Some Oscillation Results for a Class of Delay Partial Difference Equations

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Abstract—**This paper is mainly to investigate the oscillation condition for the solution of a class of delay partial difference equations. Since the oscillation condition is equivalent to the regions where the corresponding characteristic equation has no positive root, by applying the envelope theory, some necessary and sufficient conditions for the oscillatory property of the solutions are obtained.**

Index Terms—**delay partial difference equation, oscillation, envelope, characteristic equation.**

I. Introduction

PARTIAL difference equations are types of difference equations that involve functions of two or more independent variables. Delay partial difference equations have numerous applications as in molecular orbits, population dynamic with spatial migrations, image processing, random walk problems, material mechanics, etc[1-7]. In recent years, the study of the qualitative analysis for the oscillatory behavior of delay partial difference equation has attracted considerable attention, see [8-11] and the references therein.

In [12], with the constraints $p > 0, q \geq 0$, by means of the the zero point theorem, B. G. Zhang and R. P. Agarwal have investigated the oscillatory behavior of following first order delay partial difference equation

$$u_{m+1,n} + u_{m,n+1} - pu_{m,n} + qu_{m-k,n-l} = 0,$$

where m, n, k, l are nonnegative integers. In [13], also with the constraints $p > 0, q \geq 0$, using the proof of contradiction, B. G. Zhang and Y. Zhou once more studied the above equation and obtained the necessary and sufficient conditions for the solutions to be oscillatory.

We will investigate in this paper the following second order delay partial difference equation

$$u_{m+2,n} + u_{m,n+2} - pu_{m,n} + qu_{m-\sigma,n-\tau} = 0, \qquad (1)$$

where p, q are real numbers, m, n, σ, τ are nonnegative integers. The purpose of this paper is to apply a new method, based on the envelope theory of the family of lines, to derive necessary and sufficient conditions for the delay partial difference equation (1) to be oscillatory without the sign constrains for parameters p and q. Before stating the main results, we give some definitions used later in this paper.

Definition 1 $\{u_{m,n}\}_{m\geq-\sigma,n\geq-\tau}$ is said to be a solution of (1) if $\{u_{m,n}\}$ satisfies (1) for $m \geq 0$ and $n \geq 0$.

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Definition 2 A solution $\{u_{m,n}\}$ of (1) is said to be eventually positive (or negative) if $u_{m,n} > 0$ (or $u_{m,n} < 0$) for large integers m and n. The solution is said to be oscillatory if it is neither eventually positive nor eventually negative. (1) is called oscillatory if all of its nontrivial solutions are oscillatory.

II. Preliminaries

In this section, we give some lemmas that will be used in the proof of the main results in section 3.

Lemma 1 [13] The following statements are equivalent:

(i) Every solution of (1) is oscillatory.

(ii) The characteristic equation of (1)

$$\lambda^2 + \mu^2 - p + q\lambda^{-\sigma}\mu^{-\tau} = 0$$

has no positive root.

Lemma 2 [14] Suppose that f is differentiable on $(0, +\infty)$ and f is not identically zero on $(0, +\infty)$ with $\lim\limits_{x\to+\infty} f(x) > 0$ or $\lim\limits_{x\to0^+} f(x) > 0$. Then

$$F(x, y) = y + f(x)$$

has no positive root on $(0, +\infty) \times (0, +\infty)$ if and only if f has no positive root on $(0, +\infty)$.

Lemma 3 [15] Suppose that f, g and v are differentiable on $(-\infty, +\infty) \times (-\infty, +\infty)$. Let Γ be the two-parameter family of lines defined by the equation

$$f(\lambda, \mu)x + g(\lambda, \mu)y = v(\lambda, \mu),$$

where λ, μ are parameters. Let Σ be the envelope of the family Γ. Then the equation (with respect to λ, μ)

$$f(\lambda, \mu)a + g(\lambda, \mu)b = v(\lambda, \mu)$$

has no real root if and only if there is no tangent line of Σ passing through the point (a, b) in xy-plane.

III. Main results

In this section, some necessary and sufficient conditions for oscillations of all solutions of equation (1) are established. Since σ, τ are nonnegative integers, to facilitate discussions, we divide the study into four mutually exclusive cases:

(i) $\sigma \geq 1$ and $\tau \geq 1$; (ii) $\sigma \geq 1$ and $\tau = 0$;
(iii) $\sigma = 0$ and $\tau \geq 1$; (iv) $\sigma = 0$ and $\tau = 0$.

