

Stability and Bifurcation in a Stage-structured Predator-prey Model with Allee Effect and Time Delay

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Abstract—In this paper, we study a stage-structured predator-prey model incorporating time delay with prey growth subject to a strong Allee effect. By analyzing the characteristic equation of the corresponding linearized system, we investigate the local asymptotic stability of the system according to the change of birth rate in prey. Using the delay as a bifurcation parameter, the model undergoes a Hopf bifurcation at the positive equilibrium when the delay crosses some critical values. On the other hand, we show that both predators and prey will go extinct if the birth rate is small or the Allee effect is large.

Index Terms—predator-prey, Allee effect, stage structure, stability, time delay.

I. INTRODUCTION

THE Predator-prey model is one of the basic models in the theoretical studies of ecology. As we all know, predator-prey model has been studied extensively [1]-[8]. In order to explain the phenomena of predator species and prey species, many scholars investigated functional responses. For example, Chen, Xie, et al[9] and Chen, Wang, et al[10] studied the dynamic behaviors incorporating functional response for predator-prey model.

In modeling the predator-prey systems, a key factor in consideration is the Allee effect. The Allee effect named after Allee[11], has significant contribution to population dynamics. The Allee effect is a biological phenomenon, wherein positive correlation between population density and the average degree of individual fitness of the population or species [12], [13]. The Allee effects occur as a result of mate limitation, evading natural enemies, inbreeding depression, raising their young, and environmental conditioning [12], [14], [15]. In addition, the Allee effect has recently being studied by many scholars, see [16]-[21] and the references cited therein.

The Allee effect was shown that the low density population can affect the birth rate of the species [22], but the coefficients of the growth rate are irrelevant to the Allee-type function. Hence, let $F(x)$ be the fertility rate of a species x [23]:

$$F(x) = \frac{ax}{A+x},$$

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where a is the per capita maximum fertility rate of species x ; A is the Allee effect constant of the species. If $A > 0$, the fertility rate of the species is zero if $x = 0$ and approaches to a if x is large enough. The value of the parameter A determines the growth rate of $F(x)$. When $A = 0$, the fertility rate $F(x) = a$ is density independent. Therefore when considering the Allee effect, the logistic equation can be rewritten in the following form

$$\dot{x} = x \left(\frac{ax}{A+x} - d - bx \right), \quad (1)$$

where d denotes the death rate of the species; the intra-specific competition intensity of the species is represented by b ; $ax/(A+x)$ is a Michaelis-Menten type function. Clearly, when $A = 0$, system (1) is reduced to the traditional logistic equation.

Zu[24] researched the following predator-prey system with Allee effect

$$\begin{aligned} \dot{x} &= x \left(\frac{ax}{A+x} - d - bx \right) - mxy, \\ \dot{y} &= nmxy - \delta y. \end{aligned} \quad (2)$$

The authors investigated the existence and local stability of the equilibria of system (2). Also they studied the unstable or stable periodic fluctuations with the influence of Allee effect.

In most of ecosystems, maturation, pregnancy and hunting occur all the time. Hence, time delay due to gestation has been greatly researched as a focus issue in predator-prey system [25]-[31], since the current birth rate of the predator is related to its consumption of prey throughout the past history. For example, Chen, Xie, et al[32] discussed the partial survival and extinction of a delayed predator-prey model with stage structure. By applying the standard comparison theorem, some novel results concern with the extinction of the system and partial survival of the predator and prey species, respectively, are obtained.

As we all know, there are always many species going through two stages, immature and mature, which reflected the different characteristics of species at each stage. Therefore, stage structure population models are more reasonable than other models and exhibits real world phenomenon. In recent years, numerous papers have been considered the predator-prey system with stage structure, see [33]-[39] and the references cited therein. For example, Li, Han, et al[40] proposed a predator-prey system with stage structure and mutual interference. The authors study the global stability of the interior equilibrium of the system. Chen, Xie, et al[41] investigated a two-species May type cooperation model with stage structure. They obtained the global extinction, partial

survival and global attractivity of the positive equilibrium. Wei and Fu[42] considered the dynamic behaviors of a predator-prey systems with Beddington-DeAngelis type functional response and stage structure for prey incorporating refuge.

However, still seldom scholars consider the stability of a stage-structured predator-prey model incorporating time delay with a strong Allee effect in prey. Hence, the main purpose of this paper is to study the stability and Hopf bifurcation of system (2) with stage structure. More precisely, we study the following model

$$\begin{aligned}\dot{x} &= x \left(\frac{ax}{A+x} - d - bx \right) - mxy_2, \\ \dot{y}_1 &= nm x(t-\tau)y_2(t-\tau) - \delta y_1 - \beta y_1, \\ \dot{y}_2 &= \beta y_1 - ey_2,\end{aligned}\quad (3)$$

where x , y_1 and y_2 denote the densities of prey species, immature predator species and mature predator species at time t , respectively; A , a , b , d , n , m , β , δ , e are positive constants; τ demonstrates the time delay because of the gestation of the predator; m is the capture rate and n is the food conversion rate of predator; η is the conversion rate of nutrients into the production of predator species; δ and e are the death rates of the immature and mature predator species.

