Novel Power Aggregation Operators Based on Einstein Operations for Interval Neutrosophic Linguistic Sets

Lihua Yang， Baolin Li *，Haitao Xu

Abstract—**Interval neutrosophic linguistic sets (INLSs) take the advantages of interval neutrosophic sets (INSs) and linguistic variables (LVs), which can be better to handle the indeterminate and inconsistent information existing in the real world, and the power operator can consider all the decision arguments and their relationship. But the traditional power aggregation operator cannot handle interval neutrosophic linguistic sets. Motivated by these, firstly, the operations of interval neutrosophic linguistic numbers (INLNs) based on Einstein operations are defined, and the Hamming distance measure for INLNs is also explored in this paper. Secondly, some novel power aggregation operators based on Einstein operations are proposed, including the interval neutrosophic linguistic power weighted average (INLPWA) operator and the interval neutrosophic linguistic power weighted geometric (INLPWG) operator, and their properties are also studied. Thirdly, an illustrative example is illustrated to show the feasibility and practicality of the proposed method.**

Index Terms—**interval neutrosophic linguistic Sets, Einstein, power aggregation operator.**

I. INTRODUCTION

Smarandache firstly proposed Neutrosophic set (NS) [1]， which is an extension of fuzzy set (FS) [2], intuitionistic fuzzy set (IFS) [3], hesitant fuzzy set (HFS) [4]. In NS, the degrees of true-membership, indeterminacy-membership, and false-membership are completely independent. In recent years, NS has been widely applied in handling multi-criteria decision-making (MCDM) problems. However, NS is defined from a philosophical perspective, and it is difficult to apply in real application from a scientific engineering point of view. Thus, wang [5, 6] proposed the concepts of single-valued neutrosophic sets (SVNSs) and interval neutrosophic sets (INSs). Ye [7, 8] defined simplified neutrosophic sets (SNSs) and single-valued neutrosophic hesitant fuzzy sets (SVNHFSs). Wang [9] described the concept of multi-valued neutrosophic sets (MVNSs). Liu [10] developed interval neutrosophic hesitant fuzzy sets (INHFSs).

In some cases, decision-making problems are too complexto be expressed by quantitative values. Considering this situation, the linguistic variables (LVs) originally introduced by Zadeh [11] has become an effective tool to express quanlitative information. However, using LVs commonly implies the truth degree of a linguistic term is 1.To overcome the drawback of utilizing LVs concerned with FS and IFS, various neutrosophic linguistic sets have been developed, such as the single-valued neutrosophic linguistic sets (SVNLSs) [12], the simplified neutrosophic linguistic sets (SNLSs) [13], the interval neutrosophic linguistic sets (INLSs) [14], the interval neutrosophic uncertain linguistic sets (INULSs) [15] and the multi-valued neutrosophic linguistic sets (MVNLSs) [16].

Additionally, due to the significance of information fusion, some related aggregation operators have also been proposed for solving MCDM problems. Wang [17] extended a series of Maclaurin symmetric mean (MSM) aggregation operators under single-valued neutrosophic linguistic environments. Tian [13] applied normalized Bonferroni mean (NBM) operator to SNLSs, and tian [18] also developed the traditional PA operator under a simplified neutrosophic uncertain linguistic environment. Ye [14, 15] extended the traditional weighted arithmetic average (WAA) operator and weighted geometric average (WGA) operator to INLSs and INULSs, respectively, and the INLWAA operator, the INLWGA operator, the INULWAA operator, and the INULWGA operator were defined. Ma [19] studied a generalized interval neutrosophic linguistic prioritized weighted harmonic mean (GINLPWHM) operator and a generalized interval neutrosophic linguistic prioritized weighted hybrid harmonic mean (GINLPWHHM) operator. Li [16] investigated the normalized Weighted Bonferroni mean Hamacher (NWBMH) operator with

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multi-valued neutrosophic linguistic sets.

The power average operator was originally defined by Yager [20] which has the well-known advantage of considering the relationship among the multi-input arguments being fused. Related studies from different perspectives have been achieved. The power geometric operator [21], generalized power average operator [22], single-valued neutrosophic power operator [23], and interval neutrosophic power generalized operators [24] were also proposed. However, existing power operator fail to handle situations in which the input arguments are interval neutrosophic linguistic numbers (INLNs). Motivated by gap in these literatures, the purpose of this paper is to develop power operators under interval neutrosophic linguistic environment to solve MCDM problems.

The paper is organized as follows. In Section Ⅱ, some concepts of INLSs and INLNs are briefly reviewed, novel operational rules based on Einstein operations and the Hamming distance measure are defined. In Section Ⅲ, the traditional power operators are extended to the interval neutrosophic linguistic environment, the INLPWA operator and the INLPWG operator based on the novel operations are proposed and some desirable properties are discussed. In Section Ⅳ, an illustrative example is performed based on the proposed method. In Section Ⅴ, some summary remarks are provided.

II. PRELIMINARIES

In this section, some basic concepts and definitions with respect to INLSs and INLNs are conducted, which will be utilized in the later analysis.

Based on interval neutrosophic sets and linguistic variables, ye [14] defined the concept of interval neutrosophic linguistic sets, which is presented as follows:

Definition 1 Let X be a set of points with generic elements in X denoted by x, an INLS A in X is defined as follows:

$$A = \left\{\left\langle x, \left[s_{\theta(x)}, \left(T_A(x), I_A(x), F_A(x)\right)\right]\right\rangle \middle| x \in X\right\},$$

Where $s_{\theta(x)} \in S$, $S = \{s_1, s_2, \cdots, s_t\}$ is an ordered and finite linguistic term set, in which s_j denotes a linguistic variable value and t is an odd value. $T_A(x) = \left[T_A^L(x), T_A^U(x)\right] \subseteq [0,1]$, $I_A(x) = \left[I_A^L(x), I_A^U(x)\right] \subseteq [0,1]$, and $F_A(x) = \left[F_A^L(x), F_A^U(x)\right] \subseteq [0,1]$, satisfying these conditions $0 \le T_A^U(x) + I_A^U(x) + F_A^U(x) \le 3$ for any x in X. $T_A(x)$, $I_A(x)$, and $F_A(x)$ denoting three degrees of x in X belonging to the linguistic variable of $s_{\theta(x)}$, that are true, indeterminacy and falsity.

Definition 2[14] Suppose there is only one element in X, then the seven tuple $\left\langle s_{\theta(x)}, \left(\left[T_A^L(x), T_A^U(x)\right], \left[I_A^L(x), I_A^U(x)\right], \left[F_A^L(x), F_A^U(x)\right]\right)\right\rangle$ is depicted as an INLN.

We know different aggregation operators are all depended on different t-norms and t-conorms. Einstein t-norm and t-conorm consist of the following equations [25],

$$a \otimes b = \frac{ab}{1+(1-a)(1-b)},\ a \oplus b = \frac{a+b}{1+ab}.$$

Then the operational laws of INLNs based on Einstein operations are defined as follows.

Definition3Let

$a_1 = \left\langle s_{\theta(a_1)}, \left(\left[T_A^L(a_1), T_A^U(a_1)\right], \left[I_A^L(a_1), I_A^U(a_1)\right], \left[F_A^L(a_1), F_A^U(a_1)\right]\right)\right\rangle$ and

$a_2 = \left\langle s_{\theta(a_2)}, \left(\left[T_A^L(a_2), T_A^U(a_2)\right], \left[I_A^L(a_2), I_A^U(a_2)\right], \left[F_A^L(a_2), F_A^U(a_2)\right]\right)\right\rangle$

be two INLNs, and $\lambda > 0$, then the operations of INLNs can be defined based on Einstein operations.