Theorem 1 Let $\sigma \geq 1$ and $\tau \geq 1$. Then every solution of (1) oscillates if and only if $p \leq 0, q \geq 0$ or $p > 0$, $q > \frac{2\sigma^{\frac{\sigma}{2}}\tau^{\frac{\tau}{2}}}{(\sigma+\tau+2)^{\frac{\sigma+\tau+2}{2}}}p^{\frac{\sigma+\tau+2}{2}}$.

Proof Since $\sigma \geq 1$ and $\tau \geq 1$, the characteristic equation of (1) is

$$\phi(p, q, \lambda, \mu) = \lambda^2 + \mu^2 - p + q\lambda^{-\sigma}\mu^{-\tau} = 0. \qquad (2)$$

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Set

$$F(p,q,\lambda,\mu)=\lambda^{\sigma}\mu^{\tau}\phi(p,q,\lambda,\mu)$$
$$=\lambda^{\sigma+2}\mu^{\tau}+\lambda^{\sigma}\mu^{\tau+2}-p\lambda^{\sigma}\mu^{\tau}+q=0. \quad (3)$$

It can be seen that (2) has no positive root if and only if (3) has no positive root. According to Lemma 1, we only need to consider the positive solutions of (3), that is, $\lambda>0$ and $\mu>0$. Now consider (p,q) as a point in pq-plane, and try to search for the exact regions including points (p,q) in pq-plane such that (3) has no positive root. Actually, $F(x,y,\lambda,\mu)=0$ can be regarded as an equation describing a two-parameter family of lines in $pq-$plane, where λ,μ are parameters. According to the envelop theory, the point (x,y) on the envelope for (3) must satisfy all the following equations

$$\begin{cases} F(x,y,\lambda,\mu)=0,\\ F_{\lambda}(x,y,\lambda,\mu)=-\sigma\lambda^{\sigma-1}\mu^{\tau}x+(\sigma+2)\lambda^{\sigma+1}\mu^{\tau}\\ \quad+\sigma\lambda^{\sigma-1}\mu^{\tau+2}=0,\\ F_{\mu}(x,y,\lambda,\mu)=-\tau\lambda^{\sigma}\mu^{\tau-1}x+\tau\lambda^{\sigma+2}\mu^{\tau-1}\\ \quad+(\tau+2)\lambda^{\sigma}\mu^{\tau+1}=0,\end{cases} \quad (4)$$

where $\lambda>0$ and $\mu>0$. Eliminating λ and μ from (4), we get the equation of the envelope

$$y(x)=\frac{2\sigma^{\frac{\sigma}{2}}\tau^{\frac{\tau}{2}}}{(\sigma+\tau+2)^{\frac{\sigma+\tau+2}{2}}}x^{\frac{\sigma+\tau+2}{2}}, \quad (5)$$

where $x>0$. Consequently,

$$y'(x)=\frac{\sigma^{\frac{\sigma}{2}}\tau^{\frac{\tau}{2}}}{(\sigma+\tau+2)^{\frac{\sigma+\tau}{2}}}x^{\frac{\sigma+\tau}{2}},$$
$$y''(x)=\frac{(\sigma+\tau)\sigma^{\frac{\sigma}{2}}\tau^{\frac{\tau}{2}}}{2(\sigma+\tau+2)^{\frac{\sigma+\tau}{2}}}x^{\frac{\sigma+\tau-2}{2}},$$

where $x>0$. Then $y(x)>0, y'(x)>0$ and $y''(x)>0$ imply that y is a positive and strictly convex function on $(0,+\infty)$. Moreover, $\lim_{x\to+\infty}y(x)=+\infty$, $\lim_{x\to0^+}y(x)=0$. The envelope defined by (5) is a strictly convex curve C in the first quadrant as described in Fig.1. It is clearly seen that when either the point (p,q) is in the second closed quadrant, namely, $p\le0,q\ge0$ or (p,q) is vertically above C, namely, $p>0,q>2\sigma^{\frac{\sigma}{2}}\tau^{\frac{\tau}{2}}p^{\frac{\sigma+\tau+2}{2}}/(\sigma+\tau+2)^{\frac{\sigma+\tau+2}{2}}$, there cannot be any tangent line of C which passes through (p,q). Since (2) is equivalent to (3) for the existence of positive solutions, by Lemma 3, (2) has no positive root if and only if $p\le0,q\ge0$ or $p>0,q>2\sigma^{\frac{\sigma}{2}}\tau^{\frac{\tau}{2}}p^{\frac{\sigma+\tau+2}{2}}/(\sigma+\tau+2)^{\frac{\sigma+\tau+2}{2}}$. Associated with Lemma 1, the proof completes.