The initial conditions for system (3) take the form

$$\begin{aligned}x(\theta) &= \phi(\theta), & y_1(\theta) &= \psi_1(\theta), & y_2(\theta) &= \psi_2(\theta), \\ \phi(\theta) &\geq 0, & \psi_1(\theta) &\geq 0, & \psi_2(\theta) &\geq 0, & \theta \in [-\tau, 0], \\ \phi(0) &> 0, & \psi_1(0) &> 0, & \psi_2(0) &> 0,\end{aligned}\quad (4)$$

where $(\phi(\theta), \psi_1(\theta), \psi_2(\theta)) \in C([-\tau, 0], \mathbb{R}_+^3)$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}_+^3$, where $\mathbb{R}_+^3 = \{(x_1, x_2, x_3) : x_i \geq 0, i = 1, 2, 3\}$.

The rest of this paper is organized as follows: The boundedness and the existence of equilibriums of system (3) are derived in the next section. In Section 3, we study the local asymptotic stability and Hopf bifurcation of system (3). We end this paper with examples and a brief discussion.

II. BOUNDEDNESS, AND EXISTENCE OF EQUILIBRIA

In this section, we study the boundedness and the existence of equilibriums of system (3).

Define

$$a_1 = Ab + d + 2\sqrt{Abd}.$$

Theorem 2.1. [43] *Let $x(t)$ be a positive solution of system (1) with the initial value $x(0) > 0$.*

(1) *If $a < a_1$, then $\lim_{t \rightarrow +\infty} x(t) = 0$.*

(2) *If $a \geq a_1$, then we obtain the following results:*

(i) *For $0 < x(0) < x_2$, $\lim_{t \rightarrow +\infty} x(t) = 0$;*

(ii) *For $x_2 < x(0)$, $\lim_{t \rightarrow +\infty} x(t) = x_1$,*

where

$$x_1 = \frac{k + \sqrt{Q_1}}{2b} \quad \text{and} \quad x_2 = \frac{k - \sqrt{Q_1}}{2b}$$

with $k = a - (Ab + d)$, $Q_1 = a^2 - 2(Ab + d)a + (Ab - d)^2$.

In the following, we show that the positivity and boundedness of solution of system (3).

Lemma 2.1. *Every solution of system (3) with the initial condition (4) are positive and ultimately bounded for all $t \geq 0$.*

Proof. It is obviously that every solution of model (3) with initial conditions (4) are positive for all $t \geq 0$. The result is a direct consequence of Nagumos theorem [44]. Let $V(t) = nx(t-\tau) + y_1(t) + y_2(t)$, calculating the derivative of $V(t)$ with respect to t along the positive solution of system (3), we have

$$\begin{aligned}\dot{V}(t) &= n\dot{x}(t-\tau) + \dot{y}_1(t) + \dot{y}_2(t) \\ &= nx(t-\tau) \left(\frac{ax(t-\tau)}{A+x(t-\tau)} - d - bx(t-\tau) \right) \\ &\quad - nm x(t-\tau)y_2(t-\tau) + nm x(t-\tau)y_2(t-\tau) \\ &\quad - \delta y_1(t) - \beta y_1(t) + \beta y_1(t) - ey_2(t) \\ &= nx(t-\tau) \left(\frac{ax(t-\tau)}{A+x(t-\tau)} - d - bx(t-\tau) \right) \\ &\quad - \delta y_1(t) - ey_2(t).\end{aligned}$$

For a small positive constant $s \leq \min\{\delta, e\}$,

$$\begin{aligned}\dot{V} + sV &= (s - \delta)y_1 + (s - e)y_2 + nx(t-\tau) \\ &\quad \times \left(s + \frac{ax(t-\tau)}{A+x(t-\tau)} - d - bx(t-\tau) \right) \\ &\leq nx(t-\tau) \left(s + \frac{ax(t-\tau)}{A+x(t-\tau)} - d - bx(t-\tau) \right).\end{aligned}$$

By Theorem 2.1, there exist some positive constants B and T , such that $\dot{V}(t) \leq B - sV(t)$ for all $t \geq T$. Thus $\dot{V}(t) + sV(t) \leq B$, that is $V(t) \leq \left(V(0) - \frac{B}{s} \right) e^{-st} + \frac{B}{s}$. Therefore, $V(t)$ is ultimately bounded, that is, each solution $z(t) = (x(t), y_1(t), y_2(t))$ of system (3) is ultimately bounded. The proof is complete.

