.

Without loss of generality, we assume that $r \equiv 2 \pmod{3}$ and $s \equiv t \equiv 1 \pmod{3}$, and $z_0 = u_j$ for some $2 \leq j \leq r - 2$. Clearly, $\chi'_\alpha(G) \geq 3$. We first show that $\chi'_\alpha(G) \geq 4$ by contradiction. Suppose that $\varphi$ is a 3-avd-coloring of $G$. Then the colors of $xu_1$, $xv_1$ and $xw_1$ are pairwise distinct. Without loss of generality, we assume that $\varphi(xu_1) = 3$, $\varphi(xv_1) = 2$ and $\varphi(xw_1) = 1$. From Lemma 1 it follows that $\varphi(yv_{s-1}) = 2$ and $\varphi(yv_{t-1}) = 1$. Thus $\varphi(yu_{s-1}) = 3$. Since the lengths of $xu_1u_2\cdots u_{r-1}y$ and $u_1u_{j+1}\cdots u_{r-1}y$ are both congruent to 1 modulo 3, it follows from Lemma 1 that $\varphi(u_ju_{j+1}) = \varphi(u_ju_{j+1}) = 3$, a contradiction.

It remains to show that $G$ has a 4-avd-coloring. By induction on the number of vertices of $G$.

When $n = 16$, $G$ must be the graph illustrated in Figure 3, and a 4-avd-coloring of $G$ is also presented.

![Figure 3: Basis step in Case 2.](image)

Suppose that the theorem is true for $\beta$-type graph with fewer than $n$ vertices, and let $G$ be a $\beta$-type graph with $n \geq 17$ vertices. We have $H(G) = H_1$.

When $G$ has a pendant vertex $v$ such that the neighbor of $v$ is not in $H(G)$. Let $v_0$ be a pendant vertex of $G$ such that $d(v_0, H(G))$ is maximum. Clearly $d(v_0, H(G)) \geq 2$. Let $w$ be the neighbor of $v_0$, and $u$ the only neighbor of $w$ which is not a pendant vertex. Set $G' = G - v_0$. Note that $G'$ has at least one pendant vertex.

(a) $G$ is an empty graph.
If $G'$ is $\alpha$-type or $\beta$-type, then $\Delta(G') = 4$ and $\Delta(G') = 3$. From Case 1, Case 2 or induction hypothesis, it follows that $G'$ has a $(\Delta(G') + 1)$-avd-coloring.

If $w$ is the vertex of maximum degree in $G$, then $u$ is not the vertex of maximum degree. We assign a color missing from the edges incident to $w$ to $wv_0$ such that the resulting coloring is proper.

If $w$ is not the vertex of maximum degree in $G$, then $w$ meets at most $\Delta(G) - 2$ colors, i.e., there are at least $2$ colors missing from the edges incident to $w$. Hence there is at least one remaining color with which to color $wv_0$ such that $w$ and $u$ are distinguished from each other.

(b) $G_\Delta$ is not an empty graph.

If $G'$ is $\alpha$-type or $\beta$-type, then $\Delta(G') = \Delta(G) = 3$. From Case 1, Case 2 or induction hypothesis, it follows that $G'$ has a $(\Delta(G) + 1)$-avd-coloring. Clearly, $w$ meets at most $\Delta(G) - 1$ colors (since $wv_0$ has not been colored), thus there are at least $2$ colors missing from the edges incident to $w$. Therefore there is at least one remaining color with which to color $wv_0$ such that $w$ and $u$ are distinguished from each other.

Subcase 3.2. The neighbor of each pendant vertex of $G$ is in $H$, and $H$ has a vertex $z$ of degree two in $H$ and at least three in $G$ such that $d_G(z) \neq d_G(z')$, where $z'$ is one neighbor of $z$ in $H$.

Let $z''$ be the other neighbor of $z$ in $H$, i.e. $N_H(z) = \{z', z''\}$, where $N_H(z)$ denotes the neighborhood of $z$ in $H$. Set $v \in N_G(z) \setminus \{z', z''\}$ and $G' = G - v$.

(a) $G_\Delta$ is not an empty graph.

Note that $\Delta(G') = \Delta(G)$. If $G'$ has no pendant vertex, then $G'$ has a $(\Delta(G) + 1)$-avd-coloring from Propositions 1-3. If $G'$ has a pendant vertex, then $G'$ has a $(\Delta(G) + 1)$-avd-coloring from Case 1, Case 2 or induction hypothesis. Since there are at least two colors missing from the edges incident to $z$, there is at least one remaining color with which to color $vz$ such that the resulting coloring distinguishes $z$ and $z''$. Clearly, $z$ and $z'$ are distinguishable ($z$ and $z'$ have distinct degree in $G$). Therefore $G$ has a $(\Delta(G) + 1)$-avd-coloring.

(b) $G_\Delta$ is an empty graph.

Type 1: $G'$ has no pendant vertex. It is easy to see that $G' = H_1$, where $i = 1, 3, 5$.

When $G' = H_1$, let $\varphi$ be a 4-avd-coloring of $H_1$ obtained from the proof of Proposition 2, and we assign one color missing from the edges incident to $z$ to $vz$. Clearly, the resulting coloring is a 4-avd-coloring of $G$.