(1) $a_1 \oplus a_2$

$$= \left\langle s_{\theta(a_1)+\theta(a_2)}, \left(\left[\frac{T^L(a_1) + T^L(a_2)}{1 + T^L(a_1)\cdot T^L(a_2)}, \frac{T^U(a_1) + T^U(a_2)}{1 + T^U(a_1)\cdot T^U(a_2)}\right], \left[\frac{I^L(a_1)\cdot I^L(a_2)}{1 + (1 - I^L(a_1))\cdot(1 - I^L(a_2))}, \frac{I^U(a_1)\cdot I^U(a_2)}{1 + (1 - I^U(a_1))\cdot(1 - I^U(a_2))}\right], \left[\frac{F^L(a_1)\cdot F^L(a_2)}{1 + (1 - F^L(a_1))\cdot(1 - F^L(a_2))}, \frac{F^U(a_1)\cdot F^U(a_2)}{1 + (1 - F^U(a_1))\cdot(1 - F^U(a_2))}\right]\right)\right\rangle;$$

(2) $a_1 \otimes a_2$

$$= \left\langle s_{\theta(a_1)\times\theta(a_2)}, \left(\left[\frac{T^L(a_1)\cdot T^L(a_2)}{1 + (1 - T^L(a_1))\cdot(1 - T^L(a_2))}, \frac{T^U(a_1)\cdot T^U(a_2)}{1 + (1 - T^U(a_1))\cdot(1 - T^U(a_2))}\right], \left[\frac{I^L(a_1) + I^L(a_2)}{1 + I^L(a_1)\cdot I^L(a_2)}, \frac{I^U(a_1) + I^U(a_2)}{1 + I^U(a_1)\cdot I^U(a_2)}\right] \left[\frac{F^L(a_1) + F^L(a_2)}{1 + F^L(a_1)\cdot F^L(a_2)}, \frac{F^U(a_1) + F^U(a_2)}{1 + F^U(a_1)\cdot F^U(a_2)}\right]\right)\right\rangle;$$

(3) $\lambda a_1 = \left\langle s_{\lambda\theta(a_1)},\right.$

$$\left(\left[\frac{(1 + T^L(a_1))^\lambda - (1 - T^L(a_1))^\lambda}{(1 + T^L(a_1))^\lambda + (1 - T^L(a_1))^\lambda}, \frac{(1 + T^U(a_1))^\lambda - (1 - T^U(a_1))^\lambda}{(1 + T^U(a_1))^\lambda + (1 - T^U(a_1))^\lambda}\right], \left[\frac{2\cdot(I^L(a_1))^\lambda}{(2 - I^L(a_1))^\lambda + (I^L(a_1))^\lambda}, \frac{2\cdot(I^U(a_1))^\lambda}{(2 - I^U(a_1))^\lambda + (I^U(a_1))^\lambda}\right], \left[\frac{2\cdot(F^L(a_1))^\lambda}{(2 - F^L(a_1))^\lambda + (F^L(a_1))^\lambda}, \frac{2\cdot(F^U(a_1))^\lambda}{(2 - F^U(a_1))^\lambda + (F^U(a_1))^\lambda}\right]\right)\right\rangle;$$

(4) $a_1^\lambda = \left\langle s_{\theta^\lambda(a_1)},\right.$

$$\left(\left[\frac{2\cdot(T^L(a_1))^\lambda}{(2 - T^L(a_1))^\lambda + (T^L(a_1))^\lambda}, \frac{2\cdot(T^U(a_1))^\lambda}{(2 - T^U(a_1))^\lambda + (T^U(a_1))^\lambda}\right], \left[\frac{(1 + I^L(a_1))^\lambda - (1 - I^L(a_1))^\lambda}{(1 + I^L(a_1))^\lambda + (1 - I^L(a_1))^\lambda}, \frac{(1 + I^U(a_1))^\lambda - (1 - I^U(a_1))^\lambda}{(1 + I^U(a_1))^\lambda + (1 - I^U(a_1))^\lambda}\right], \left[\frac{(1 + F^L(a_1))^\lambda - (1 - F^L(a_1))^\lambda}{(1 + F^L(a_1))^\lambda + (1 - F^L(a_1))^\lambda}, \frac{(1 + F^U(a_1))^\lambda - (1 - F^U(a_1))^\lambda}{(1 + F^U(a_1))^\lambda + (1 - F^U(a_1))^\lambda}\right]\right)\right\rangle.$$

Theorem 1 For any three INLNs a_1, a_2, a_3, and any real numbers $\lambda, \lambda_1, \lambda_2 \ge 0$, then the following equations can be true.

(1) $a_1 \oplus a_2 = a_2 \oplus a_1$;

(2) $a_1 \otimes a_2 = a_2 \otimes a_1$;

(3) $\lambda\left(a_1 \oplus a_2\right) = \lambda a_1 \oplus \lambda a_2$;

(4) $\lambda_1 a_1 \oplus \lambda_2 a_1 = (\lambda_1 + \lambda_2)a_1$;

(5) $a_1^{\lambda_1} \otimes a_1^{\lambda_2} = a_1^{\lambda_1+\lambda_2}$;

(6) $a_1^{\lambda} \otimes a_2^{\lambda} = \left(a_1 \otimes a_2\right)^{\lambda}$;

(7) $\left(a_1 \oplus a_2\right) \oplus a_3 = a_1 \oplus \left(a_2 \oplus a_3\right)$;

(8) $\left(a_1 \otimes a_2\right) \otimes a_3 = a_1 \otimes \left(a_2 \otimes a_3\right)$.

Theorem 1 can be easily proven based on Definition 3.

Definition 4Let a_1 and a_2 be any two INLNs, then the Hamming distance between a_1 and a_2 can be defined as follows:

$$\begin{aligned} d(a_1, a_2) = \frac{1}{2(\iota+2)}\Big(&\left|\theta(a_1)\times T^L(a_1) - \theta(a_2)\times T^L(a_2)\right| \\ &+\left|\theta(a_1)\times T^U(a_1) - \theta(a_2)\times T^U(a_2)\right| \\ &+\left|\theta(a_1)\times I^L(a_1) - \theta(a_2)\times I^L(a_2)\right| \\ &+\left|\theta(a_1)\times I^U(a_1) - \theta(a_2)\times I^U(a_2)\right| \\ &+\left|\theta(a_1)\times F^L(a_1) - \theta(a_2)\times F^L(a_2)\right| \\ &+\left|\theta(a_1)\times F^U(a_1) - \theta(a_2)\times F^U(a_2)\right|\Big)\end{aligned} \tag{1}$$

Where ι is the numbers of the values in S.

The Hamming distance defined above can satisfy the following three conditions.

(1)$d(a_1, a_1) = 0$,

(2)$d(a_1, a_2) = d(a_2, a_1)$, $d(a_1, a_2) \in [0, 1]$,

(3)$d(a_1, a_2) + d(a_2, a_3) \geq d(a_1, a_3)$.

III. Novel Power Operators for INLNs

Based on the operational rules in Definition 3, the INLPWA operator and INLPWG operator are defined, and some properties of two operators are also discussed in this section.

A. INLPWA operator

Definition 5Let $a_i (i = 1, 2, \cdots, n)$ be a collection of INLNs, and $\omega = (\omega_1, \omega_2, , \cdots, \omega_n)$ be the weighted vector of a_i, $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$. The INLPWA operator is defined as follows.

$$INLPWA(a_1, a_2, \cdots a_n) = \frac{\overset{n}{\underset{i=1}{\oplus}} \omega_i (1 + S(a_i))a_i}{\sum_{i=1}^{n} \omega_i (1 + S(a_i))} \tag{2}$$

Where $S(a_i) = \sum_{\substack{j=1 \\ j\neq i}}^{n} \omega_j Supp(a_i, a_j)$, and $Supp(a_i, a_j)$ is the support for a_i and a_j, which meets the following conditions:

(1)$Supp(a_i, a_j) \in [0,1]$,

(2)$Supp(a_i, a_j) = Supp(a_j, a_i)$,

(3)$Supp(a_i, a_j) \geq Supp(a_p, a_q) iff d(a_i, a_j) \leq d(a_p, a_q)$.