Obviously, system (3) always has a trivial equilibrium $E_0(0, 0, 0)$. If $a > a_1$, then system (3) has two boundary equilibria $E_1(x_1, 0, 0)$ and $E_2(x_2, 0, 0)$.

Further, if $a > a_2$, then model (3) has a coexistence equilibrium $E^*(x^*, y_1^*, y_2^*)$, where

$$x^* = \frac{e(\beta + \delta)}{nm\beta}, \quad y_1^* = \frac{ey_2^*}{\beta}, \quad y_2^* = \frac{ax^* - (A + x^*)(d + bx^*)}{m(A + x^*)}$$

with $a_2 = Ab + d + \frac{Ad}{x^*} + bx^*$.

Obviously,

$$a_2 = Ab + d + \frac{Ad}{x^*} + bx^* \geq Ab + d + 2\sqrt{Abd} = a_1, \quad (5)$$

where $a_2 = a_1$ if and only if $A = \frac{b(x^*)^2}{d}$.

The above analysis can be summarized in Table I.

TABLE I
EQUILIBRIA OF SYSTEM (3).

$0 < a < a_1$		E_0 exists
$a > a_1$		E_0, E_1, E_2 exist
$A \neq \frac{b(x^*)^2}{d}$	$a_1 < a \leq a_2$	E_0, E_1, E_2 exist
	$a > a_2$	E_0, E_1, E_2, E^* exist
$A = \frac{b(x^*)^2}{d}$	$a > a_1$	E_0, E_1, E_2, E^* exist

Define $A_1 = \frac{a+d-2\sqrt{ad}}{b}$, $A_2 = \frac{(a-d-bx^*)x^*}{bx^*+d}$ and $x_0^* = \frac{\sqrt{d}(\sqrt{a}-\sqrt{d})}{b}$. By simple computation, we have

$$A_1 - A_2 = \frac{[bx^* - \sqrt{d}(\sqrt{a}-\sqrt{d})]^2}{b(bx^*+d)} \geq 0, \text{ for all } 0 < x^* < \frac{a-d}{b},$$

particularly $A_1 = A_2$ if and only if $x^* = x_0^*$. According to the Table I and simple analysis, we get the existence of equilibria of system (3) with the change of the Allee effect (see Table II).

TABLE II
EQUILIBRIA OF SYSTEM (3).

$a \leq d$	$A > 0$		E_0 exists
	$x^* \geq \frac{a-d}{b}$	$0 < A < A_1$	E_0, E_1, E_2 exist
$a > d$	$x_0^* \neq x^* < \frac{a-d}{b}$	$A_2 \leq A < A_1$	E_0, E_1, E_2 exist
		$0 < A < A_2$	E_0, E_1, E_2, E^* exist
	$x^* = x_0^* < \frac{a-d}{b}$	$0 < A < A_1$	E_0, E_1, E_2, E^* exist

Tables I and II show that if the birth rate a is relatively small or the Allee effect A is relatively large, then system (3) only has one trivial equilibrium E_0 , that is the predators and prey will be extinct. If the birth rate a is relatively large or the Allee effect A is relatively small, then system (3) has a positive equilibrium E^* , which guarantees the coexistence of system (3).

III. LOCAL STABILITY AND HOPF BIFURCATION

A. $E_0 = (0, 0, 0)$

Theorem 3.1. *The equilibrium point E_0 of system (3) is locally asymptotically stable.*

Proof. The variational matrix of system (3) at the equilibrium point E_0 is

$$V(E_0) = \begin{bmatrix} \lambda + d & 0 & 0 \\ 0 & \lambda + (\beta + \delta) & 0 \\ 0 & -\beta & \lambda + e \end{bmatrix}.$$

Clearly, the characteristic equation of the equilibrium point E_0 always has three negative real roots: $\lambda = -d$, $\lambda = -(\beta + \delta)$, $\lambda = -e$. The proof is complete.

It follows from Theorem 2.1 that we have

Theorem 3.2. *Assume that $a < a_1$, then $E_0(0, 0, 0)$ of system (3) is globally asymptotically stable.*

When $a < a_1$, E_0 is globally asymptotically stable, which implies that both predators and prey will become extinct when their population densities lie in the attraction region of E_0 . In particular, if the population density of prey becomes low, then both prey and predators will extinct.

B. $E_1 = (x_1, 0, 0)$

Theorem 3.3.

(1) Assume that $A \neq \frac{b(x^*)^2}{d}$ hold.

(i) If $a_1 < a < a_2$, then E_1 of system (3) is a locally asymptotically stable equilibrium point.

(ii) If $a > a_2$, then the equilibrium point E_1 of system (3) is unstable.