When $G' = H_1$, let $\varphi$ be a 3-avd-coloring of $H_1$ obtained from the proof of Proposition 1 except the cases $r \equiv s \equiv 1 \pmod{3}$, $t \equiv 0 \pmod{3}$ and $r \equiv 2 \pmod{3}$, $s \equiv t \equiv 1 \pmod{3}$. We assign one color missing from the edges incident to $z$ to $vz$ such that the coloring obtained is proper. Clearly the resulting coloring is a 3-avd-coloring of $G$ (since $d_G(z') = d_G(z') = 2$ and $d_G(z) = 3$). So there remains to consider the cases $r \equiv s \equiv 1 \pmod{3}$, $t \equiv 0 \pmod{3}$ and $r \equiv 2 \pmod{3}$, $s \equiv t \equiv 1 \pmod{3}$.

(i) $r \equiv s \equiv 1 \pmod{3}$, $t \equiv 0 \pmod{3}$.

If $\varphi = u_j$ ($2 \leq j \leq r - 2$), then the colorings of $P_2$ and $P_3$ are $(231)^{t\equiv0\pmod{3}}$ and $(231)^{t\equiv0\pmod{3}}$, respectively. The coloring of $P_3$ is $(312)^{1\pmod{3}}3^{(j \equiv 1 \pmod{3})}$, $(312)^{1\pmod{3}}3^{(j \equiv 1 \pmod{3})}$ or $(312)^{3\pmod{3}1\pmod{3}}$ (if $j \equiv 2 \pmod{3}$), where $P_1$, $P_2$ and $P_3$ are defined as Proposition 1. Then we properly color $u_j v$, and we obtain a 3-avd-coloring of $G$.

The case that $z = v_j$ ($2 \leq j \leq s - 2$) can be disposed by a similar manner.

If $z = w_j$ ($2 \leq j \leq t - 2$), then the colorings of $P_2$ and $P_3$ are $(231)^{t\equiv0\pmod{3}}$ and $(231)^{t\equiv0\pmod{3}}$, respectively. The coloring of $P_3$ is $(312)^{1\pmod{3}}3^{(j \equiv 1 \pmod{3})}$, $(312)^{1\pmod{3}}3^{(j \equiv 1 \pmod{3})}$ or $(312)^{3\pmod{3}1\pmod{3}}$ (if $j \equiv 2 \pmod{3}$). Then we properly color $w_j v$, and we obtain a 3-avd-coloring of $G$.

(ii) $r \equiv 2 \pmod{3}$, $s \equiv t \equiv 1 \pmod{3}$.

If $z = u_j$ ($2 \leq j \leq r - 2$), then the colorings of $P_2$ and $P_3$ are $(231)^{t\equiv0\pmod{3}}$ and $(231)^{t\equiv0\pmod{3}}$, respectively. The coloring of $P_1$ is $(312)^{1\pmod{3}}3^{(j \equiv 1 \pmod{3})}$ or $(312)^{3\pmod{3}1\pmod{3}}$ (if $j \equiv 2 \pmod{3}$). Note that $G$ is not a $\beta$-type graph, thus $j \neq 1 \pmod{3}$. Then we properly color $u_j v$.

The case that $z = w_j$ ($2 \leq j \leq t - 2$) can be disposed by a similar manner.

When $G' = H_3$, then $r \equiv 1 \pmod{3}$ and $s \equiv 1 \pmod{3}$ cannot both hold.

(i) $r \equiv 1 \pmod{3}$ and $s \equiv 1 \pmod{3}$.

Clearly that $z = u_j$ ($2 \leq j \leq r - 2$). We cyclically color $xw_1, w_1 w_2, \ldots, w_{s-1} y$ by colors $1, 2$ and $3$. Suppose that the color of $w_{t-1} y$ is $a$, where $a \in \{1, 2, 3\}$.

If $s \equiv 0 \pmod{3}$, then the coloring of $yv_1, v_1 w_2, \ldots, v_{s-1} y$ is $[(a + 2) a (a + 1) j]$ if $s \equiv 2 \pmod{3}$, then the coloring of $yv_1, v_1 w_2, \ldots, v_{s-1} y$ is $[(a + 1) (a + 2) a j]$.

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1)(a + 2), where addition is taken modulo 3.

The coloring of \( xu_1, u_1u_2, \ldots, u_{r-1}u_r, u_rx \) is \((231)\frac{j}{3}, (312)\frac{j+1}{3}(132)\frac{j+2}{3}\) or \((231)\frac{j+2}{3}23\) depending on \(j \equiv 0, 1, \) or \(2 \pmod{3}\). Then we properly color \( ev_j \), and we obtain a 3-avd-coloring of \( G \).

(ii) \( r \not\equiv 1 \pmod{3} \) and \( s \not\equiv 1 \pmod{3} \).

Let \( \varphi \) be a 3-avd-coloring of \( H_s \) obtained from the proof of Proposition 3. Then we properly color \( vz \), and we obtain a 3-avd-coloring of \( G \).

Type 2: \( G' \) has a pendant vertex and \( G' \) is an \( \alpha \)-type graph.

In this case it is obvious that \( 3 \leq \Delta(G) \leq 4 \) and \( H(G) = H_s \). Without loss of generality, we assume that \( r \equiv 1 \pmod{3} \). Set \( C = xu_1u_2 \cdots u_{r-1}x \).