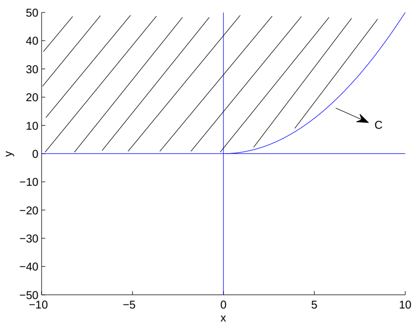

Fig. 1. A Sample Envelope Curve of (3) for $\sigma=1$ and $\tau=1$

Theorem 2 Let $\sigma\ge1$ and $\tau=0$. Then every solution of (1) oscillates if and only if $p\le0,q\ge0$ or $p>0$, $q>\frac{2\sigma^{\frac{\sigma}{2}}}{(\sigma+2)^{\frac{\sigma+2}{2}}}p^{\frac{\sigma+2}{2}}$.

Proof For $\sigma\ge1$ and $\tau=0$, the characteristic equation of (1) is

$$\phi(p,q,\lambda,\mu)=\lambda^2+\mu^2-p+q\lambda^{-\sigma}=0. \quad (6)$$

Set

$$f(p,q,\lambda)=\lambda^2-p+q\lambda^{-\sigma}=0. \quad (7)$$

Then $\lim_{\lambda\to+\infty}f(p,q,\lambda)>0$ and $f(p,q,\lambda)$ is differentiable with regard to $\lambda>0$. Lemma 2 implies that (6) has no positive root if and only if (7) has no positive root. Set

$$F(p,q,\lambda)=\lambda^{\sigma}f(p,q,\lambda)=\lambda^{\sigma+2}-p\lambda^{\sigma}+q=0. \quad (8)$$

Then we see that (8) is equivalent to (7) for the existence of positive root. To investigate oscillatory solutions of (1), by Lemma 1, attention will be restricted to the case where $\lambda>0$. Consider (p,q) as a point in pq-plane and search for the exact regions including (p,q) in pq-plane such that (8) has no positive root. Actually, $F(x,y,\lambda)=0$ can be considered as an equation describing a one-parameter family of lines in $pq-$plane, where λ is the parameter. According to the theory of envelopes, the point (x,y) on the envelope for (8) must satisfy all the following equations

$$\begin{cases} F(x,y,\lambda)=\lambda^{\sigma+2}-\lambda^{\sigma}x+y=0,\\ F_{\lambda}(x,y,\lambda)=(\sigma+2)\lambda^{\sigma+1}-\sigma\lambda^{\sigma-1}x=0,\end{cases} \quad (9)$$

where $\lambda>0$. Eliminating λ from (9), we obtain the equation of the envelope

$$y(x)=\frac{2\sigma^{\frac{\sigma}{2}}}{(\sigma+2)^{\frac{\sigma+2}{2}}}x^{\frac{\sigma+2}{2}}, \quad (10)$$

where $x>0$. Then

$$y'(x)=\frac{\sigma^{\frac{\sigma}{2}}}{(\sigma+2)^{\frac{\sigma}{2}}}x^{\frac{\sigma}{2}},$$
$$y''(x)=\frac{\sigma^{\frac{\sigma+2}{2}}}{2(\sigma+2)^{\frac{\sigma}{2}}}x^{\frac{\sigma-2}{2}},$$

where $x>0$. Consequently, $y(x)>0, y'(x)>0$ and

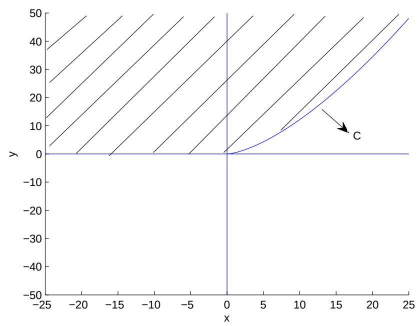

Fig. 2. A Sample Envelope Curve of (8) for $\sigma=1$ and $\tau=0$

$y''(x)>0$ for $x\in(0,+\infty)$. Hence, y is a positive and strictly convex function on $(0,+\infty)$. Moreover, $\lim_{x\longrightarrow+\infty}y(x)=+\infty$, $\lim_{x\longrightarrow0^+}y(x)=0$, the envelope defined by (10) is a strictly convex curve C in the first quadrant as described in