(2) If $A = \frac{b(x^*)^2}{d}$ and $a > a_1$, then E_1 of system (3) is unstable.

Proof. The characteristic equation of the equilibrium point E_1 is

$$\left(\lambda - \frac{2Aax_1 + ax_1^2}{(A + x_1)^2} + d + 2bx_1 \right) \times [\lambda^2 + P_1\lambda + P_0 + Q_0e^{-\lambda\tau}] = 0, \quad (6)$$

where $P_1 = \beta + \delta + e$, $P_0 = e(\beta + \delta)$, $Q_0 = -nm\beta x_1$. First, let

$$F_1(\lambda) = \lambda - \frac{2Aax_1 + ax_1^2}{(A + x_1)^2} + d + 2bx_1 = 0. \quad (7)$$

Solving the equation (7), we have

$$\begin{aligned} \lambda_1 &= \frac{2Aax_1 + ax_1^2}{(A + x_1)^2} - d - 2bx_1 \\ &= -\frac{2}{(Ab + \sqrt{Q_1} + a - d)^2} \left[(a - d)Q_1 + (a^2 - (Ab + 2d)a - Abd + d^2)\sqrt{Q_1} \right]. \end{aligned} \quad (8)$$

Obviously, $(a - d)Q_1 > 0$. Further, we consider the following equation

$$f_1(a) = a^2 - (Ab + 2d)a - Abd + d^2. \quad (9)$$

Since

$$f_1(a_1) = f_1(Ab + d + 2\sqrt{Abd}) = 2Abd + 2Ab\sqrt{Abd} > 0,$$

and the symmetry axis of $f_1(a)$ is $\bar{a} = \frac{Ab}{2} + d < a_1$, the inequality $f_1(a) > 0$ holds for all $a > a_1$. Therefore, we have $\lambda_1 < 0$, then Eq.(7) has only one negative real root. Next, we consider the following equation

$$F_2(\lambda) = \lambda^2 + P_1\lambda + P_0 + Q_0e^{-\lambda\tau}. \quad (10)$$

By calculation, $e(\beta + \delta) < nm\beta x_1$ is equivalent to $a > a_2$. Thus, if $a > a_2$, it is easy to show that, for λ real, $F_2(0) = e(\beta + \delta) - nm\beta x_1 < 0$, $\lim_{\lambda \rightarrow +\infty} F_2(\lambda) = +\infty$. Hence, $F_2(\lambda) = 0$ has at least one positive real root. Therefore, if $a > a_2$, the equilibrium E_1 is unstable.

When $\tau = 0$, Eq.(10) turns to

$$\lambda^2 + P_1\lambda + P_0 + Q_0 = 0. \quad (11)$$

If $a_1 < a < a_2$, then $P_0 + Q_0 > 0$. By the Routh-Hurwitz criterion, this implies that the boundary equilibrium E_1 is locally asymptotically stable.

For $\tau > 0$, we investigate the existence of purely imaginary roots of (10). If $i\omega_1$ ($\omega_1 > 0$) is a solution of (10) if and only if ω_1 satisfies

$$-\omega_1^2 + P_1\omega_1i + P_0 + Q_0(\cos(\tau\omega_1) - i\sin(\tau\omega_1)) = 0.$$

Separating the real and imaginary parts, we obtain

$$\begin{aligned} P_1\omega_1 &= Q_0\sin(\tau\omega_1), \\ \omega_1^2 - P_0 &= Q_0\cos(\tau\omega_1), \end{aligned} \quad (12)$$

which implies

$$\omega_1^4 + (P_1^2 - 2P_0)\omega_1^2 + P_0^2 - Q_0^2 = 0. \quad (13)$$

Note that if $a < a_2$, which yields

$$\begin{aligned} P_1^2 - 2P_0 &= (\beta + \delta + e)^2 - 2e(\beta + \delta) \\ &= e^2 + (\beta + \delta)^2 > 0, \\ P_0^2 - Q_0^2 &= [e(\beta + \delta) + nm\beta x_1] \\ &\quad \times [e(\beta + \delta) - nm\beta x_1] > 0. \end{aligned}$$

By Theorem 3.4.1 in [45], if $a_1 < a < a_2$, then all the roots of (10) have negative real parts for all $\tau \geq 0$, this implies that the boundary equilibrium E_1 locally asymptotically stable for all $\tau \geq 0$. The proof is complete.

C. $E_2 = (x_2, 0, 0)$

Theorem 3.4. Let $a > a_1$, then E_2 of system (3) is unstable.