When \( \Delta(G) = 3 \), we have \( z = u_j (2 \leq j \leq r - 2) \). Note that \( C \cup \{v_j, xu_1\} \) satisfies the conditions of Lemma 3. It follows from Lemma 3 that \( C \cup \{v_j, xu_1\} \) has a \( \xi \)-coloring \( \varphi \) such that the color of \( xu_1 \) is 1. We cyclically color \( w_1u_2, w_2u_3, \ldots, w_{r-1}y \) by colors 2, 3 and 1. Assume that the color of \( w_{r-1}y \) is \( a \), where \( a \in \{1, 2, 3\} \).

If \( s \equiv 0 \pmod{3} \), then the coloring of \( C' = yw_1u_2 \cdots w_{r-1}y \) is \([(a + 2)a(a + 1)]\frac{j}{3} \) (starting from \( yw_1 \) in clockwise), where addition is taken modulo 3.

If \( s \equiv 1 \pmod{3} \), then there exists some vertex \( v_i \) of \( C' \) such that \( d_G(v_i) = 3 \). Let \( e \) be the pendant edge incident to \( v_i \). By Lemma 3, \( C' \cup \{v_i, yw_1y\} \) has a \( \xi \)-coloring such that the color of \( w_{r-1}y \) is \( a \).

If \( s \equiv 2 \pmod{3} \), then the coloring of \( C' \) is \([(a+1)(a+2)]\frac{j}{3} \) \( (a + 1)(a + 2) \) (starting from \( yw_1 \) in clockwise), where addition is taken modulo 3.

Finally we properly color all the uncolored pendant edges and obtain a 3-avd-coloring of \( G \).

When \( \Delta(G) = 4 \), then \( G \) has exactly one vertex of maximum degree. Clearly \( z \) is just the vertex of maximum degree in \( G \), and any two 3-vertices are not adjacent in \( G \). We cyclically color the edges of \( xu_1u_2 \cdots u_{r-1}x \) by colors 1, 2 and 3, and assign color 4 to \( u_{r-1}x \). Starting from \( xu_1 \), we cyclically color the edges of \( xu_1u_2 \cdots w_{r-1}y \) by colors 2, 3 and 1. Assume that the color of \( w_{r-1}y \) is \( a \), then we cyclically color the edges of \( yw_1u_2 \cdots w_{r-1}y \) by colors \( a + 1, a, a + 2 \), and assign color 4 to \( v_iy \), where addition is taken modulo 3. Finally we properly color all the pendant edges. It is not difficult to verify, whether \( z = w_j \) \((1 \leq j \leq t - 1)\) or \( z = v_j \) \((1 \leq j \leq s - 1)\), that the resulting coloring is a 4-avd-coloring of \( G \).

Type 3: \( G' \) has a pendant vertex and \( G' \) is a \( \beta \)-type graph.

Without loss of generality, we assume that \( r = 2 \pmod{3} \) and \( s = t = 1 \pmod{3} \), and \( s_0 = u_j \) for some \( 1 \leq j \leq r - 1 \). Clearly \( 3 \leq \Delta(G) \leq 4 \). Set \( P_1 = xu_1u_2 \cdots u_{r-1}y, P_2 = xu_1u_2 \cdots v_{r-1}y \) and \( P_3 = xu_1u_2 \cdots u_{r-1}y \).

When \( \Delta(G) = 4 \), then \( z = u_j \) is the only vertex of maximum degree. The colorings of \( P_2, P_3 \) and \( P_1 \) are \((231)\frac{1}{3}23 \) and \((231)\frac{1}{3}23(312)\frac{1}{3}23 \) respectively. Two pendant edges incident to \( u_j \) are assigned colors 1 and 2. It is obvious that the resulting coloring is a 4-avd-coloring of \( G \).

When \( \Delta(G) = 3 \), let \( e \) be the pendant edge incident to \( u_j \). There are four cases to consider.

If \( z = u_i \) \((2 \leq i \leq j - 2)\), then the colorings of \( P_2 \) and \( P_3 \) are \((231)\frac{1}{3}23 \) and \((231)\frac{1}{3}23(312)\frac{1}{3}23 \). Set \( P_1 = xu_1u_2 \cdots u_{i-1}u_i \). The colorings of \( P_1, P_2' \) and \( P_3' \) are given as follows (see Table 2):

<table>
<thead>
<tr>
<th>Conditions</th>
<th>( P_1' )</th>
<th>( P_2' )</th>
<th>( P_3' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i \equiv 0 \pmod{3} )</td>
<td>((312)\frac{1}{3}23 )</td>
<td>((123)\frac{1}{3}23 )</td>
<td>((312)\frac{1}{3}23 )</td>
</tr>
<tr>
<td>( i \equiv 1 \pmod{3} )</td>
<td>((312)\frac{1}{3}23 )</td>
<td>((231)\frac{1}{3}23 )</td>
<td>((312)\frac{1}{3}23 )</td>
</tr>
<tr>
<td>( i \equiv 2 \pmod{3} )</td>
<td>((312)\frac{1}{3}23 )</td>
<td>((123)\frac{1}{3}23 )</td>
<td>((312)\frac{1}{3}23 )</td>
</tr>
</tbody>
</table>

Finally we properly color \( e \) and \( v_i, v \), and we obtain a 3-avd-coloring of \( G \).

The case that \( z = u_i \) \((j + 2 \leq i \leq r - 2)\) can be dealt with in a similar manner as the above case.