Here $d(a_i, a_j)$ is the Hamming distance between a_i and a_j in Definition 4.

Theorem2Let $a_i (i = 1, 2, \cdots, n)$ be a collection of INLNs, and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ be the weighted vector for a_i, $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$. Based on the operational rules in Definition 3 and Equation (2), we can derived the following result, and the aggregated result utilizing INLPWA operator is still an INLN.

$$INLPWA(a_1, a_2, \cdots a_n) = \left\langle s_{\sum_{i=1}^{n}\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}\cdot\theta(a_i)}, \left(\left[\frac{\prod_{i=1}^{n}(1+T^L(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}} - \prod_{i=1}^{n}(1-T^L(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}{\prod_{i=1}^{n}(1+T^L(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}} + \prod_{i=1}^{n}(1-T^L(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}, \frac{\prod_{i=1}^{n}(1+T^U(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}} - \prod_{i=1}^{n}(1-T^U(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}{\prod_{i=1}^{n}(1+T^U(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}} + \prod_{i=1}^{n}(1-T^U(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}\right], \left[\frac{2\prod_{i=1}^{n}(I^L(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}{\prod_{i=1}^{n}(2-I^L(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}} + \prod_{i=1}^{n}(I^L(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}, \frac{2\prod_{i=1}^{n}(I^U(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}{\prod_{i=1}^{n}(2-I^U(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}} + \prod_{i=1}^{n}(I^U(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}\right], \left[\frac{2\prod_{i=1}^{n}(F^L(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}{\prod_{i=1}^{n}(2-F^L(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}} + \prod_{i=1}^{n}(F^L(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}, \frac{2\prod_{i=1}^{n}(F^U(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}{\prod_{i=1}^{n}(2-F^U(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}} + \prod_{i=1}^{n}(F^U(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}\right]\right)\right\rangle \tag{3}$$

Proof Let $\xi_i = \frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}$

(1) If n=2, based on the operations (1) and (3) in Definition 3.

$$INLPWA(a_1,a_2)=\xi_1 a_1\oplus\xi_2 a_2$$

$$=\left\langle s_{\xi_1\theta(a_1)+\xi_2\theta(a_2)},\right.$$

$$\left(\left[\frac{\frac{(1+T^L(a_1))^{\xi_1}-(1-T^L(a_1))^{\xi_1}}{(1+T^L(a_1))^{\xi_1}+(1-T^L(a_1))^{\xi_1}}+\frac{(1+T^L(a_2))^{\xi_2}-(1-T^L(a_2))^{\xi_2}}{(1+T^L(a_2))^{\xi_2}+(1-T^L(a_2))^{\xi_2}}}{1+\frac{(1+T^L(a_1))^{\xi_1}-(1-T^L(a_1))^{\xi_1}}{(1+T^L(a_1))^{\xi_1}+(1-T^L(a_1))^{\xi_1}}\cdot\frac{(1+T^L(a_2))^{\xi_2}-(1-T^L(a_2))^{\xi_2}}{(1+T^L(a_2))^{\xi_2}+(1-T^L(a_2))^{\xi_2}}},\right.\right.$$

$$\left.\frac{\frac{(1+T^U(a_1))^{\xi_1}-(1-T^U(a_1))^{\xi_1}}{(1+T^U(a_1))^{\xi_1}+(1-T^U(a_1))^{\xi_1}}+\frac{(1+T^U(a_2))^{\xi_2}-(1-T^U(a_2))^{\xi_2}}{(1+T^U(a_2))^{\xi_2}+(1-T^U(a_2))^{\xi_2}}}{1+\frac{(1+T^U(a_1))^{\xi_1}-(1-T^U(a_1))^{\xi_1}}{(1+T^U(a_1))^{\xi_1}+(1-T^U(a_1))^{\xi_1}}\cdot\frac{(1+T^U(a_2))^{\xi_2}-(1-T^U(a_2))^{\xi_2}}{(1+T^U(a_2))^{\xi_2}+(1-T^U(a_2))^{\xi_2}}}\right],$$

$$\left[\frac{\frac{2\cdot(I^L(a_1))^{\xi_1}}{(2-I^L(a_1))^{\xi_1}+(I^L(a_1))^{\xi_1}}\cdot\frac{2\cdot(I^L(a_2))^{\xi_2}}{(2-I^L(a_2))^{\xi_2}+(I^L(a_2))^{\xi_2}}}{1+(1-\frac{2\cdot(I^L(a_1))^{\xi_1}}{(2-I^L(a_1))^{\xi_1}+(I^L(a_1))^{\xi_1}})\cdot(1-\frac{2\cdot(I^L(a_2))^{\xi_2}}{(2-I^L(a_2))^{\xi_2}+(I^L(a_2))^{\xi_2}})},\right.$$

$$\left.\frac{\frac{2\cdot(I^U(a_1))^{\xi_1}}{(2-I^U(a_1))^{\xi_1}+(I^U(a_1))^{\xi_1}}\cdot\frac{2\cdot(I^U(a_2))^{\xi_2}}{(2-I^U(a_2))^{\xi_2}+(I^U(a_2))^{\xi_2}}}{1+(1-\frac{2\cdot(I^U(a_1))^{\xi_1}}{(2-I^U(a_1))^{\xi_1}+(I^U(a_1))^{\xi_1}})\cdot(1-\frac{2\cdot(I^U(a_2))^{\xi_2}}{(2-I^U(a_2))^{\xi_2}+(I^U(a_2))^{\xi_2}})}\right],$$

$$\left[\frac{\frac{2\cdot(F^L(a_1))^{\xi_1}}{(2-F^L(a_1))^{\xi_1}+(F^L(a_1))^{\xi_1}}\cdot\frac{2\cdot(F^L(a_2))^{\xi_2}}{(2-F^L(a_2))^{\xi_2}+(F^L(a_2))^{\xi_2}}}{1+(1-\frac{2\cdot(F^L(a_1))^{\xi_1}}{(2-F^L(a_1))^{\xi_1}+(F^L(a_1))^{\xi_1}})\cdot(1-\frac{2\cdot(F^L(a_2))^{\xi_2}}{(2-F^L(a_2))^{\xi_2}+(F^L(a_2))^{\xi_2}})},\right.$$

$$\left.\left.\left.\frac{\frac{2\cdot(F^U(a_1))^{\xi_1}}{(2-F^U(a_1))^{\xi_1}+(F^U(a_1))^{\xi_1}}\cdot\frac{2\cdot(F^U(a_2))^{\xi_2}}{(2-F^U(a_2))^{\xi_2}+(F^U(a_2))^{\xi_2}}}{1+(1-\frac{2\cdot(F^U(a_1))^{\xi_1}}{(2-F^U(a_1))^{\xi_1}+(F^U(a_1))^{\xi_1}})\cdot(1-\frac{2\cdot(F^U(a_2))^{\xi_2}}{(2-F^U(a_2))^{\xi_2}+(F^U(a_2))^{\xi_2}})}\right]\right)\right\rangle$$

$$=\left\langle s_{\sum_{i=1}^{2}\xi_i\cdot\theta(a_i)},\right.$$

$$\left(\left[\frac{(1+T^L(a_1))^{\xi_1}(1+T^L(a_2))^{\xi_2}-(1-T^L(a_1))^{\xi_1}(1-T^L(a_2))^{\xi_2}}{(1+T^L(a_1))^{\xi_1}(1+T^L(a_2))^{\xi_2}+(1-T^L(a_1))^{\xi_1}(1-T^L(a_2))^{\xi_2}},\right.\right.$$