Proof. The characteristic equation of the equilibrium point E_2 is

$$\left(\lambda - \frac{2Aax_2 + ax_2^2}{(A + x_2)^2} + d + 2bx_2 \right) [\lambda^2 + (\beta + \delta + e)\lambda + e(\beta + \delta) - nm\beta x_2 e^{-\lambda\tau}] = 0. \quad (14)$$

Solving the following equation

$$G_1(\lambda) = \lambda - \frac{2Aax_2 + ax_2^2}{(A + x_2)^2} + d + 2bx_2 = 0, \quad (15)$$

we have

$$\begin{aligned} \lambda &= \frac{2Aax_2 + ax_2^2}{(A + x_2)^2} - d - 2bx_2 \\ &= -\frac{2}{(Ab + \sqrt{Q_1} + a - d)^2} \left[(a - d)Q_1 \right. \\ &\quad \left. + (-a^2 + Aba + 2da + Abd - d^2) \sqrt{Q_1} \right] \\ &= -G_2(Q_1) \times \frac{2}{(Ab + \sqrt{Q_1} + a - d)^2}, \end{aligned} \quad (16)$$

where

$$G_2(Q_1) = \frac{(a - d)Q_1 + (-a^2 + Aba + 2da + Abd - d^2) \sqrt{Q_1}}{(Ab + \sqrt{Q_1} + a - d)^2}. \quad (17)$$

Let $\sqrt{Q_1} = t > 0$. It follows from Eq.(17) that

$$g(t) = (a - d)t^2 + (-a^2 + Aba + 2da + Abd - d^2)t. \quad (18)$$

Let $g(t) = 0$, then

$$t_1 = 0 \text{ and } t_2 = \frac{a^2 - (Ab + 2d)a - Abd + d^2}{a - d} = \frac{f_1(a)}{a - d}, \quad (19)$$

where $f_1(a)$ is defined by (9). Similar to the analysis of Theorem 3.3, we obtain $f_1(a) > 0$, thus $t_2 = f_1(a)/(a - d) > 0$.

Hence, if $0 < t < t_2$, then $g(t) < 0$; if $t_2 \leq t$, then $g(t) \geq 0$. That is, if $0 < Q_1 < t_2^2$, then $G_2(Q_1) < 0$; if $t_2^2 \leq Q_1$, then $G_2(Q_1) \geq 0$. Clearly, the inequality $t_2^2 \leq Q_1$ does not hold, due to $Q_1 - t_2^2 = -\frac{4A^2b^2ad}{(a-d)^2} < 0$, which is a contradiction. Thus $\lambda > 0$, which implies Eq.(14) has at least one positive root and E_2 is unstable. The proof of the theorem is complete.

We summarize the results of Theorems 3.3 and 3.4 in Table III. On the other hand, by the analysis of section 2 and Theorems 3.3 and 3.4, we also obtain the local asymptotic stability of equilibria $E_i, i = 0, 1, 2$ (see Table IV).

TABLE III
EQUILIBRIA $E_i, i = 0, 1, 2$ OF SYSTEM (3).

$0 < a < a_1$		E_0 GAS; E_1 and E_2 do not exist
$A = \frac{b(x^*)^2}{d}$	$a > a_1$	E_0 LAS; E_1 unstable; E_2 unstable
$A \neq \frac{b(x^*)^2}{d}$	$a_1 < a < a_2$	E_0 LAS; E_1 LAS; E_2 unstable
	$a > a_2$	E_0 LAS; E_1 unstable; E_2 unstable

TABLE IV
EQUILIBRIA $E_i, i = 0, 1, 2$ OF SYSTEM (3) WITH $a > d$.

$x^* \geq \frac{a-d}{b}$	$A \neq \frac{b(x^*)^2}{d}$	$0 < A < A_1$	$E_{0,1}$ LAS; E_2 unstable
	$A = \frac{b(x^*)^2}{d}$	$0 < A < A_1$	E_0 LAS; $E_{1,2}$ unstable
$x^* < \frac{a-d}{b}$ $x^* \neq x_0^*$	$A \neq \frac{b(x^*)^2}{d}$	$0 < A < A_2$	E_0 LAS; $E_{1,2}$ unstable
		$A_2 < A < A_1$	$E_{0,1}$ LAS; E_2 unstable
$x^* = x_0^*$	$A = \frac{b(x^*)^2}{d}$	$0 < A < A_1$	E_0 LAS; $E_{1,2}$ unstable
		$A > A_1$	E_0 GAS; $E_{1,2}$ inexistent

