On Maximal Incidence Energy of Graphs with Given Connectivity

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Abstract—**Let G be a graph of order n. The incidence energy $IE(G)$ of graph G, IE for short, is defined as the sum of all singular values of its incidence matrix. In this paper, we determine the maximal incidence energy $IE(G)$ among all connected graphs with the connectivity κ and edge connectivity κ' and $K_{\kappa'} \vee (K_1 \cup K_{n-\kappa'-1})$ has the maximal incidence energy.**

Index Terms—**Signless Laplacian matrix, Incidence energy, Connectivity, Edge connectivity**

I. Introduction

LET $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G) = \{e_1, e_2, ..., e_m\}$, the adjacency matrix $A(G) = (a_{ij})$ of G is an $n \times n$(vertex-vertex) symmetric matrix with $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. Denote the degree of vertex v_i by $d(v_i)$, the signless Laplacian matrix $Q(G)$ of G is defined as $Q(G) = D(G) + A(G)$, where $D(G) = diag(d(v_1)), d(v_2), ..., d(v_n))$ is the diagnoal matrix of degree of G. Since $Q(G)$ are symmetric matrix, their eigenvalues are real numbers. So, we can assume that $q_1(G) \geq q_2(G) \geq ... \geq q_n(G)$ are the signless Laplacian eigenvalues of G. Let $I(G)$ be the (vertex-edge) incidence matrix of the graph G, the (i, j)-entry of $I(G)$ is 0 if v_i is not incident with e_j and 1 if v_i is incident with e_j. Jooyandeh et al. [1] introduced the incidence energy IE of G, which is defined as the sum of the singular values of the incidence matrix of G. Gutman et al. [2] showed that

$$IE = IE(G) = \sum_{i=1}^{n} \sqrt{q_i(G)}.$$

Some basic properties of IE may be found in $[1-3]$.

Let G be a connected graph with n vertices and m edges. Let $S(G)$ be the subdivision graph of G, that is, $S(G)$ is obtained from G by inserting a new vertex in each edge. Clearly, $S(G)$ is a bipartite graph with $n + m$ vertices and $2m$ edges. We denote $\sigma(G, x)$ the characteristic polynomial $det(xI - A(G))$ of G. It is well known [4] that if G is a bipartite graph, then

$$\sigma(G, x) = det(xI_n - A(G)) = \sum_{t=0}^{\lfloor n/2 \rfloor} (-1)^t b(G, t) x^{n-2t}, \tag{1}$$

where $b(G, 0) = 1$ and $b(G, t) \geq 0$ for all $t = 1, 2, ..., \lfloor n/2 \rfloor$. This expression for $\sigma(G, x)$ induce a quasi-order relation (i.e. reflexive and transitive relation) on the set of all bipartite graphs with n vertices: If G_1 and G_2 are bipartite graphs with characteristic polynomials in the form

$$G_1 \succeq G_2 \Leftrightarrow b(G_1, t) \geq b(G_2, t)$$

for all $t = 0, 1, 2, ..., \lfloor n/2 \rfloor$. If $G_1 \succeq G_2$ and there exist t such that $b(G_1, t) > b(G_2, t)$, then we write $G_1 \succ G_2$.

Gutman [5] introduced this quasi-order relation in order to compare the energies of a pair of graphs. It is known $[5, 6]$ that for the bipartite graph $S(G)$, $E(S(G))$ can be also expressed as the Coulson integral formula

$$E(S(G)) = \frac{2}{\pi} \int_0^{+\propto} \frac{1}{x^2} In[1 + \sum_{i=1}^{\lfloor (n+m)/2 \rfloor} b_{2i}(S(G)) x^{2i}] dx. \tag{2}$$

Thus for $m \geq n$, we have [7]

$$IE(G) = \frac{1}{\pi} \int_0^{+\propto} \frac{1}{x^2} In[1 + \sum_{i=1}^{n} (-1)^i p_i(G) x^{2i}] dx, \tag{3}$$

where $p_i(G) = (-1)^i b_{2i}(S(G))$.