$$\left.\frac{(1+T^U(a_1))^{\xi_1}(1+T^U(a_2))^{\xi_2}-(1-T^U(a_1))^{\xi_1}(1-T^U(a_2))^{\xi_2}}{(1+T^U(a_1))^{\xi_1}(1+T^U(a_2))^{\xi_2}+(1-T^U(a_1))^{\xi_1}(1-T^U(a_2))^{\xi_2}}\right],$$

$$\left[\frac{2\cdot(I^L(a_1))^{\xi_1}\cdot(I^L(a_2))^{\xi_2}}{(2-I^L(a_1))^{\xi_1}\cdot(2-I^L(a_2))^{\xi_2}+(I^L(a_1))^{\xi_1}\cdot(I^L(a_2))^{\xi_2}},\right.$$

$$\left.\frac{2\cdot(I^U(a_1))^{\xi_1}\cdot(I^U(a_2))^{\xi_2}}{(2-I^U(a_1))^{\xi_1}\cdot(2-I^U(a_2))^{\xi_2}+(I^U(a_1))^{\xi_1}\cdot(I^U(a_2))^{\xi_2}}\right],$$

$$\left[\frac{2\cdot(F^L(a_1))^{\xi_1}\cdot(F^L(a_2))^{\xi_2}}{(2-F^L(a_1))^{\xi_1}\cdot(2-F^L(a_2))^{\xi_2}+(F^L(a_1))^{\xi_1}\cdot(F^L(a_2))^{\xi_2}},\right.$$

$$\left.\left.\left.\frac{2\cdot(F^U(a_1))^{\xi_1}\cdot(F^U(a_2))^{\xi_2}}{(2-F^U(a_1))^{\xi_1}\cdot(2-F^U(a_2))^{\xi_2}+(F^U(a_1))^{\xi_1}\cdot(F^U(a_2))^{\xi_2}}\right]\right)\right\rangle$$

(2) If Equation (3) holds for n=k, then

$$INLPWA(a_1,a_2,\cdots a_k)=\left\langle s_{\sum_{i=1}^{k}(\xi_i\cdot\theta(a_i))},\right.$$

$$\left(\left[\frac{\prod_{i=1}^{k}(1+T^L(a_i))^{\xi_i}-\prod_{i=1}^{k}(1-T^L(a_i))^{\xi_i}}{\prod_{i=1}^{k}(1+T^L(a_i))^{\xi_i}+\prod_{i=1}^{k}(1-T^L(a_i))^{\xi_i}},\right.\right.$$

$$\left.\frac{\prod_{i=1}^{k}(1+T^U(a_i))^{\xi_i}-\prod_{i=1}^{k}(1-T^U(a_i))^{\xi_i}}{\prod_{i=1}^{k}(1+T^U(a_i))^{\xi_i}+\prod_{i=1}^{k}(1-T^U(a_i))^{\xi_i}}\right],$$

$$\left[\frac{2\prod_{i=1}^{k}(I^L(a_i))^{\xi_i}}{\prod_{i=1}^{k}(2-I^L(a_i))^{\xi_i}+\prod_{i=1}^{k}(I^L(a_i))^{\xi_i}},\right.$$

$$\left.\frac{2\prod_{i=1}^{k}(I^U(a_i))^{\xi_i}}{\prod_{i=1}^{k}(2-I^U(a_i))^{\xi_i}+\prod_{i=1}^{k}(I^U(a_i))^{\xi_i}}\right],$$

$$\left[\frac{2\prod_{i=1}^{k}(F^L(a_i))^{\xi_i}}{\prod_{i=1}^{k}(2-F^L(a_i))^{\xi_i}+\prod_{i=1}^{k}(F^L(a_i))^{\xi_i}},\right.$$

$$\left.\left.\left.\frac{2\prod_{i=1}^{k}(F^U(a_i))^{\xi_i}}{\prod_{i=1}^{k}(2-F^U(a_i))^{\xi_i}+\prod_{i=1}^{k}(F^U(a_i))^{\xi_i}}\right]\right)\right\rangle$$

If n=k+1, by the operations (1) and (3) in Definition 3.

$$INLPWA(a_1, a_2, \cdots a_k, a_{k+1}) = \left\langle s_{\sum_{i=1}^{k}(\xi_i \cdot \theta(a_i)) + \xi_{k+1} \cdot \theta(a_{k+1})}, \right.$$

$$\left(\left[\frac{\frac{\prod_{i=1}^{k}(1+T^L(a_i))^{\xi_i} - \prod_{i=1}^{k}(1-T^L(a_i))^{\xi_i}}{\prod_{i=1}^{k}(1+T^L(a_i))^{\xi_i} + \prod_{i=1}^{k}(1-T^L(a_i))^{\xi_i}} + \frac{(1+T^L(a_{k+1}))^{\xi_{k+1}} - (1-T^L(a_{k+1}))^{\xi_{k+1}}}{(1+T^L(a_{k+1}))^{\xi_{k+1}} + (1-T^L(a_{k+1}))^{\xi_{k+1}}}}{1 + \frac{\prod_{i=1}^{k}(1+T^L(a_i))^{\xi_i} - \prod_{i=1}^{k}(1-T^L(a_i))^{\xi_i}}{\prod_{i=1}^{k}(1+T^L(a_i))^{\xi_i} + \prod_{i=1}^{k}(1-T^L(a_i))^{\xi_i}} \cdot \frac{(1+T^L(a_{k+1}))^{\xi_{k+1}} - (1-T^L(a_{k+1}))^{\xi_{k+1}}}{(1+T^L(a_{k+1}))^{\xi_{k+1}} + (1-T^L(a_{k+1}))^{\xi_{k+1}}}}, \right.\right.$$

$$\left.\frac{\frac{\prod_{i=1}^{k}(1+T^U(a_i))^{\xi_i} - \prod_{i=1}^{k}(1-T^U(a_i))^{\xi_i}}{\prod_{i=1}^{k}(1+T^U(a_i))^{\xi_i} + \prod_{i=1}^{k}(1-T^U(a_i))^{\xi_i}} + \frac{(1+T^U(a_{k+1}))^{\xi_{k+1}} - (1-T^U(a_{k+1}))^{\xi_{k+1}}}{(1+T^U(a_{k+1}))^{\xi_{k+1}} + (1-T^U(a_{k+1}))^{\xi_{k+1}}}}{1 + \frac{\prod_{i=1}^{k}(1+T^U(a_i))^{\xi_i} - \prod_{i=1}^{k}(1-T^U(a_i))^{\xi_i}}{\prod_{i=1}^{k}(1+T^U(a_i))^{\xi_i} + \prod_{i=1}^{k}(1-T^U(a_i))^{\xi_i}} \cdot \frac{(1+T^U(a_{k+1}))^{\xi_{k+1}} - (1-T^U(a_{k+1}))^{\xi_{k+1}}}{(1+T^U(a_{k+1}))^{\xi_{k+1}} + (1-T^U(a_{k+1}))^{\xi_{k+1}}}}\right],$$

$$\left[\frac{\frac{2\prod_{i=1}^{k}(I^L(a_i))^{\xi_i}}{\prod_{i=1}^{k}(2-I^L(a_i))^{\xi_i} + \prod_{i=1}^{k}(I^L(a_i))^{\xi_i}} \cdot \frac{2(I^L(a_{k+1}))^{\xi_{k+1}}}{(2-I^L(a_{k+1}))^{\xi_{k+1}} + (I^L(a_{k+1}))^{\xi_{k+1}}}}{1 + \left(1 - \frac{2\prod_{i=1}^{k}(I^L(a_i))^{\xi_i}}{\prod_{i=1}^{k}(2-I^L(a_i))^{\xi_i} + \prod_{i=1}^{k}(I^L(a_i))^{\xi_i}}\right)\left(1 - \frac{2(I^L(a_{k+1}))^{\xi_{k+1}}}{(2-I^L(a_{k+1}))^{\xi_{k+1}} + (I^L(a_{k+1}))^{\xi_{k+1}}}\right)}, \right.$$