If \( z = v_i \) \((2 \leq i \leq s - 2)\), then the colorings of \( P_1 \) and \( P_3 \) are \((231)\frac{1}{3}23 \) and \((312)\frac{1}{3}23 \). Set \( P_2' = xu_1u_2 \cdots v_{i-1}v_i \). Then the colorings of \( P_2' \) and \( P_2'' \) are given as follows (see Table 3):

<table>
<thead>
<tr>
<th>Conditions</th>
<th>The coloring of ( P_1' )</th>
<th>The coloring of ( P_2' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i \equiv 0 \pmod{3} )</td>
<td>((231)\frac{1}{3}23 )</td>
<td>((123)\frac{1}{3}23 )</td>
</tr>
<tr>
<td>( i \equiv 1 \pmod{3} )</td>
<td>((231)\frac{1}{3}23 )</td>
<td>((312)\frac{1}{3}23 )</td>
</tr>
<tr>
<td>( i \equiv 2 \pmod{3} )</td>
<td>((231)\frac{1}{3}23 )</td>
<td>((123)\frac{1}{3}23 )</td>
</tr>
</tbody>
</table>

Finally we properly color \( e \) and \( v_i, v \), and we obtain a 3-avd-coloring of \( G \).

The case that \( z = u_j \) \((2 \leq j \leq t - 2)\) can be dealt with in a similar manner as the above case.

Type 4: \( G' \) has a pendant vertex and \( G' \) is a \( \gamma \)-type graph.

By induction hypothesis, \( G' \) has a \( \Delta(G) \)-avd-coloring. If \( z \) is the vertex of maximum degree, then we properly color \( zv \). If \( z \) is not the vertex of maximum degree, then there are at least two colors missing from the edges incident to \( z \). Hence we can assign one missing color to \( zv \) such
that $z$ and $z''$ are distinguishable. Clearly the resulting coloring is a $\Delta(G)$-avd-coloring of $G$.

**Subcase 3.3.** The neighbor of each pendant vertex of $G$ is in $H$, and $H$ has a vertex $z$ of degree two in $H$ and degree at least 3 in $G$. For any such $z$, $d_G(z') = d_G(z'') = d_G(x)$, where $N_H(z) = \{z', z''\}$.

(i) $H = H_1$.

In view of the symmetry of three paths from $x$ to $y$ in $H$, $G$ must be one of the following three graphs, where $\Delta(G) = k + 2$ and $k \geq 1$ (see Figure 5).

![Diagram](https://via.placeholder.com/150)

Figure 5: Illustrations in Subcase 3.3(i).

We just show that $G_{11}$ (see Figure 5(a)) has a $(\Delta(G) + 1)$-avd-coloring. The other cases can be dealt with in a similar manner. Note that in $G_{11}$ the number of pendant edges incident to $x$ or $y$ is $k - 1$, and the number of pendant edges incident to each of the other vertices of $H_1$ is $k$. Suppose that $k \geq 2$.

If $r + s$ is even, then we alternately color the edges of cycle $xu_1u_2 \cdots u_{r-1}yv_{s-1} \cdots v_2v_1x$ starting from $xu_1$ by colors $k + 2$ and $k + 3$. The uncolored edges incident to $x$, $u_1$, $u_2$, \ldots, $u_{r-1}$, $y$, $v_{s-1}$, \ldots, $v_1$ are alternately colored by $\{1, 2, \ldots, k\}$ and $\{2, 3, \ldots, k + 1\}$ such that the colors of $xu_1$ and $yv_{s-1}$ are $1$ and $2$ respectively.

If $r + s$ is odd, the coloring of cycle $xu_1u_2 \cdots u_{r-1}yv_{s-1} \cdots v_2v_1x$ starting from $xu_1$ is $[k + 2](k + 3)$.

The uncolored edges incident to $u_1$, $u_2$, \ldots, $u_{r-1}$, $y$, $v_{s-1}$, \ldots, $v_1$ are alternately colored by $\{1, 2, \ldots, k\}$ and $\{2, 3, \ldots, k + 1\}$ such that the color of $w_{r-1}y$ is $2$.

Then we cyclically color $w_1w_2$, $w_2w_3$, \ldots, $w_{t-2}w_{t-1}$ starting from $w_1w_2$ by colors $3$, $4$ and $1$. The pendant edges incident to $w_{t-1}$, $w_{t-2}$, \ldots, $w_2$ are colored such that the missing color of these vertices are alternately $k + 3$ and $2$. We color the pendant edges incident to $w_1$ such that the missing color of $w_1$ is $k + 2$. It is not difficult to verify that the resulting coloring is a $(\Delta(G) + 1)$-avd-coloring of $G_{11}$.

When $\Delta(G) = 3$ (i.e. $k = 1$), we assign colors $1$, $3$ and $4$ to $xu_1$, $xv_1$ and $xv_1$, respectively. The edges of $u_1u_2 \cdots u_{r-1}yv_{s-1} \cdots v_2v_1$ are cyclically colored starting from $u_1v_2$ by $2$, $3$ and $1$. The pendant edge incident to each $u_i$ ($2 \leq i \leq r - 1$) or $v_j$ ($3 \leq j \leq s - 1$) is colored by $4$, and the pendant edge incident to $u_1$ is colored by $3$. If the colors of $v_3v_2$ and $v_2v_1$ are $1$ and $2$ respectively, then we assign colors $4$ and $3$ to the pendant edges incident to $v_2$ and $v_1$, respectively. If the colors of $v_3v_2$ and $v_2v_1$ are $2$ and $3$ respectively, then we assign colors $1$ and $2$ to the pendant edges incident to $v_2$ and $v_1$, respectively. If the colors of $v_3v_2$ and $v_2v_1$ are $3$ and $1$ respectively, then we assign colors $4$ and $2$ to the pendant edges incident to $v_2$ and $v_1$, respectively. Denote by $c(e)$ the color that has been assigned to $e$.