From $Eq.(2)$ and $Eq.(3)$ we know that for two bipartite graphs $S(G_1)$ and $S(G_2)$,

$$S(G_1) \preceq S(G_2) \Rightarrow IE(G_1) \leq IE(G_2),$$

$$S(G_1) \prec S(G_2) \Rightarrow IE(G_1) < IE(G_2).$$

For two nonadjacent vertices v_i and v_j, we use $G + e$ to denote the graph obtained by inserting a new edge $e = v_i v_j$ in G. For two vertex disjoint graph G_1 and G_2, we denote by $G_1 \cup G_2$ the graph which consists of two connected components G_1 and G_2. The join of G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{u_i u_j : u_i \in V(G_1), u_j \in V(G_2)\}$. The connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal from G yields a disconnected graph or a trivial graph. The edge-connectivity κ' is defined analogously.

It is both interesting and significant to determine the graph with extremal energies among a given class of graphs. Numerous results on this subject have been put forward, for details see [8-13]. Zhou and Trinajstić [9] determined the extremal Kirchhoff index of graphs with respect to given matching number. In [10], Xu characterized the extremal Laplacian-energy-like with given matching number. Rojo [8] obtained the extremal incidence energy with respect to

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the connectivity. In [14], Hu et al. determined the maximal energy among all subdivisions of graphs with n vertices and chromatic number k. Zhang [18] characterize the graphs with the maximum incidence energies among all graphs with given chromatic number and given pendent vertex number. Inspired by those works, in this paper, we determine the maximal incidence energy among all graphs with n vertices and the connectivity κ and edge-connectivity κ'.

II. The signless Laplacian characteristic polynomial of $G_1 \vee (G_2 \cup G_3)$ for regular graph

In this section, we determine the signless Laplacian characteristic polynomials of $G = G_1 \vee (G_2 \cup G_3)$ with the help of the *coronal* of a matrix. The M-*coronal* $T_M(x)$ of an $n\times n$ matrix M is defined [15, 16] to be the sum of the entries of the matrix $(xI_n - M)^{-1}$, that is $T_M(x) = j_n^T(xI_n - M)^{-1}j_n$, where j_n denotes the column vector of dimension n with all the entries equal one.

It is well known [15, *Proposition*2] that, if M is an $n\times n$ matrix with each row sum equal to a constant t, then

$$T_M(x) = \frac{n}{x-t}. \quad (4)$$

Theorem 2.1 Let $G_i(i = 1, 2, 3)$ be three graphs on n_i vertices. Also let $T_{Q_i}(\lambda)(i = 1, 2, 3)$ be the Q_i-*coronal* of G_i. Then the signless Laplacian characteristic polynomial of the matrix $Q(G_1 \vee (G_2 \cup G_3))$ is

$$\begin{aligned} &P_Q(G_1 \vee (G_2 \cup G_3)) \\ =\;& P_Q(G_1, x-n_2-n_3)P_Q(G_2, x-n_1)P_Q(G_3, x-n_1) \\ & (1 - T_{Q(G_3)}(x-n_1)T_{Q(G_1)}(x-n_2-n_3) \\ & -T_{Q(G_2)}(x-n_1)T_{Q(G_1)}(x-n_2-n_3)). \end{aligned}$$

Proof With a proper labeling of vertices, the signless Laplacian characteristic polynomial of $Q(G) = Q(G_1\vee(G_2\cup G_3))$ is given by