D. $E^* = (x^*, y_1^*, y_2^*)$

The characteristic equation at E^* is

$$\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 + (q_1\lambda + q_0)e^{-\lambda\tau} = 0, \quad (20)$$

where

$$\begin{aligned} p_2 &= \beta + \delta + e + bx^* - \frac{Aax^*}{(A + x^*)^2}, \\ p_1 &= (\beta + \delta) \left(bx^* - \frac{Aax^*}{(A + x^*)^2} \right) \\ &\quad + e \left(\beta + \delta + bx^* - \frac{Aax^*}{(A + x^*)^2} \right), \\ p_0 &= e(\beta + \delta) \left(bx^* - \frac{Aax^*}{(A + x^*)^2} \right), \\ q_1 &= -e(\beta + \delta), \\ q_0 &= -e(\beta + \delta) \left(d + 2bx^* - \frac{2Aax^* + ax^{*2}}{(A + x^*)^2} \right). \end{aligned}$$

When $\tau = 0$, Eq.(20) turns to

$$\lambda^3 + p_2\lambda^2 + (p_1 + q_1)\lambda + p_0 + q_0 = 0. \quad (21)$$

If $a < a_3 = \frac{b(A+x^*)^2}{A}$, we can obtain

$$\begin{aligned} p_2 &= \beta + \delta + e + bx^* - \frac{Aax^*}{(A + x^*)^2} > 0, \\ p_1 + q_1 &= (\beta + \delta + e)(bx^* - \frac{Aax^*}{(A + x^*)^2}) > 0, \\ p_0 + q_0 &= e(\beta + \delta)my_2^* > 0. \end{aligned}$$

Note that $a_3 - a_2 = \frac{b(A+x^*)^2}{A} - (Ab + d + \frac{Ad}{x^*} + bx^*) = \frac{(A+x^*)(b(x^*)^2 - Ad)}{Ax^*}$. Let $A < \frac{b(x^*)^2}{d}$, then $a_3 > a_2$. Accordingly, we see that if $A < \frac{b(x^*)^2}{d}$, $a_2 < a < a_3$ and $p_2(p_1 + q_1) > (p_0 + q_0)$ hold, by the Routh-Hurwitz criterion, the positive equilibrium E^* is locally asymptotically stable.

For $\tau > 0$. If $i\omega (\omega > 0)$ is a solution of (20) if and only if ω satisfies $-\omega^3 i - p_2\omega^2 + p_1\omega i + p_0 + (q_1\omega i + q_0)(\cos(\tau\omega) - i\sin(\tau\omega)) = 0$. Separating the real and imaginary parts, we have

$$\begin{aligned} \omega^3 - p_1\omega &= q_1\omega \cos(\tau\omega) - q_0 \sin(\tau\omega), \\ p_2\omega^2 - p_0 &= q_1\omega \sin(\tau\omega) + q_0 \cos(\tau\omega), \end{aligned} \quad (22)$$

which implies

$$\omega^6 + (p_2^2 - 2p_1)\omega^4 + (p_1^2 - 2p_0p_2 - q_1^2)\omega^2 + p_0^2 - q_0^2 = 0, \quad (23)$$

where

$$p_2^2 - 2p_1 = (\beta + \delta)^2 + e^2 + \left(bx^* - \frac{Aax^*}{(A+x^*)^2}\right)^2 > 0,$$

$$p_1^2 - 2p_0p_2 - q_1^2 = [(\beta + \delta)^2 + e^2] \times \left(bx^* - \frac{Aax^*}{(A+x^*)^2}\right)^2 > 0,$$

$$p_0^2 - q_0^2 = emy_2^*(\beta + \delta)(p_0 - q_0).$$

$$\text{Denote } a_4 = \frac{(3bx^* + d)(A+x^*)^2}{x^*(3A+x^*)}.$$

Then we have

$$p_0 - q_0 = \frac{e(\beta + \delta)x^*(3A+x^*)^2}{(A+x^*)^2}(a_4 - a).$$

Note that, let $A < \frac{b(x^*)^2}{d}$, we derive that

$$a_3 - a_4 = \frac{b(A+x^*)^2}{A} - \frac{(3bx^* + d)(A+x^*)^2}{x^*(3A+x^*)}$$

$$= \frac{(A+x^*)^2(bx^{*2} - Ad)}{x^*(3A+x^*)}$$

$$> 0, \quad (24)$$

and

$$a_4 - a_2 = \frac{(3bx^* + d)(A+x^*)^2}{x^*(3A+x^*)}$$

$$- (Ab + d + \frac{Ad}{x^*} + bx^*)$$

$$= \frac{2(A+x^*)(bx^{*2} - Ad)}{x^*(3A+x^*)}$$

$$> 0. \quad (25)$$

Therefore, let $A < \frac{b(x^*)^2}{d}$, if $a_2 < a \leq a_4$, then $p_0^2 - q_0^2 > 0$. Thus if $a_2 < a \leq a_4$ and $p_2(p_1 + q_1) > (p_0 + q_0)$, this implies that (20) has no positive real roots. Therefore, by Theorem 3.4.1 in [45], if $A < \frac{b(x^*)^2}{d}$, $a_2 < a \leq a_4$ and $p_2(p_1 + q_1) > (p_0 + q_0)$ are satisfied, then all the roots of (20) have negative real parts for all $\tau \geq 0$. Hence the positive equilibrium $E^* = (x^*, y_1^*, y_2^*)$ is locally asymptotically stable for all $\tau \geq 0$.