$$\left.\frac{\frac{2\prod_{i=1}^{k}(I^U(a_i))^{\xi_i}}{\prod_{i=1}^{k}(2-I^U(a_i))^{\xi_i} + \prod_{i=1}^{k}(I^U(a_i))^{\xi_i}} \cdot \frac{2(I^U(a_{k+1}))^{\xi_{k+1}}}{(2-I^U(a_{k+1}))^{\xi_{k+1}} + (I^U(a_{k+1}))^{\xi_{k+1}}}}{1 + \left(1 - \frac{2\prod_{i=1}^{k}(I^U(a_i))^{\xi_i}}{\prod_{i=1}^{k}(2-I^U(a_i))^{\xi_i} + \prod_{i=1}^{k}(I^U(a_i))^{\xi_i}}\right)\left(1 - \frac{2(I^U(a_{k+1}))^{\xi_{k+1}}}{(2-I^U(a_{k+1}))^{\xi_{k+1}} + (I^U(a_{k+1}))^{\xi_{k+1}}}\right)}\right],$$

$$\left[\frac{\frac{2\prod_{i=1}^{k}(F^L(a_i))^{\xi_i}}{\prod_{i=1}^{k}(2-F^L(a_i))^{\xi_i} + \prod_{i=1}^{k}(F^L(a_i))^{\xi_i}} \cdot \frac{2(F^L(a_{k+1}))^{\xi_{k+1}}}{(2-F^L(a_{k+1}))^{\xi_{k+1}} + (F^L(a_{k+1}))^{\xi_{k+1}}}}{1 + \left(1 - \frac{2\prod_{i=1}^{k}(F^L(a_i))^{\xi_i}}{\prod_{i=1}^{k}(2-F^L(a_i))^{\xi_i} + \prod_{i=1}^{k}(F^L(a_i))^{\xi_i}}\right)\left(1 - \frac{2(F^L(a_{k+1}))^{\xi_{k+1}}}{(2-F^L(a_{k+1}))^{\xi_{k+1}} + (F^L(a_{k+1}))^{\xi_{k+1}}}\right)}, \right.$$

$$\left.\left.\left.\frac{\frac{2\prod_{i=1}^{k}(F^U(a_i))^{\xi_i}}{\prod_{i=1}^{k}(2-F^U(a_i))^{\xi_i} + \prod_{i=1}^{k}(F^U(a_i))^{\xi_i}} \cdot \frac{2(F^U(a_{k+1}))^{\xi_{k+1}}}{(2-F^U(a_{k+1}))^{\xi_{k+1}} + (F^U(a_{k+1}))^{\xi_{k+1}}}}{1 + \left(1 - \frac{2\prod_{i=1}^{k}(F^U(a_i))^{\xi_i}}{\prod_{i=1}^{k}(2-F^U(a_i))^{\xi_i} + \prod_{i=1}^{k}(F^U(a_i))^{\xi_i}}\right)\left(1 - \frac{2(F^U(a_{k+1}))^{\xi_{k+1}}}{(2-F^U(a_{k+1}))^{\xi_{k+1}} + (F^U(a_{k+1}))^{\xi_{k+1}}}\right)}\right]\right)\right\rangle$$

$$= \left\langle s_{\sum_{i=1}^{k+1}(\xi_i \cdot \theta(a_i))}, \right.$$

$$\left(\left[\frac{\prod_{i=1}^{k+1}(1+T^L(a_i))^{\xi_i} - \prod_{i=1}^{k+1}(1-T^L(a_i))^{\xi_i}}{\prod_{i=1}^{k+1}(1+T^L(a_i))^{\xi_i} + \prod_{i=1}^{k+1}(1-T^L(a_i))^{\xi_i}}, \frac{\prod_{i=1}^{k+1}(1+T^U(a_i))^{\xi_i} - \prod_{i=1}^{k+1}(1-T^U(a_i))^{\xi_i}}{\prod_{i=1}^{k+1}(1+T^U(a_i))^{\xi_i} + \prod_{i=1}^{k+1}(1-T^U(a_i))^{\xi_i}} \right], \right.$$

$$\left[\frac{2\prod_{i=1}^{k+1}(I^L(a_i))^{\xi_i}}{\prod_{i=1}^{k+1}(2-I^L(a_i))^{\xi_i} + \prod_{i=1}^{k+1}(I^L(a_i))^{\xi_i}}, \frac{2\prod_{i=1}^{k+1}(I^U(a_i))^{\xi_i}}{\prod_{i=1}^{k+1}(2-I^U(a_i))^{\xi_i} + \prod_{i=1}^{k+1}(I^U(a_i))^{\xi_i}} \right],$$

$$\left. \left. \left[\frac{2\prod_{i=1}^{k+1}(F^L(a_i))^{\xi_i}}{\prod_{i=1}^{k+1}(2-F^L(a_i))^{\xi_i} + \prod_{i=1}^{k+1}(F^L(a_i))^{\xi_i}}, \frac{2\prod_{i=1}^{k+1}(F^U(a_i))^{\xi_i}}{\prod_{i=1}^{k+1}(2-F^U(a_i))^{\xi_i} + \prod_{i=1}^{k+1}(F^U(a_i))^{\xi_i}} \right] \right) \right\rangle$$

Eq. (3) holds for n=k+1. Thus, Eq. (3) holds for all n.

The INLPWA operator has the following properties.

(1) Commutativity:

Let $a_i = \left\langle s_{\theta(a_i)}, \left(\left[T^L(a_i), T^U(a_i)\right], \left[I^L(a_i), I^U(a_i)\right], \left[F^L(a_i), F^U(a_i)\right]\right)\right\rangle$ be a collection of INLNs, if $a_i^*(i=1,2,\cdots,n)$ is any permutation of $a_i\left(i=1,2,\cdots,n\right)$, then

$INLPWA(a_1, a_2, \cdots, a_n) = INLPWA(a_1^*, a_2^*, \cdots, a_n^*)$.

(2) Idempotency:

Let $a_i = \left\langle s_{\theta(a_i)}, \left(\left[T^L(a_i), T^U(a_i)\right], \left[I^L(a_i), I^U(a_i)\right], \left[F^L(a_i), F^U(a_i)\right]\right)\right\rangle$ $(i=1,2,\cdots,n)$ be a collection of INLNs, and $a = \left\langle s_{\theta(a)}, \left(\left[T^L(a), T^U(a)\right], \left[I^L(a), I^U(a)\right], \left[F^L(a), F^U(a)\right]\right)\right\rangle$ be a INLN, if $a_i = a\left(i = 1, 2, \cdots, n\right)$, then

$INLPWA(a_1, a_2, \cdots, a_n) = a.$

(3) Boundness:

Let $a_i = \left\langle s_{\theta(a_i)}, \left(\left[T^L(a_i), T^U(a_i)\right], \left[I^L(a_i), I^U(a_i)\right], \left[F^L(a_i), F^U(a_i)\right]\right)\right\rangle$ $\left(i=1,2,\cdots,n\right)$ and $a_i^* = \left\langle s_{\theta(a_i^*)}, \left(\left[T^L(a_i^*), T^U(a_i^*)\right], \left[I^L(a_i^*), I^U(a_i^*)\right], \left[F^L(a_i^*), F^U(a_i^*)\right]\right)\right\rangle$ $\left(i=1,2,\cdots,n\right)$ be two collections of INLNs. If $\theta(a_i) \le \theta(a_i^*)$, $T^L(a_i) \le T^L(a_i^*)$, $T^U(a_i) \le T^U(a_i^*)$, $I^L(a_i) \ge I^L(a_i^*)$, $I^U(a_i) \ge I^U(a_i^*)$, $F^L(a_i) \ge F^L(a_i^*)$, $F^U(a_i) \ge F^U(a_i^*)$, then $INLPWA(a_1, a_2, \cdots, a_n) \le INLPWA(a_1^*, a_2^*, \cdots, a_n^*)$.