Let $M = xI_{n_1} - Q(G_1) - (n_2+n_3)I_{n_1}$, $N = xI_{n_2} - Q(G_2) - n_1 I_{n_2}$, $T = xI_{n_3} - Q(G_3) - n_1 I_{n_3}$, then

$$\begin{aligned} &P_Q(G) \\ =\;& det\begin{pmatrix} M & -j_{n_1\times n_2} & -j_{n_1\times n_3} \\ -j_{n_2\times n_1} & N & 0_{n_2\times n_3} \\ -j_{n_3\times n_1} & 0_{n_3\times n_2} & T \end{pmatrix} \\ =\;& det(xI_{n_3} - Q(G_3) - n_1 I_{n_3})det(B) \\ =\;& P_{Q(G_3)}(x-n_1)det(B), \end{aligned}$$

where

$$B = \begin{pmatrix} M & -j_{n_1\times n_2} \\ -j_{n_2\times n_1} & N \end{pmatrix} - \begin{pmatrix} -j_{n_1\times n_3} \\ 0_{n_2\times n_3} \end{pmatrix} ((x-n_1)I_{n_3} - Q(G_3))^{-1} \begin{pmatrix} -j_{n_3\times n_1} & 0_{n_3\times n_2} \end{pmatrix}$$

is the Schur complement of $\lambda I_{n_3} - Q(G_3) - n_1 I_{n_3}$. Thus the result follows.

Corollary 2.2 Let $G = K_s \vee (K_{n_1} \cup K_{n_2})$, where K_s, $K_{n_i}(i = 1, 2)$ denote the complete graph on s and n_i vertices, respectively. Then the characteristic polynomial of the signless Laplacian matrix of G is

$$\begin{aligned} &P_Q(G, x) \\ &= (x-n+2)^{s-1}\prod_{i=1}^{2}(x-s-n_i+2)^{n_i-1}[(n+s-2-x) \\ &\prod_{i=1}^{2}(2n_i-2+s-x) - (\sum_{j=1}^{2} sn_j \prod_{i=1,i\neq j}^{2}(2n_i-2+s-x))]. \end{aligned}$$

Proof Since the sum of all entries on every row of signless Laplacian matrix of K_n is $2(n-1)$, (4) implies that $T_Q(x) = \frac{n}{x-2(n-1)}$, using Theorem 2.1, we can get it immediately.

III. Results

First, we define the relation $\succ (\prec, \succeq, \preceq)$ as follows.

Definition 3.1. ([13]) We say p is partial larger than q if $|p| > |q|$, denoted by $p \succ q$. Similarly, we have $p \prec q$, $p \succeq q$, $p \preceq q$.

Definition 3.2. ([13]) Let $p(x) = \sum_{i=0}^{n} p_i x^i$ and $q(x) = \sum_{i=0}^{n} q_i x^i$. If $|p_i| \geq |q_i|$(resp. $|p_i| \leq |q_i|$)for each $0 \leq i \leq n$, then we call $p(x) \succeq q(x)$ (resp. $p(x) \preceq q(x)$). If $p(x) \succeq q(x)$(resp. $p(x) \preceq q(x)$), and there exist a $j \in \{0, 1, 2, ..., n\}$ such that $p_i \succ q_i$(resp. $p_j \prec q_j$), we call $p(x) \succ q(x)$(resp. $p(x) \prec q(x)$).

By the definition above, the following result is immediate.

Lemma 3.3 ([14]) Suppose $a_i \geq b_i \geq 0$ for $i = 1, 2, ..., n$. Then

$$\prod_{i=1}^{n}(x-a_i) \succeq \prod_{i=1}^{n}(x-b_i),$$

furthermore, if there exist a $j \in \{1, 2, ..., n\}$ such that $a_j > b_j$, then

$$\prod_{i=1}^{n}(x-a_i) \succ \prod_{i=1}^{n}(x-b_i).$$

Theorem 3.4 Let n, a and k be three positive integers and $2 \leq a \leq \lfloor\frac{n-k}{2}\rfloor$. Then

$$(x-n+3)^{n-k-2} \succ (x-k-a+2)^{a-1}(x-n+a+2)^{n-k-a-1}.$$

Proof Note that $2 \leq a \leq \lfloor\frac{n-k}{2}\rfloor$ and $n-1 > n-a \geq a+k$. By Lemma 3.3,

$$(x-n+3)^{n-k-2} \succ (x-k-a+2)^{a-1}(x-n+a+2)^{n-k-a-1}.$$

The result follows.