If $t = 2$, then there are three cases to consider. When $c(yv_{s-1}) = 2$ and $c(yv_{s-1}) = 3$, we exchange the colors of $u_{r-1}y$ and the pendant edge incident to $u_{r-1}$ (i.e. we recolor $u_{r-1}y$ by color $4$, and the pendant edge incident to $u_{r-1}$ by $2$). Then we assign colors $2$ and $1$ to $u_1y$ and the pendant edge incident to $w_1$, respectively. When $c(yv_{s-1}) = 3$ and $c(yv_{s-1}) = 1$, we assign colors $2$ and $4$ to $w_1y$ and the pendant edge incident to $w_1$, respectively. When $c(yv_{s-1}) = 1$ and $c(yv_{s-1}) = 2$, we assign colors $4$ and $2$ to $w_1y$ and the pendant edge incident to $w_1$, respectively.

If $t \geq 3$, then there are three cases to consider. When $c(yv_{s-1}) = 2$ and $c(yv_{s-1}) = 3$, we assign color $1$ to $w_{t-3}w_{t-2}$ and $w_{t-2}w_{t-1}$ are $2$ and $4$ respectively, then the pendant edge incident to each $w_i$ ($1 \leq i \leq t - 2$) is colored by $1$, and the pendant edge incident to $w_{t-1}$ is colored by $3$. If the colors of $w_{t-3}w_{t-2}$ and $w_{t-2}w_{t-1}$ are $4$ and $3$ respectively, then the pendant edge incident to each $w_i$ ($1 \leq i \leq t - 3$) is colored by $1$, and the pendant edge incident to $w_{t-2}$ or $w_{t-1}$ is colored by $2$, and the pendant edge incident to $w_{t-1}$ is colored by $4$.

When $c(yv_{s-1}) = 3$ and $c(yv_{s-1}) = 1$, we assign color $2$ to $w_{t-1}y$ and cyclically color $w_1w_2$, $w_2w_3$, \ldots, $w_{t-2}w_{t-1}$ by $1$, $4$ and $3$. If the colors of $w_{t-3}w_{t-2}$ and $w_{t-2}w_{t-1}$ are $1$ and $4$ respectively, then the pendant edge incident to each $w_i$ ($1 \leq i \leq t - 2$) is colored by $2$, and the pendant edge incident to $w_{t-1}$ is colored by $3$. If the colors of $w_{t-3}w_{t-2}$ and $w_{t-2}w_{t-1}$ are $4$ and $3$ respectively, then the pendant edge incident to each $w_i$ ($1 \leq i \leq t - 3$) is colored by $2$, and the pendant edge incident to $w_{t-2}$ or
$w_{t-1}$ is colored by 1 or 4, respectively. If the colors of $w_{1-3}w_{1-2}$ and $w_{1-2}w_{1-1}$ are 3 and 1 respectively, then the pendant edge incident to each $w_i \ (1 \leq i \leq t-2)$ is colored by 2, and the pendant edge incident to $w_{t-1}$ is colored by 4.

When $c(yu_{u-1}) = 1$ and $c(yv_{u-1}) = 2$, we assign color 3 to $w_{u-1}y$ and cyclically color $w_1w_2, w_2w_3, \ldots, w_{u-2}w_{u-1}$ by 2, 1 and 4. Suppose that $t \geq 4$. If the colors of $w_{1-3}w_{1-2}$ and $w_{1-2}w_{1-1}$ are 2 and 1 respectively, then the pendant edge incident to each $w_i \ (2 \leq i \leq t-2)$ is colored by 3, and the pendant edge incident to $w_1$ or $w_{t-1}$ is colored by 4. If the colors of $w_{1-3}w_{1-2}$ and $w_{1-2}w_{1-1}$ are 1 and 4 respectively, then the pendant edge incident to each $w_i \ (2 \leq i \leq t-2)$ is colored by 3, and the pendant edge incident to $w_1$ or $w_{t-1}$ is colored by 4 or 2, respectively. If the colors of $w_{1-3}w_{1-2}$ and $w_{1-2}w_{1-1}$ are 4 and 2 respectively, then the pendant edge incident to each $w_i \ (2 \leq i \leq t-3)$ is colored by 3, and the pendant edge incident to $w_1$ or $w_{t-1}$ is colored by 4. The pendant edge incident to $w_{t-1}$ is colored by 1. If $t = 3$, then the pendant edge incident to $w_1$ or $w_2$ is colored by 1 or 4, respectively.

It is not difficult to see that the resulting coloring is a 4-avd-coloring of $G_{11}$.

(ii) $H = H_2$.

In view of the symmetry of graph, $G$ must be one of the following two cases, where $\Delta(G) = k + 2$ and $k \geq 1$ (see Figure 6).

![Figure 6: Illustrations in Subcase 3.3(ii).](image)

We just show that $G_{21}$ (see Figure 6(a)) has a $(\Delta(G) + 1)$-avd-coloring, and the case $G_{22}$ (see Figure 6(b)) can be dealt with in a similar manner. Note that in $G_{21}$ the number of pendant edges incident to $x$ or $y$ is $k - 1$, and the number of pendant edges incident to each of the other vertices of $H$ is $k$.