B. INLPWG operator

Definition 6 Let $a_i(i = 1, 2, \cdots, n)$ be a collection of INLNs, $a_i = \left\langle s_{\theta(a_i)}, \left(\left[T^L(a_i), T^U(a_i)\right], \left[I^L(a_i), I^U(a_i)\right], \left[F^L(a_i), F^U(a_i)\right]\right)\right\rangle$ and $\omega = \left(\omega_1, \omega_2, \cdots, \omega_n\right)$ be the weighted vector for a_i, $\omega_i \in \left[0, 1\right]$ and $\sum_{i=1}^{n}\omega_i = 1$. Then the operator of INLPWG can be achieved, and the aggregation result is still an INLN.

$$INLPWG(a_1, a_2, \cdots a_n) = \bigotimes_{i=1}^{n}(a_i)^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}} \tag{4}$$

Where $S(a_i) = \sum_{j=1, j\ne i}^{n}\omega_j Supp(a_i, a_j)$, satisfying the following conditions.

(1) $Supp(a_i, a_j) \in [0,1]$

(2) $Supp(a_i, a_j) = Supp(a_j, a_i)$

(3) $Supp(a_i, a_j) \ge Supp(a_p, a_q)$. If $d(a_i, a_j) < d(a_p, a_q)$,

Here $d(a_i, a_j)$ is the Hamming distance between a_i and a_j defined in Definition 4.

Based on the operations in Definition 3 and Eq. (4), we can derive the following Theorem 3.

Theorem 3 Let $a_i(i = 1, 2, \cdots, n)$ be a collection of INLNs, $a_i = \left\langle s_{\theta(a_i)}, \left(\left[T^L(a_i), T^U(a_i)\right], \left[I^L(a_i), I^U(a_i)\right], \left[F^L(a_i), F^U(a_i)\right]\right)\right\rangle$ and $\omega = \left(\omega_1, \omega_2, \cdots, \omega_n\right)$ be the weighted vector for a_i, $\omega_i \in \left[0, 1\right]$ and $\sum_{i=1}^{n}\omega_i = 1$. Then the aggregated result of INLPWG is also an INLN.

$$
\begin{aligned}
INLPWG(a_1, a_2, \cdots a_n) = \Bigg\langle & s_{\prod_{i=1}^{n} \theta(a_i)^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}, \\
& \left(\left[\frac{2\prod_{i=1}^{n}(T^L(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}{\prod_{i=1}^{n}(2-T^L(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}+\prod_{i=1}^{n}(T^L(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}},\right.\right. \\
& \left.\frac{2\prod_{i=1}^{n}(T^U(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}{\prod_{i=1}^{n}(2-T^U(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}+\prod_{i=1}^{n}(T^U(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}\right], \\
& \left[\frac{\prod_{i=1}^{n}(1+I^L(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}-\prod_{i=1}^{n}(1-I^L(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}{\prod_{i=1}^{n}(1+I^L(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}+\prod_{i=1}^{n}(1-I^L(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}},\right. \\
& \left.\frac{\prod_{i=1}^{n}(1+I^U(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}-\prod_{i=1}^{n}(1-I^U(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}{\prod_{i=1}^{n}(1+I^U(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}+\prod_{i=1}^{n}(1-I^U(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}\right], \\
& \left[\frac{\prod_{i=1}^{n}(1+F^L(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}-\prod_{i=1}^{n}(1-F^L(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}{\prod_{i=1}^{n}(1+F^L(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}+\prod_{i=1}^{n}(1-F^L(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}},\right. \\
& \left.\left.\frac{\prod_{i=1}^{n}(1+F^U(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}-\prod_{i=1}^{n}(1-F^U(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}{\prod_{i=1}^{n}(1+F^U(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}+\prod_{i=1}^{n}(1-F^U(a_i))^{\frac{\omega_i(1+S(a_i))}{\sum_{i=1}^{n}\omega_i(1+S(a_i))}}}\right]\right)\Bigg\rangle
\end{aligned}
\tag{5}
$$

Where $S(a_i) = \sum_{j=1, j\neq i}^{n} \omega_j Supp(a_i, a_j)$, satisfying the conditions in Definition 6.

Similarly, the INLPWG operator Eq. (5) can be proved using the mathematical induction, and the INLPWG operator also has the properties of commutativity, dempotency and boundness.

Ⅳ. ILLUSTRATIVE EXAMPLE

In this section, we will use the novel operators to deal with the multi-criteria decision-making problems under the interval neutrosophic linguistic environment, where the alternative values are in the form of INLNs and the criteria weights are in the form of crisp values.

Next, we will consider the same decision-making problem adapted from Ye [14].

An investment company wants to expand its business. Four alternatives will be chosen, A_1 represents an auto corporation, A_2 represents a food corporation, A_3 represents a computer company corporation, and A_4 represents a weapon corporation. Each alternative is evaluated under three criteria, C_1 denotes risk, C_2 denotes growth, and C_3 denotes the environment impact. The corresponding weighted vector is $\omega = \{0.35, 0.25, 0.4\}$.

The expert gives values for the satisfaction, indeterminacy and dissatisfaction regarding the alternative A_i corresponding to the criteria C_j under the linguistic term set S. Therefore, the assessment value is given in the form of the INLN, and the linguistic term set is employed as $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$
$= \{\text{extremely poor,very poor,poor,medium,good,very good,extremely good}\}$.

The interval neutrosophic linguistic decision matrix $B = [b_{ij}]_{4\times 3}$ is shown as follows.

$$B = [b_{ij}]_{4\times 3} = \begin{bmatrix}
\langle s_5, ([0.4, 0.5], [0.2, 0.3], [0.3, 0.4])\rangle & \langle s_6, ([0.4, 0.6], [0.1, 0.2], [0.2, 0.4])\rangle & \langle s_5, ([0.2, 0.3], [0.1, 0.2], [0.5, 0.6])\rangle \\
\langle s_6, ([0.5, 0.7], [0.1, 0.2], [0.2, 0.3])\rangle & \langle s_5, ([0.6, 0.7], [0.1, 0.2], [0.2, 0.3])\rangle & \langle s_5, ([0.5, 0.7], [0.2, 0.2], [0.1, 0.2])\rangle \\
\langle s_6, ([0.3, 0.5], [0.1, 0.2], [0.3, 0.4])\rangle & \langle s_5, ([0.5, 0.6], [0.1, 0.3], [0.3, 0.4])\rangle & \langle s_4, ([0.5, 0.6], [0.1, 0.3], [0.1, 0.3])\rangle \\
\langle s_4, ([0.7, 0.8], [0.0, 0.1], [0.1, 0.2])\rangle & \langle s_4, ([0.5, 0.7], [0.1, 0.2], [0.2, 0.3])\rangle & \langle s_6, ([0.3, 0.4], [0.1, 0.2], [0.1, 0.2])\rangle
\end{bmatrix}$$

Step1. Calculate the supports $Supp(b_{ij}, b_{ip})$.

As an example, $Supp(b_{11}, b_{12})$ can be obtained as follows:

$$\begin{aligned}
Supp(b_{11}, b_{12}) &= 1 - d(b_{11}, b_{12}) \\
&= 1 - d(\langle s_5, ([0.4, 0.5], [0.2, 0.3], [0.3, 0.4])\rangle, \\
&\quad \langle s_6, ([0.4, 0.6], [0.1, 0.2], [0.2, 0.4])\rangle) \\
&= 0.8389
\end{aligned}$$

Where $d(b_{11}, b_{12})$ is the Hamming distance defined in equation (1).