Theorem 3.5 Let n, a_i and b_i be positive integers and $a_1 - a_2 \geq 2$, where $n-s = a_1 + a_2$. If $b_1 = a_1 - 2$, $b_2 = a_2 + 2$, then

$$(n+s-2-x)\prod_{i=1}^{2}(2b_i+s-x-2)-(\sum_{j=1}^{2} sb_j(\prod_{i=1,i\neq j}^{t}(2b_i+s-x-2)))$$

$$\succ (n+s-2-x)\prod_{i=1}^{2}(2a_i+s-x-2)-(\sum_{j=1}^{2} sa_j(\prod_{i=1,i\neq j}^{2}(2a_i+s-x-2))).$$

Proof Note that

$$\begin{aligned} &(2a_1+s-2-x)(2a_2+s-2-x) \\ =\;& x^2 - 2(a_1+a_2+s-2)x + 4a_1a_2 + 2s(a_1+a_2) \\ & -4(a_1+a_2) + (s-2)^2. \end{aligned}$$

and

$$\begin{aligned}&(2b_1+s-2-x)(2b_2+s-2-x)=\\ =\;& x^2-2(b_1+b_2+s-2)x+4b_1b_2+2s(b_1+b_2)\\ &-4(b_1+b_2)+(s-2)^2.\end{aligned}$$

So

$$(2b_1+s-2-x)(2b_2+s-2-x)-(2a_1+s-2-x)(2a_2+s-2-x)$$
$$=-8s(a_1-a_2-2). \quad (5)$$

Again,

$$(-sa_1)(2a_2+s-x-2)+(-sa_2)(2a_1+s-x-2)$$
$$=s(a_1+a_2)x-4sa_1a_2+2s(a_1+a_2)-s^2(a_1+a_2)$$

and

$$(-sb_1)(2b_2+s-x-2)+(-sb_2)(2b_1+s-x-2)$$
$$=s(b_1+b_2)x-4sb_1b_2+2s(b_1+b_2)-s^2(b_1+b_2).$$

Thus

$$[(-sb_1)(2b_2+s-x-2)+(-sb_2)(2b_1+s-x-2)]$$
$$-[(-sa_1)(2a_2+s-x-2)+(-sa_2)(2a_1+s-x-2)]$$
$$=-8s(a_1-a_2-2). \quad (6)$$

Hence by (5) and (6)

$$(n+s-2-x)\prod_{i=1}^{2}(2b_i+s-x-2)$$
$$-(n+s-2-x)\prod_{i=1}^{2}(2a_i+s-x-2)$$
$$=(n+s-2-x)[-8s(a_1-a_2-2)] \quad (7)$$

$$-\sum_{j=1}^{2}(sb_j\prod_{i=1,i\neq j}^{2}(2b_i-2+s-x))$$
$$+\sum_{j=1}^{2}(sa_j\prod_{i=1,i\neq j}^{2}(2a_i-2+s-x))$$
$$=[-8s(a_1-a_2-2)]. \quad (8)$$

Hence, by (7) and (8),

$$[(n+s-2-x)\prod_{i=1}^{2}(2b_i+s-x-2)$$
$$-(\sum_{j=1}^{2}sb_j(\prod_{i=1,i\neq j}^{2}(2b_i+s-x-2)))]$$
$$-[(n+s-2-x)\prod_{i=1}^{2}(2a_i+s-x-2)$$
$$-(\sum_{j=1}^{2}sa_j(\prod_{i=1,i\neq j}^{t}(2a_i+s-x-2)))]$$
$$=(n+s-1-x+a_i)[-8s(a_1-a_2-2)].$$

Thus by Definition 3.2

$$(n+s-2-x)\prod_{i=1}^{2}(2b_i+s-x-2)$$
$$-(\sum_{j=1}^{2}sb_j(\prod_{i=1,i\neq j}^{2}(2b_i+s-x-2)))$$
$$\succ (n+s-2-x)\prod_{i=1}^{2}(2a_i+s-x-2)$$
$$-(\sum_{j=1}^{2}sa_j(\prod_{i=1,i\neq j}^{2}(2a_i+s-x-2))).$$

Hence we have finished the proof of the Theorem.