Fig.2. It can be easily seen that there are two cases for the point (p,q) through which there cannot be any tangent line of C passes. The first case is when (p,q) is in the second closed quadrant, namely, $p\le 0, q\ge 0$. The second case is when (p,q) is vertically above C in the first quadrant, namely, $p>0, q>2\sigma^{\frac{\sigma}{2}}p^{\frac{\sigma+2}{2}}/(\sigma+2)^{\frac{\sigma+2}{2}}$. Meanwhile, if the point (p,q) is lied in somewhere else except the above two domains, such a tangent line of C exists. Lemma 1 and Lemma 3 imply the statement of this theorem. The proof is accomplished.

Theorem 3 Let $\sigma=0$ and $\tau\ge 1$. Then every solution of (1) oscillates if and only if $p\le 0, q\ge 0$ or $p>0$, $q>\frac{2\tau^{\frac{\tau}{2}}}{(\tau+2)^{\frac{\tau+2}{2}}}p^{\frac{\tau+2}{2}}$.

Proof Since (1) has the symmetric property, in this case, when replacing λ with μ and replacing σ with τ, the discussion is almost the same as in the proof of Theorem 2, here we omit it.

Theorem 4 Let $\sigma=0$ and $\tau=0$. Then every solution of (1) oscillates if and only if $p\le q$.

Proof Because $\sigma=0$ and $\tau=0$, (1) can be rewritten as

$$u_{m+2,n}+u_{m,n+2}+(q-p)u_{m,n}=0. \tag{11}$$

The characteristic equation of (11) is

$$\lambda^2+\mu^2+(q-p)=0. \tag{12}$$

It is obvious that (12) does not have any positive root if and only if $q-p\ge 0$. By Lemma 1, every solution of (1) oscillates if and only if $p\le q$. This completes the proof of the theorem.

IV. Illustrative Examples

In this section, we give some examples to illustrate the results obtained in Section 3.

Example 1 Consider the delay partial difference equation

$$u_{m+2,n}+u_{m,n+2}-0.1u_{m,n}+0.8u_{m-1,n-1}=0. \tag{13}$$

Here $\sigma=1$, $\tau=1$, $p=-0.1$, $q=0.8$. Since $p=-0.1<0$, $q=0.8>0$, by Theorem 1, every solution of (13) is oscillatory. The oscillatory behavior of (13) is demonstrated by Fig.3.

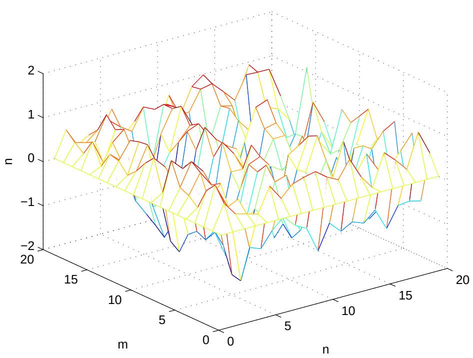

Fig. 3. The Oscillatory Behavior of Solutions for Equation (13)

Example 2 Consider the delay partial difference equation

$$u_{m+2,n}+u_{m,n+2}+0.2u_{m,n}+0.9u_{m-1,n-1}=0. \tag{14}$$

Here $\sigma=1$, $\tau=1$, $p=0.2$, $q=0.9$. Since $p=0.2>0$, $q=0.9>\frac{1}{8}(0.2)^2=\frac{2\sigma^{\frac{\sigma}{2}}\tau^{\frac{\tau}{2}}}{(\sigma+\tau+2)^{\frac{\sigma+\tau+2}{2}}}p^{\frac{\sigma+\tau+2}{2}}$, by Theorem 1, every solution of (14) is oscillatory. The oscillatory character of (14) is demonstrated by Fig.4.