If $a_4 < a < a_3$ holds, this implies that $p_0^2 - q_0^2 < 0$, then there exists a unique positive root ω_0 satisfying (23). From (22), we have

$$\cos(\tau\omega_0) = \frac{q_1\omega_0^4 + (p_2q_0 - p_1q_1)\omega_0^2 - p_0q_0}{q_1^2\omega_0^2 + q_0^2}. \quad (26)$$

Denote

$$\tau_{0n} = \frac{1}{\omega_0} \arccos \frac{q_1\omega_0^4 + (p_2q_0 - p_1q_1)\omega_0^2 - p_0q_0}{q_1^2\omega_0^2 + q_0^2}$$

$$+ \frac{2n\pi}{\omega_0}, \quad n = 0, 1, 2, \dots \quad (27)$$

By Theorem 3.4.1 in Kuang [45], we see that if $p_0^2 - q_0^2 < 0$ hold, then E^* remains stable for $\tau < \tau_0 := \tau_{00}$.

We now claim that

$$\left\{ \frac{d(\text{Re}\lambda)}{d\tau} \right\}_{\tau=\tau_0} > 0.$$

This will show that there exists at least one eigenvalue with a positive real part for $\tau > \tau_0$. Moreover, the conditions for the existence of a Hopf bifurcation are then satisfied yielding a periodic solution. To this end, differentiating Eq.(20) with respect to τ , it follows that

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = -\frac{3\lambda^2 + 2p_2\lambda + p_1}{\lambda(\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0)} + \frac{q_1}{\lambda(q_1\lambda + q_0)} - \frac{\tau}{\lambda}.$$

Hence, a direct calculation shows that

$$\text{sgn} \left\{ \frac{d(\text{Re}\lambda)}{d\tau} \right\}_{\lambda=i\omega_0}$$

$$= \text{sgn} \left\{ \text{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1} \right\}_{\lambda=i\omega_0}$$

$$= \text{sgn} \left\{ -\frac{(p_1 - 3\omega_0^2)(\omega_0^2 - p_1) + 2p_2(p_0 - p_2\omega_0^2)}{(\omega_0^3 - p_1\omega_0)^2 + (p_0 - p_2\omega_0^2)^2} \right.$$

$$\left. - \frac{q_1^2}{q_1^2\omega_0^2 + q_0^2} \right\}.$$

We derive from (22) that

$$(\omega_0^3 - p_1\omega_0)^2 + (p_0 - p_2\omega_0^2)^2 = q_1^2\omega_0^2 + q_0^2.$$

$$\text{Therefore, we have } \text{sgn} \left\{ \frac{d(\text{Re}\lambda)}{d\tau} \right\}_{\lambda=i\omega_0} =$$

$$\text{sgn} \left\{ \frac{3\omega_0^4 + 2(p_2^2 - 2p_1)\omega_0^2 + p_1^2 - 2p_0p_2 - q_1^2}{q_1^2\omega_0^2 + q_0^2} \right\} > 0.$$

Thus, the transversal condition holds and a Hopf bifurcation occurs at $\omega = \omega_0$, $\tau = \tau_0$. Now, let us summarize our results as follows:

Theorem 3.5. (1) If $A \geq \frac{b(x^*)^2}{d}$, E^* of system (3) is unstable.

(2) Let $A < \frac{b(x^*)^2}{d}$ and $p_2(p_1 + q_1) > (p_0 + q_0)$ hold.

(i) If $a_2 < a \leq a_4$, then the positive equilibrium E^* of system (3) is locally asymptotically stable for all $\tau \geq 0$.

(ii) If $a_4 < a < a_3$, then there exists a $\tau_0 > 0$ such that E^* is locally asymptotically stable when $\tau \in [0, \tau_0)$. Furthermore, system (3) undergoes a Hopf bifurcation at E^* when $\tau = \tau_0$.

IV. NUMERIC EXAMPLES

In system (3), let $A = 0.1$, $b = 2$, $\beta = 0.5$, and $d = m = n = \delta = e = 1$, then $\frac{b(x^*)^2}{d} = 18$, $a_1 = Ab + d + 2\sqrt{Abd} = 2.0944$, $a_2 = Ab + d + \frac{Ad}{x^*} + bx^* = 7.2333$, $a_3 = \frac{b(A+x^*)^2}{A} = 192.2$ and $a_4 = \frac{(3bx^* + d)(A+x^*)^2}{x^*(3A+x^*)} = 18.4434$.