When $\Delta(G) \geq 4$ (i.e. $k \geq 2$), then we alternately color $xu_1, u_1u_2, \ldots, u_{t-2}y, yv_{u-2}, v_{u-2}v_{u-1}, \ldots, v_2v_1$ by colors $k + 3$ and $k + 2$ starting from $xu_1$. Assign color 1 to $xv_1$. We alternately assign colors $\{1,2,\ldots,k\}$ and $\{2,3,\ldots,k+1\}$ to the pendant edges incident to $u_1, u_2, \ldots, u_{t-2}, y, v_{u-2}, \ldots, v_2$. Note that the number of pendant edges incident to $x$ is $k - 1$, so we consider here that $xy$ is a “pendant incident” edge to $y$ and color it by 2. The pendant edges incident to $x$ are colored by $\{3,4,\ldots,k+1\}$. If $r + s$ is even, then the pendant edges incident to $v_1$ are colored by $\{2,3,\ldots,k+2\}$. If $r + s$ is odd, then the pendant edges incident to $v_1$ are colored by $\{2,3,\ldots,k+1\}$. It is not difficult to verify that the resulting coloring is a $(\Delta(G) + 1)$-coloring of $G_{21}$.

When $\Delta(G) = 3$ (i.e. $k = 1$), we assign colors 1, 2, 3, 2 and 4 to $xu_1, xu_1, xy, yu_{u-2}$ and $yu_{u-2}$ respectively. The edges of $u_1u_2 \ldots u_{t-2}$ are cyclically colored by colors 4, 3 and 1 starting from $u_1u_2$. We assign color 3 to the pendant edge incident to $u_1$. If the color of $u_{u-3}u_{u-2}$ is 3 and the color of $u_{u-3}u_{u-1}$ is 1, then the pendant edge incident to $u_{u-3}$ is colored by 2 and the pendant edge incident to $u_{u-2}$ is colored by 4. If the colors of $u_{u-3}u_{u-1}$ and $u_{u-3}u_{u-2}$ are 1 and 4 respectively, then the pendant edges incident to $u_{u-3}$ and $u_{u-2}$ are colored by 3 and 1, respectively. If the colors of $u_{u-3}u_{u-2}$ and $u_{u-3}u_{u-1}$ are 4 and 3 respectively, then the pendant edges incident to $u_{u-3}$ and $u_{u-2}$ are colored by 2 and 1 respectively. Then each pendant edge incident to $u_2, u_3, \ldots, u_{t-1}$ is colored by 2, respectively. The edges incident to $v_1, v_2, \ldots, v_{u-2}$ are colored in a similar manner. It is not difficult to see that the resulting coloring is a 4-avd-coloring of $G$.

(iii) $H = H_3$.

By the symmetry of graph, $G$ must be one of the following two cases, where $\Delta(G) = k + 2$ and $k \geq 2$ (see Figure 7).

![Figure 7: Illustrations in Subcase 3.3(iii).](image)

We just show that $G_{31}$ (see Figure 7(a)) has a $(\Delta(G) + 1)$-avd-coloring. The case $G_{32}$ (see Figure 7(b)) can be dealt with in a similar manner. Note that in $G_{31}$ the number of pendant edges incident to $x$ is $k - 2$, and the number of pendant edges incident to each of the other vertices of $H_3$ is $k$.

We assign colors 1, 2, 3 and 4 to the edges $xu_1, xu_1, xu_{u-1}$ and $xv_{u-1}$ respectively. We alternately color the edges of $u_1u_2 \ldots u_{t-1}$ starting from $u_1u_2$ by colors $k + 3$ and $k + 2$. If the color of $u_{u-2}u_{u-1}$ is 3 and $k + 3$, then the pendant edges incident to $u_{u-3}$ are colored by $\{1,2,4,5,\ldots,k+1\}$; if the color of $u_{u-2}u_{u-1}$ is $k + 2$, then the pendant edges incident to $u_{u-2}$ are colored by $\{1,4,5,\ldots,k+1,3\}$. We alternately color the pendant edges incident to $u_2, u_3, \ldots, u_{t-2}$ starting from $u_2$ by colors $\{1,2,\ldots,k\}$ and $\{2,3,\ldots,k+1\}$. The pendant edges incident to $u_1$ are colored by $\{2,3,\ldots,k+1\}$. Assign colors $\{5,6,\ldots,k+2\}$ to the pendant edges incident to $x$ (if $k = 2$, then $G$ has no pendant edge incident to $x$).
We alternately color the edges of \( v_1 v_2 \cdots v_{s-1} \) starting from \( v_1 v_2 \) by colors 1 and 2. The pendant edges incident to \( v_1 \) and \( v_{s-1} \) are colored by \( \{4,5,\cdots,k+3\} \) and \( \{3,5,6,\cdots,k+3\} \) respectively. We alternately color the pendant edges incident to \( v_2, v_3, \cdots, v_{s-2} \) starting from \( v_2 \) by colors \( \{3,4,\cdots,k+2\} \) and \( \{4,5,\cdots,k+3\} \). It is not difficult to verify that the resulting coloring is a \((\Delta(G)+1)\)-avd-coloring of \( G_{41} \).

(iv) \( H = H_4 \).