Lemma 3.6([16]) Let G be a non-complete connected graph of order n and $e\in E(\overline{G})$. Then

$$\begin{aligned}\sigma(Q(G+e),x) &= det(xI_n-Q(G+e))\\ &\succ det(xI_n-Q(G))=\sigma(Q(G),x),\end{aligned}$$

where $\overline{G}$ denote the complement of a graph G.

Lemma 3.7([17]) Let G be a graph with n vertices and m edges. Then

$$\sigma(S(G),x)=x^{m-n}\sigma(Q(G),x^2)=x^{m-n}det(x^2I_n-Q(G)).$$

Now we present our main result.

Theorem 3.8 Let G be a connected graph with n vertices and connectivity κ. Then

$$\sigma(Q(G),x)\preceq(x-n+2)^{\kappa-1}(x-n+3)^{n-\kappa-2}$$
$$[(n+\kappa-2-x)(\kappa-x)(2n-4-\kappa-x)$$
$$-\kappa(2n-\kappa-4-x)-\kappa(n-\kappa-1)(\kappa-x)].$$

The equality holds if and only if $G=K_\kappa\vee(K_1\cup K_{n-\kappa-1})$.

Proof Let G_0 be a graph having the maximal coefficients of the signless Laplacian characteristic polynomial among all connected graphs of order n with connectivity κ. Then there is a vertex subset $X_0\subset V(G_0)$ and $|V_0|=\kappa$ such that $G_0-V_0=G_1\cup G_2\cup...\cup G_t$, where $G_1,G_2,...,G_t$ $(t\geq 2)$ connected components of G_0-V_0. By Lemma 3.6, $t=2$, G_1 and G_2 and $G[U]$ are complete, and each vertex of G_1 and G_2 is adjacent to each vertex x in V_0. Let $n_i=|G_i|$ for $i=1,2$. Then $G_0=K_\kappa\vee(K_{n_1}\cup K_{n_2})$ and $n_1+n_2=n-\kappa$. Assume that $n_1\leq n_2$. By Corollary 2.2,

$$P_Q(G_0,x)=(x-n+2)^{\kappa-1}\prod_{i=1}^{2}(x-\kappa-n_i+2)^{n_i-1}$$
$$[(n+\kappa-2-x)\prod_{i=1}^{2}(2n_i-2+\kappa-x)$$
$$-(\sum_{j=1}^{2}\kappa n_j\prod_{i=1,i\neq j}^{2}(2n_i-2+\kappa-x))].$$

If $2\leq n_1\leq\lfloor\frac{n-\kappa}{2}\rfloor$, then by Lemma 3.4,

$$(x-n+3)^{n-\kappa-2}\succ(x-\kappa-a+2)^{a-1}(x-n+a+2)^{n-\kappa-a-1}.$$

By Theorem 3.5

$$(n+\kappa-2-x)(\kappa-x)(2n-4-\kappa-x)$$
$$-\kappa(2n-\kappa-4-x)-\kappa(n-\kappa-1)(\kappa-x)]$$
$$\succ (n+\kappa-2-x)\prod_{i=1}^{2}(2a_i+\kappa-x-2)$$
$$-(\sum_{j=1}^{2}\kappa a_j(\prod_{i=1,i\neq j}^{2}(2a_i+\kappa-x-2))),$$

where $b_1=1, b_2=n_1+n_2-1=n-\kappa-1$.