Then, $Supp(b_{ij}, b_{ip})$ $(i = 1, 2, 3, 4;\ j, p = 1, 2, 3;\ j \neq p)$ can be calculated.

$Supp(b_{11}, b_{12}) = Supp(b_{12}, b_{11}) = 0.8389;$
$Supp(b_{11}, b_{13}) = Supp(b_{13}, b_{11}) = 0.7222;$
$Supp(b_{12}, b_{13}) = Supp(b_{13}, b_{12}) = 0.6833;$
$Supp(b_{21}, b_{22}) = Supp(b_{22}, b_{21}) = 0.9167;$
$Supp(b_{21}, b_{23}) = Supp(b_{23}, b_{21}) = 0.8167;$
$Supp(b_{22}, b_{23}) = Supp(b_{23}, b_{22}) = 0.8889;$
$Supp(b_{31}, b_{32}) = Supp(b_{32}, b_{31}) = 0.9000;$
$Supp(b_{31}, b_{33}) = Supp(b_{33}, b_{31}) = 0.8000;$
$Supp(b_{32}, b_{33}) = Supp(b_{33}, b_{32}) = 0.8111;$
$Supp(b_{41}, b_{42}) = Supp(b_{42}, b_{41}) = 0.8444;$
$Supp(b_{41}, b_{43}) = Supp(b_{43}, b_{41}) = 0.7889;$
$Supp(b_{42}, b_{43}) = Supp(b_{43}, b_{42}) = 0.9222.$

Step2. Calculate the weights of ξ_{ij}.

The weighted support $S(b_{ij})$ can be obtained using the weights $\omega_j\ (j = 1, 2, 3)$ of the criteria $C_j\ (j = 1, 2, 3)$

$$S(b_{ij}) = \sum_{p=1, p\neq j}^{3} \omega_p Supp(b_{ij}, b_{ip})\ (i = 1, 2, 3, 4;\ p = 1, 2, 3)$$

Then, the weights $\xi_{ij}\ (i = 1, 2, 3, 4;\ j = 1, 2, 3)$ associated with the INLN b_{ij} can be calculated by the following formula:

$$\xi_{ij} = \frac{\omega_j (1+S(b_{ij}))}{\sum_{j=1}^{3} \omega_j (1+S(b_{ij}))}$$

As an example, $S(b_{11})$ can be calculated as follows:

$$\begin{aligned}
S(b_{11}) &= \omega_2 \cdot Supp(b_{11}, b_{12}) + \omega_3 \cdot Supp(b_{11}, b_{13}) \\
&= 0.25 \cdot 0.8389 + 0.4 \cdot 0.7222 \\
&= 0.4986;
\end{aligned}$$

Then,

$$(S(b))_{4\times 3} = \begin{bmatrix}
0.4986 & 0.5669 & 0.4236 \\
0.5558 & 0.6764 & 0.5081 \\
0.5450 & 0.6394 & 0.4828 \\
0.5267 & 0.6644 & 0.5067
\end{bmatrix}$$

Therefore, as an example, ξ_{11} can be calculated as follows:

$$\xi_{11} = \frac{\omega_1 (1+S(b_{11}))}{\sum_{j=1}^{3} \omega_j (1+S(b_{1j}))} = \frac{0.5245}{1.4857} = 0.3530$$

Then,

$$\xi_{4\times 3}=\begin{bmatrix}0.3530 & 0.2637 & 0.3833\\ 0.3475 & 0.2675 & 0.3850\\ 0.3503 & 0.2655 & 0.3842\\ 0.3440 & 0.2679 & 0.3880\end{bmatrix}$$

Step3. Calculate the comprehensive evaluate value of each alternative.

Utilize the INLPWA operator in Eq. (3) to aggregate all the values of each alternative. Then, the comprehensive value b_1 of alternative A_1 can be obtained as follows:

$$b_1 = INLPWA(b_{11}, b_{12}, b_{13})$$

$$= \left\langle s_{\xi_{11}\theta(b_{11})+\xi_{12}\theta(b_{12})+\xi_{13}\theta(b_{13})}, \left(\left[\frac{\prod_{j=1}^{3}(1+T^L(b_{1j}))^{\xi_{1j}}-\prod_{j=1}^{3}(1-T^L(b_{1j}))^{\xi_{1j}}}{\prod_{j=1}^{3}(1+T^L(b_{1j}))^{\xi_{1j}}+\prod_{j=1}^{3}(1-T^L(b_{1j}))^{\xi_{1j}}}, \frac{\prod_{j=1}^{3}(1+T^U(b_{1j}))^{\xi_{1j}}-\prod_{j=1}^{3}(1-T^U(b_{1j}))^{\xi_{1j}}}{\prod_{j=1}^{3}(1+T^U(b_{1j}))^{\xi_{1j}}+\prod_{j=1}^{3}(1-T^U(b_{1j}))^{\xi_{1j}}}\right], \left[\frac{2\prod_{j=1}^{3}(I^L(b_{1j}))^{\xi_{1j}}}{\prod_{j=1}^{3}(2-I^L(b_{1j}))^{\xi_{1j}}+\prod_{j=1}^{3}(I^L(b_{1j}))^{\xi_{1j}}}, \frac{2\prod_{j=1}^{3}(I^U(b_{1j}))^{\xi_{1j}}}{\prod_{j=1}^{3}(2-I^U(b_{1j}))^{\xi_{1j}}+\prod_{j=1}^{3}(I^U(b_{1j}))^{\xi_{1j}}}\right], \left[\frac{2\prod_{j=1}^{3}(F^L(b_{1j}))^{\xi_{1j}}}{\prod_{j=1}^{3}(2-F^L(b_{1j}))^{\xi_{1j}}+\prod_{j=1}^{3}(F^L(b_{1j}))^{\xi_{1j}}}, \frac{2\prod_{j=1}^{3}(F^U(b_{1j}))^{\xi_{1j}}}{\prod_{j=1}^{3}(2-F^U(b_{1j}))^{\xi_{1j}}+\prod_{j=1}^{3}(F^U(b_{1j}))^{\xi_{1j}}}\right]\right)\right\rangle$$

$$= \langle s_{5.2637}, ([0.3266, 0.4584], [0.1283, 0.2314], [0.3324, 0.4702])\rangle$$

Similarly,

$$b_2 = INLPWA(b_{21}, b_{22}, b_{23}) = \langle s_{5.3475}, ([0.5283, 0.7000], [0.1311, 0.2000], [0.1538, 0.2574])\rangle$$

$$b_3 = INLPWA(b_{31}, b_{32}, b_{33}) == \langle s_{4.9661}, ([0.4344, 0.5668], [0.1000, 0.2610], [0.1996, 0.3589])\rangle$$

$$b_4 = INLPWA(b_{41}, b_{42}, b_{43}) = \langle s_{4.7761}, ([0.5122, 0.6497], [0.0000, 0.1582], [0.1208, 0.2234])\rangle$$

Step4. Calculate the Hamming distance between an alternative A_j and the ideal solution/negative ideal solution.

The ideal solution is given as $y^+ = \langle s_{\max\theta(x)}, ([1,1],[0,0],[0,0])\rangle$, and the negative ideal solution is given as $y^- = \langle s_{\min\theta(x)}, ([0,0],[1,1],[1,1])\rangle$, the distance measure is given in the following.

$$d_j^+ = d(b_j, y^+), d_j^- = d(b_j, y^-).$$

$d_1^+ = 0.8881, d_1^- = 0.3833;$

$d_2^+ = 0.6334, d_2^- = 0.4161;$

$d_3^+ = 0.7552, d_3^- = 0.3646;$

$d_4^+ = 0.6028, d_4^- = 0.4046.$

Step5. Get the relative closeness coefficient.

$$R_j = \frac{d_j^+}{d_j^+ + d_j^-}\ j = 1, 2, 3, 4.$$

Thus,

$R_1 = 0.6985, R_2 = 0.6035,$

$R_3 = 0.6744, R_4 = 0.5984.$

Step6. Rank the alternatives.