By the symmetry of graph, \( G \) must be one of the following two cases, where \( \Delta(G) = k+2 \) and \( k \geq 1 \) (see Figure 8).

![Figure 8: Illustrations in Subcase 3.3(iv).](image)

We just show that \( G_{41} \) (see Figure 8(a)) has a \((\Delta(G)+1)\)-avd-coloring. The case that \( G = G_{42} \) (see Figure 8(b)) can be dealt with in a similar manner. Note that in \( G_{41} \) the number of pendant edges incident to \( x \) and \( y \) are \( k-1 \) and \( l-1 \), respectively. The number of pendant edges incident to each of \( u_i \) (\( 1 \leq i \leq r-1 \)) is \( k \), and the number of pendant edges incident to each \( v_j \) (\( 1 \leq j \leq s-1 \)) is \( l \). We assume that \( k \geq l \).

If \( r \) and \( s \) are both even, then we alternately color the edges of cycle \( xu_1u_2\cdots u_{r-1}x \) by colors \( k+3 \) and \( k+2 \). The pendant edges incident to \( u_1, u_2, \cdots, u_{r-1} \) are alternately colored by \( \{1,2,\cdots,k\} \) and \( \{2,3,\cdots,k+1\} \). Assign color 2 to \( xy \) and color the pendant edges incident to \( x \) by \( \{3,4,\cdots,k+1\} \). Then we alternately color \( yv_1, v_1v_2, \cdots, v_{s-2}v_{s-1} \) by colors \( l+2 \) and \( l+3 \). Assign color 1 to \( yv_{s-1} \). The pendant edges incident to \( v_1, v_2, \cdots, v_{s-2} \) are alternately colored by \( \{1,2,\cdots,l\} \) and \( \{2,3,\cdots,l+1\} \). The pendant edges incident to \( v_{s-1} \) and \( y \) are colored by \( \{2,3,\cdots,l+3\} \) and \( \{3,4,\cdots,l+1\} \), respectively.

If \( r \) is even and \( s \) is odd, then the edges incident to \( x, u_1, \cdots, u_{r-1} \) are colored as the same as the above case. The edges of \( yu_1u_2\cdots u_{r-1}y \) are alternately colored by \( l+2 \) and \( l+3 \), and assign color 1 to \( yu_{r-1} \). The pendant edges incident to \( u_1, u_2, \cdots, u_{r-2} \) are alternately colored by \( \{2,3,\cdots,l+1\} \) and \( \{1,2,\cdots,l\} \). The pendant edges incident to \( u_{r-1} \) and \( y \) are colored by \( \{2,3,\cdots,l+2\} \) and \( \{3,4,\cdots,l+1\} \), respectively.

If \( r \) is odd and \( s \) is even, then the edges of \( xu_1u_2\cdots u_{r-1}x \) are alternately colored by \( k+3 \) and \( k+2 \). Assign color 1 to \( xu_{r-1} \). The pendant edges incident to \( u_1, u_2, \cdots, u_{r-2} \) are alternately colored by \( \{1,2,\cdots,k\} \) and \( \{2,3,\cdots,k+1\} \). The pendant edges incident to \( u_{r-1} \) and \( x \) are colored by \( \{2,3,\cdots,k+1\} \) and \( \{3,4,\cdots,k+1\} \), respectively. We assign color 2 to \( xy \). The edges of cycle \( yu_1u_2\cdots u_{s-1}y \) are alternately colored starting from \( yu_1 \) by colors \( l+2 \) and \( l+3 \). The pendant edges incident to \( v_1, v_2, \cdots, v_{s-1} \) are alternately colored by \( \{1,2,\cdots,l\} \) and \( \{2,3,\cdots,l+1\} \), and the pendant edges incident to \( y \) are colored by \( \{3,4,\cdots,l+1\} \), respectively.

If \( r \) and \( s \) are both odd, then the edges incident to \( x, u_1, \cdots, u_{r-1} \) are colored as the same as the above case (i.e. the case that \( r \) is odd and \( s \) is even). The edges of \( yu_1u_2\cdots u_{s-1}y \) are alternately colored by \( l+2 \) and \( l+3 \). Assign color 1 to \( yu_{s-1} \). The pendant edges incident to \( v_1, v_2, \cdots, v_{s-2} \) are alternately colored by \( \{2,3,\cdots,l+1\} \) and \( \{1,2,\cdots,l\} \). The pendant edges incident to \( v_{s-1} \) and \( y \) are colored by \( \{3,4,\cdots,l+2\} \) and \( \{3,4,\cdots,l+1\} \), respectively.

It is not difficult to verify that the resulting coloring is a \((\Delta(G)+1)\)-avd-coloring of \( G_{41} \).

(v) \( H = H_5 \).

By the symmetry of graph, \( G \) must be one of the following six cases, where \( \Delta(G) = k+2 \) and \( k \geq 1 \) (see Figure 9).

![Figure 9: Illustrations in Subcase 3.3(v).](image)
We just show that $G_{51}$ has a $(\Delta(G)+1)$-avd-coloring, the other cases can be dealt with in a similar manner. Note that in $G_{51}$, the number of pendant edges incident to $x$ or $y$ is $k-1$, and the number of pendant edges incident to each of the other vertices of $H_2$ is $k$. Suppose that $k \geq 2$.