Hence, we find the coefficients of $P_Q(G_0,x)$ with $n_1+n_2=n-\kappa$ and $n_1\le n_2$ is maximum if and only if $n_1=1, n_2=n-\kappa-1$. It follows that $G_0=K_\kappa\vee(K_1\cup K_{n-\kappa-1})$, then

$\sigma(Q(G_0),x)\preceq(x-n+2)^{\kappa-1}(x-n+3)^{n-\kappa-2}[(n+\kappa-2-x)(\kappa-x)(2n-4-\kappa-x)-\kappa(2n-\kappa-4-x)$

$-\kappa(n-\kappa-1)(\kappa-x)]$.

This completes the proof of Theorem.

Lemma 3.9([14]) Let G_1 and G_2 be two bipartite graphs with n_1 and n_2 vertices, respectively. For any two positive integers p_1 and p_2 satisfying $n_1+p_1=n_2+p_2$, then

$$x^{p_1}\sigma(G_1,x)\succeq x^{p_2}\sigma(G_2,x)\Rightarrow E(G_1)\succeq E(G_2);$$
$$x^{p_1}\sigma(G_1,x)\succ x^{p_2}\sigma(G_2,x)\Rightarrow E(G_1)\succ E(G_2).$$

By Lemma 3.7 and Theorem 3.8 and Lemma3.9, the following result is obvious.

Theorem 3.10 Let G be a simple graph of order n whose connectivity is κ. Then

$$IE(G)\le IE(K_\kappa\vee(K_1\cup K_{n-\kappa-1})),$$

with equality holds if and only if $G=K_\kappa\vee(K_1\cup K_{n-\kappa-1})$.

Theorem 3.11 Let G be a connected graph with n vertices and edge connectivity κ'. Then

$$\sigma(Q(G),x)\preceq(x-n+2)^{\kappa'-1}(x-n+3)^{n-\kappa'-2}$$
$$[(n+\kappa'-2-x)(\kappa'-x)(2n-4-\kappa'-x)$$
$$-\kappa'(2n-\kappa'-4-x)-\kappa'(n-\kappa'-1)(\kappa'-x)].$$

The equality holds if and only if $G=K_{\kappa'}\vee(K_1\cup K_{n-\kappa'-1})$.

Proof Let G be a connected graph of order n with connectivity κ and edge-connectivity κ', then $\kappa\le\kappa'$. Note that $(x-n+2)^{\kappa-\kappa'}\preceq(x-n+3)^{\kappa-\kappa'}$ and

$[(n+\kappa-2-x)(\kappa-x)(2n-4-\kappa-x)-\kappa(2n-\kappa-4-x)$

$-\kappa(n-\kappa-1)(\kappa-x)]\preceq$

$$[(n+\kappa'-2-x)(\kappa'-x)(2n-4-\kappa'-x)$$
$$-\kappa'(2n-\kappa'-4-x)-\kappa'(n-\kappa'-1)(\kappa'-x)].$$

By Theorem 3.8,

$\sigma(Q(G),x)\preceq(x-n+2)^{\kappa-1}(x-n+3)^{n-\kappa-2}[(n+\kappa-2-x)$

$$(\kappa-x)(2n-4-\kappa-x)$$
$$-\kappa(2n-\kappa-4-x)-\kappa(n-\kappa-1)(\kappa-x)]$$

$\preceq(x-n+2)^{\kappa'-1}(x-n+3)^{n-\kappa'-2}[(n+\kappa'-2-x)(\kappa'-x)$

$$(2n-4-\kappa'-x)-\kappa'(2n-\kappa'-4-x)$$
$$-\kappa'(n-\kappa'-1)(\kappa'-x)].$$

And the equality holds if and only if $G=K_\kappa\vee(K_1\cup K_{n-\kappa-1})$ and $\kappa=\kappa'$, that is $G=K_{\kappa'}\vee(K_1\cup K_{n-\kappa'-1})$. The result follows.

Similarly Theorem 3.10, we get the following result.

Theorem 3.12 Let G be a simple graph of order n whose edge connectivity is κ'. Then

$$IE(G)\le IE(K_{\kappa'}\vee(K_1\cup K_{n-\kappa'-1}),$$

with equality holds if and only if $G=K_{\kappa'}\vee(K_1\cup K_{n-\kappa'-1})$.