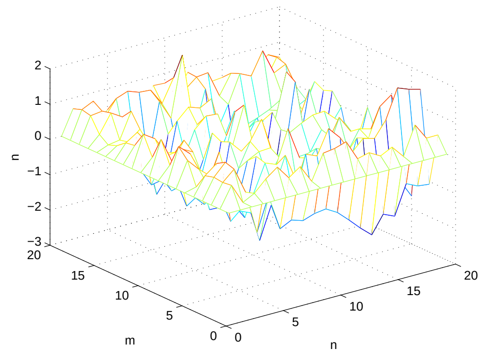

Fig. 4. The Oscillatory Behavior of Solutions for Equation (14)

Example 3 Consider the delay partial difference equation

$$u_{m+2,n}+u_{m,n+2}-0.2u_{m,n}+0.6u_{m-1,n}=0. \tag{15}$$

Here $\sigma=1$, $\tau=0$, $p=-0.2$, $q=0.6$. Since $p=-0.2<0$, $q=0.6>0$, by Theorem 2, every solution of (15) is oscillatory. The oscillatory behavior of (15) is demonstrated by Fig.5.

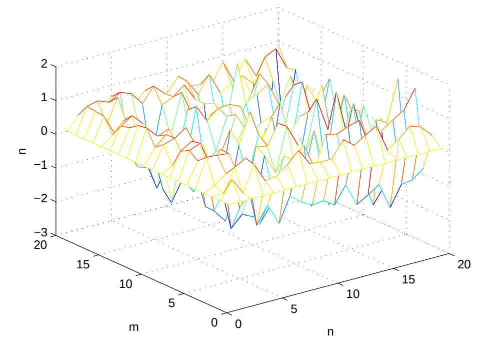

Fig. 5. The Oscillatory Behavior of Solutions for Equation (15)

Example 4 Consider the delay partial difference equation

$$u_{m+2,n}+u_{m,n+2}+0.04u_{m,n}+0.08u_{m-1,n}=0. \tag{16}$$

Here $\sigma=1$, $\tau=0$, $p=0.04$, $q=0.08$. Since $p=0.04>0$, $q=0.08>\frac{2}{3^{\frac{3}{2}}}0.04^{\frac{3}{2}}=\frac{2\sigma^{\frac{\sigma}{2}}}{(\sigma+2)^{\frac{\sigma+2}{2}}}p^{\frac{\sigma+2}{2}}$, by Theorem 2, every solution of (16) is oscillatory. The oscillatory behavior of (16) is demonstrated by Fig.6.

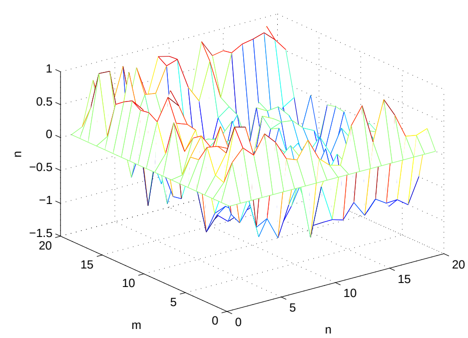

Fig. 6. The Oscillatory Behavior of Solutions for Equation (16)

Example 5 Consider the delay partial difference equation

$$u_{m+2,n} + u_{m,n+2} - 0.5u_{m,n} + 0.5u_{m,n} = 0. \qquad (17)$$

Here $\sigma = 0$, $\tau = 0$, $p = 0.5$, $q = 0.5$. Since $p = 0.5 \leq q = 0.5$, by Theorem 4, every solution of (17) is oscillatory. The oscillatory behavior of (17) is demonstrated by Fig.7.

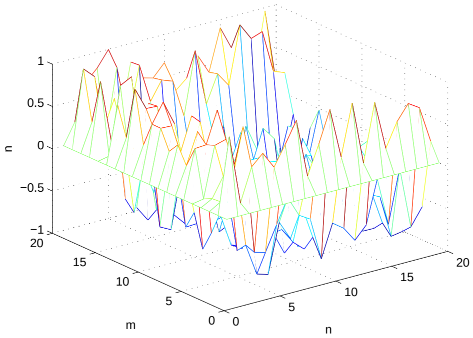

Fig. 7. The Oscillatory Behavior of Solutions for Equation (17)

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