We alternately color the edges of $u_1u_2 \cdots u_{r-1}x$ $w_1w_2 \cdots w_{r-1}y$ starting from $u_1u_2$ by colors $k+2$ and $k+3$. The pendant edges incident to $u_2, u_3, \cdots, u_{r-1}, x, w_1, \cdots, w_{r-1}, y, v_1, \cdots, v_{s-2}$ are alternately colored by $\{1,2, \cdots, k\}$ and $\{2,3, \cdots, k+1\}$. Note here that the number of pendant incident to $x$ or $y$ is $k-1$, we consider $xu_1$ or $yu_{s-1}$ the “pendant edges” incident to $x$ or $y$, respectively. The colors of $xu_1$ and $yu_{s-1}$ are both equal to 2. The pendant edges incident to $u_1$ or $v_{s-1}$ are colored by $\{1,3,4, \cdots, k+1\}$.

When $\Delta(G) = 3$ (i.e. $k = 1$), we assign colors 3, 2 and 1 to $xu_1$, $xw_{r-1}$ and $xy$, respectively. The edges of $u_1u_2 \cdots u_{r-1}$ are cyclically colored by 4, 1 and 3, and the pendant edge incident to each $u_i$ ($1 \leq i \leq r-3$) is colored by 2. If the colors of $u_{r-3}u_{r-2}$ and $u_{r-2}u_{r-1}$ are 4 and 1 respectively, then we assign colors 3 and 4 to the pendant edges incident to $u_{r-2}$ and $u_{r-1}$, respectively. If the colors of $u_{r-3}u_{r-2}$ and $u_{r-2}u_{r-1}$ are 3 and 4 respectively, then we assign colors 2 and 4 to the pendant edges incident to $u_{r-2}$ and $u_{r-1}$, respectively. If the colors of $u_{r-3}u_{r-2}$ and $u_{r-2}u_{r-1}$ are 3 and 4, respectively, then we assign colors 1 and 3 to the pendant edges incident to $u_{r-2}$ and $u_{r-1}$, respectively. We assign colors 2, 3 and 1 to $w_1w_2, w_2w_3, \cdots, w_{s-1}y$ cyclically, and color 4 to the pendant edge incident to each $w_i$ ($1 \leq i \leq s-1$). We assign two colors in $\{1,2,3\}\{c(u_{r-1}y)\}$ to $yu_1$ and $yu_{s-1}$. Then we color the edges incident to $v_1, v_2, \cdots, v_{s-1}$ by a similar manner and obtain a 4-avd-coloring of $G_{51}$.

**Subcase 3.4.** The neighbor of each pendant vertex of $G$ is in $H$, and the degree of every vertex of degree 2 in $H$ is 2 in $G$. Clearly, $G$ must be one of the following graphs illustrated in Figure 10.

Here we assume that the pendant vertices adjacent to $x$ are $x_1, x_2, \cdots, x_k$ and the pendant vertices adjacent to $y$ are $y_1, y_2, \cdots, y_l$, where $k \geq l \geq 1$.

(i) $G = G_1$.

Clearly $\Delta(G) = k+3$. From Proposition 1, $H_1$ has a 4-avd-coloring $\phi$ using colors 1, 2, 3 and 4. Then we assign colors 5, 6, \cdots, $k+3$ to $xx_1, xx_2, \cdots, xx_k$, respectively. Similarly, we assign colors 5, 6, \cdots, $k+3$ to $yy_1, yy_2, \cdots, yy_l$, respectively. Then we assign colors in $\{1,2,3,4\}\{\phi(xx_1), \phi(xx_2), \phi(xx_3)\}$ and $\{1,2,3,4\}\{\phi(yy_1), \phi(yy_2), \phi(yy_3)\}$ to $xx_1$ and $yy_1$, respectively. Clearly, the resulting coloring is a $(\Delta(G)+1)$-avd-coloring of $G$.

(ii) $G = G_2$.

By Proposition 2, $H_2$ has a 4-avd-coloring using colors 1, 2, 3 and 4. We assign colors 5, 6, \cdots, $k+4$ to $xx_1, xx_2, \cdots, xx_k$, respectively. Similarly, we assign colors 5, 6, \cdots, $k+4$ to $yy_1, yy_2, \cdots, yy_l$, respectively. It is obvious that $x$ and $y$ are distinguished from each other, and the resulting coloring is a $(\Delta(G)+1)$-avd-coloring of $G$.

(iii) $G = G_3$.

By Proposition 2, $H_3$ has a 4-avd-coloring using colors 1, 2, 3 and 4. We assign colors 5, 6, \cdots, $k+4$ to the edges $xx_1, xx_2, \cdots, xx_k$, respectively. It is obvious that the resulting coloring is a $(\Delta(G)+1)$-avd-coloring of $G$.

(iv) $G = G_4$. This case can be dealt with in a similar manner as (ii), and we may obtain a $(\Delta(G)+1)$-avd-coloring $(k \neq l)$ or $(\Delta(G)+1)$-avd-coloring of $G$ $(k = l)$.

(v) $G = G_5$. This case can be dealt with in a similar manner as (i), and we may obtain a $(\Delta(G)+1)$-avd-coloring of $G$.

Since we have dealt with all cases, the theorem is proved.

4 Conclusion and Future Work

From Propositions 1-3 and Theorem 1, we prove that $\chi'_d(G) \leq \Delta(G) + 1$ for bicyclic graphs $G$. This implies that Conjecture 1 holds for all bicyclic graphs. We will investigate the AVDPEC Conjecture for other graphs (such as tricyclic graphs) in the future.
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