According to the relative closeness coefficient, the final ranking order of the alternatives is $A_4 \succ A_2 \succ A_3 \succ A_1$. The smaller R_j is, the better the alternative A_j is. Apparently, the best alternative is A_4 while the worst alternative is A_1.

The method proposed in this paper is compared with the method that was conducted in Ye [14]. For the same MCDM problem under interval neutrosophic linguistic environment, if the aggregation operators defined by Ye are applied, either the INLWAA operator or the INLWGA operator, the final ranking of four alternatives is always $A_2 \succ A_4 \succ A_3 \succ A_1$. However, if the method in this paper is used, the final ranking is $A_4 \succ A_2 \succ A_3 \succ A_1$. Clearly, the worst alternative is the same, while the best alternative is different. There are two reasons. Firstly, different aggregation results are obtained due to different aggregation operators are used in the two methods. Secondly, to rank INLNs, the score function is defined and used in Ye [14], while the extended TOPSIS method is utilized in this paper. The main advantage of the method outlined in this paper is due to its ability to consider the relationship among the multi-input arguments being fused. Meanwhile, TOPSIS method had been proved to be an effective ranking method for MCDM problem. In this paper, we extend the traditional TOPSIS method only dealing with the real numbers to the interval neutrosophic linguistic environment. Therefore, these reasons lead to the final ranking result in this paper is different from the other method, and is more precise and reliable.

V. CONCLUSION

In this paper, the interval neutrosophic linguistic sets combine the advantages of both the interval neutrosophic set and the linguistic variables, and it can easily express the indeterminate and inconsistent information in real decision making in real world. Therefore, it is meaningful to study

MCDM problems with INLSs. However, the conventional PWA operator and PWG operator fail in handling INLSs. Thus, the main contributions of the paper are: firstly, the novel operational rules of INLNs based on Einstein operations were proposed under interval neutrosophic linguistic environment. Then, the Hamming distance of INLNs were originally established. Secondly, the traditional power operators were extended to INLNs environment and were more generation. Two novel operators, the INLPWA operator and INLPWG operator were proposed, and their properties were also investigated. Finally, an illustrative example was demonstrated to verify the effectiveness and practicality of the proposed method comparing with the other method, and the extended TOPSIS method was also conducted to rank the alternatives in MCDM problem.

In future research, we shall extend the proposed method to other domains, such as medical diagnosis, pattern recognition and group decision making.

REFERENCES

[1] F. Smarandache, A unifying field in logics. neutrosophy: Neutrosophic probability, set and logic, American Research Press, Rehoboth, 1999.

[2] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338-353, 1965.

[3] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87-96, 1986.

[4] V. Torra, "Hesitant fuzzy sets," *Int. J. Intell.* Syst. vol.25, pp. 529–539, 2010.

[5] H.Wang, F. Smarandache, Y. Q. Zhang, and R. Sunderraman, "Single valued neutrosophic sets," *Multispace and Multistructure*, vol. 4, pp. 410－413, 2010.

[6] H. Wang, F. Smarandache, Y. Q. Zhang, and R. Sunderraman, Interval Neutrosophic Sets and Logic:Theory and Applications in Computing, Hexis, Phoenix, Ariz, USA, 2005.

[7] J.Ye, "A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets," *Journal of Intelligent & Fuzzy Systems*, vol.26, no. 5, pp. 2459-2466, 2014.

[8] J.Ye, "Multiple-attribute decision-making method under a single-valued neutrosophic hesitant fuzzy environment," J Intell Syst , vol.24, no. 1, pp. 23–36, 2015.

[9] J.Q. Wang and X.E. Li, "An application of the TODIM method with multi-valued neutrosophic set," Control Decis , vol.30, no. 6, pp. 1139–1142, 2015.

[10] P.D.Liu and L.L.Shi, "The generalized hybrid weighted average operator based on interval neutrosophic hesitant set and its application to multiple attribute decision making," , neural computing & applications,vol.26, no. 2, pp. 457–471, 2015.

[11] L.A.Zadeh, "The concept of a linguistic variable and its application to approximate reasoning," Inf Sci,vol. 8, no. 3, pp. 199–249, 1975.

[12] J .Ye, "An extended TOPSIS method for multiple attribute group decision making based on single valued neutrosophic linguistic numbers," J Intell Fuzzy Syst, vol. 28, no.1, pp. 247–255, 2015.

[13] Z.P.Tian, J.Wang, H.Y.Zhang, "Simplified Neutrosophic Linguistic Normalized Weighted Bonferroni Mean Operator and Its Application to Multi-Criteria Decision-Making Problems," Filomat , vol. 30, no.12, pp. 3339–3360,2016.

[14] J.Ye, "Some aggregation operators of interval neutrosophic linguistic numbers for multiple attribute decision making," J Intell Fuzzy Syst , vol. 27, no.5, pp. 2231–2241,2014.

[15] S.Broumi, J.Ye and F.Smarandache. "An extended TOPSIS method for multiple attribute decision making based on interval neutrosophic uncertain linguistic variables," Neutrosophic sets and systems, vol. 8, pp. 22–30,2015.

[16] B.L.Li, J.R.Wang, L.H.Yang, and X.T.Li, "Multiple criteria decision making approach with multi-valued neutrosophic linguistic normalized weighted Bonferroni mean Hamacher operator," Mathematical problems in Engineering, 2018.Doi:10.1155/2018/2432167.

[17] J.Q.Wang, Y.Yang and L.Li, "Multi-criteria decision-making method based on single-valued neutrosophic linguistic Maclaurin symmetric mean operators," *Neural Computing &Applications*, vol. 4 pp. 1-19, 2016.

[18] Z.P.Tian, J. Wang, H.Y. Zhang, and J.Q. Wang, "Multi-criteria decision-making based on generalized prioritized aggregation operators under simplified neutrosophic uncertain linguistic environment," Int J Mach Learn Cybern, 2016.Doi: 10.1007/s13042-016-0552-9

[19] Y.X.Ma, J.Q.Wang and J.Wang, "An interval neutrosophic linguistic multi-criteria group decision-making method and its application in selecting medical treatment options," *Neural Computing &Applications*, pp. 1-21, 2017.

[20] R. R. Yager, "The power average operator," IEEE Trans. on Systems, Man, Cybernetics-Part A:Systems and Humans, vol. 31, pp. 724-731,2001.

[21] Z. S. Xu and R. R. Yager, "Power geometric operators and their use in group decision making," IEEE Trans. on Fuzzy Systems, vol. 18, pp. 94-105, 2010.

[22] L. G. Zhou, H. Y. Chen, and J. P. Liu, "Generalized power aggregation operators and their applications in group decision making," Computers & Industrial Engineering, vol. 62, pp. 989-999,2012.

[23] L.H.Yang and B.L.Li, "A Multi-Criteria Decision-Making Method Using Power Aggregation Operators for Single-valued Neutrosophic Sets," *International Journal of Database Theory and Application*. vol. 9, no. 2, pp. 23-32, 2016.

[24] P.D.Liu and G.Tang,"Some power generalized aggregation operators based on the interval neutrosophic sets and their application to decision making," *Journal of Intelligent and Fuzzy Systems*, vol. 30, no.5, pp. 2517-2528, 2016.

[25] B.L.Li, J.R.Wang, L.H.Yang and X.T.Li, "A Novel Generalized Simplified Neutrosophic Number Einstein Aggregation Operator," *IAENG International Journal of Applied Mathematics*, vol. 48, no.1, pp. 67-72, 2018.