IV. Conclusion

In this paper, we use quasi-order relation to compare the incidence energy of two graphs with respect to the connectivity and edge connectivity and further obtain the maximal incidence energy among all connected graphs with given connectivity and edge connectivity.

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References

[1] M. Jooyandeh, D. Kiani, and M. Mirzakhah, "Incidence energy of a graph," *MATCH. Commun. Math. Comput. Chem.*, vol. 62, no. 3, pp. 561-572, 2009.

[2] I. Gutman, D. Kiani, and M. Mirzakhah, "On incidence energy of graphs," *MATCH. Commun. Math. Comput. Chem.*, vol. 62, no. 3, pp. 573-580, 2009.

[3] I. Gutman, D. Kiani, M.Mirzakhah and B.Zhou, "On incidence energy of a graph," *Linear Algebra Appl.*, vol. 431, no. 8, pp. 1223-1233, 2009.

[4] D.M. Cvetkovic, M. Doob, H. Sachs, Spectra of Graphs , *Academic Press*, New York, 1980.

[5] I. Gutman, "Acyclic systems with extremal Huckel π-electron energy," *Theoret. Chim. Acta (Berlin)*, vol. 45, no. 2, pp. 79-87, 1977.

[6] I. Gutman, O. E. Polansky, Mathematicl Concepts in Organic Chemistry, *Springer*, Berlin, 1986.

[7] N. Biggs, Algebraic Graph Theory, 2nd edn, Cambridge, Cambridge University Press, 1993.

[8] O. Rojo, E. Lenes, "A sharp upper bound on the incidence energy of graphs in terms of connectivity," *Linear Algebra Appl.*, vol. 438, no. 3, pp. 1485-1493, 2013.

[9] B. Zhou and N. Trinajstić, "The Kirchhoff index and the matching number," *Int. J. of Quantum Chem.*, vol. 109, no. 13, pp. 2978-2981, 2009.

[10] K.X. Xu, K.C. Das, "Extremal Laplacian-energy-like invariant of graphs with given matching number," *Electron Journal of Linear Alg. Appl.*, vol. 26, no. 1, pp. 131-140, 2013.

[11] G. Zhong, M.X. Yi, and J. Huang, "Numerical method for solving fractional convection diffusion equations with time-space variable coefficients," *IAENG International Journal of Applied Mathematics*, vol. 48, no. 1, pp. 62-66, 2018.

[12] M. Asgari, "Numerical solution for solving a system of fractional integro-differential equations," *IAENG International Journal of Applied Mathematics*, vol. 45, no. 2, pp. 85-91, 2015.

[13] W. Qiu, W.G. Yan, "Coefficients of Laplacian characteristi polynomials of graphs," *Linear Algebra Appl.*, 436 (2012), 2474-2479. vol. 38, no. 4, pp. 1349-1359, 2015.

[14] M.L. Hu, W.G. Yan, W. Qiu, "Maximal energy of subdivision of graphs with a fixed chromatic number," *Bull. Malay. Math. Sci. Soc.*, vol. 38, no. 4, pp. 1349-1359, 2015.

[15] S.Y. Cui, G.X. Tian, "The spectrum and the signless Laplacian spectrum of coronae," *Linear Algebra Appl.*, vol. 437, no. 7, pp. 1692-1703, 2012.

[16] C. McLeman, E. McNicholas, "Spectra of coronae," *Linear Algebra Appl.*, vol. 435, no. 5, pp. 998-1007, 2011.

[17] S. Shinoda, "On the characteristic polynomial of the adjacency matrix of the subdivision graph of a graph," *Discrete Appl. Math.*, vol. 2, no. 4, pp. 349-351, 1980.

[18] J.B. Zhang, H.B. Zhang and X.D. Liu, "Graphs with extremal incidence energy," *Filomat*, vol. 29, no. 6, pp. 1251-1258, 2015.