(i) If $a = 2$, $\tau = 10$. It is easy to show that $a < a_1$. By Theorem 3.2, $E_0(0, 0, 0)$ is globally asymptotically stable. A numerical simulation illustrates this fact (Fig. 1).

(ii) If $a = 6$, $\tau = 10$. It is easy to show that $a_1 < a < a_2$, $A \neq \frac{b(x^*)^2}{d}$, and the predator-extinction equilibrium $E_1 = (2.3683, 0, 0)$. By Theorem 3.3, E_1 is locally asymptotically stable. A numerical simulation illustrates this fact (Figs. 2, 3).

(iii) If $a = 18$, $\tau = 10$, then $A < \frac{b(x^*)^2}{d}$, $p_2(p_1 + q_1) = 107.9200 > 15.6290 = (p_0 + q_0)$, $a_2 < a < a_4$, and the positive equilibrium $E^* = (3, 20.8387, 10.4194)$. By Theorem 3.5, E^* is locally asymptotically stable. A numerical simulation illustrates this fact (Fig. 4).

(iv) If $a = 21$, then $A < \frac{b(x^*)^2}{d}$, $p_2(p_1 + q_1) = 104.8101 > 19.9839 = (p_0 + q_0)$, $a_4 < a < a_3$, and the

positive equilibrium $E^* = (3, 26.6452, 13.3226)$. Further, by calculation, we have $\tau_0 = 2.0387$. By Theorem 3.5, when $\tau < \tau_0$, then E^* is locally asymptotically stable (Fig. 5); When $\tau > \tau_0$, then the positive equilibrium E^* of model (3) is unstable, it yields a periodic solution (Fig. 6).

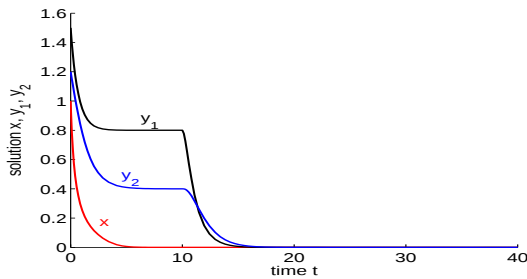


Fig. 1. $a = 2, \tau = 10, E_0$ of System (3) is Globally Asymptotically Stable.

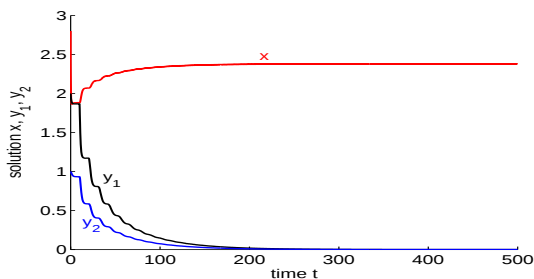


Fig. 2. $a = 6, \tau = 10, E^* = (3, 20.8387, 10.4194)$ of System (3) is Locally Asymptotically Stable.

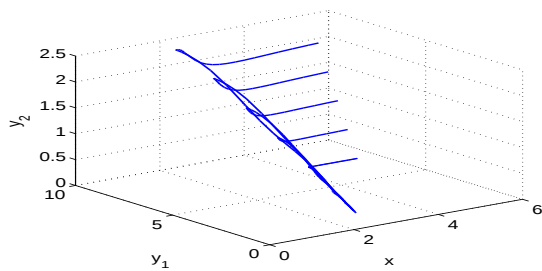


Fig. 3. $a = 6, \tau = 10, E^* = (3, 20.8387, 10.4194)$ of System (3) with Different Initial Values is Locally Asymptotically Stable.

V. CONCLUSION

In this paper, we investigate the stability and Hopf bifurcation of a stage-structured predator-prey model incorporating time delay with prey growth subject to a strong Allee effect. According to the change of the birth rate a , we investigate the stability of the equilibria of system (3), and find birth rate plays an important role in the dynamic behaviors of system (3). Theorem 3.5 shows that the system (3) undergoes a Hopf bifurcation at the coexistence equilibrium. That is, if $\tau < \tau_0$, the positive equilibrium E^* is stable, but it can lose its stability and a Hopf bifurcation occurs at the

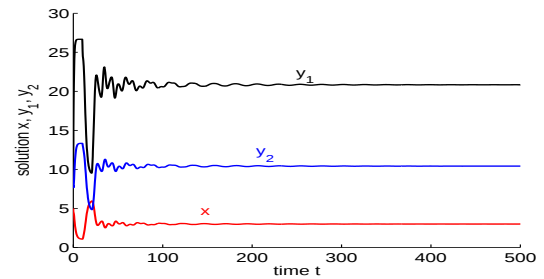


Fig. 4. $a = 18, \tau = 10, E^* = (3, 20.8387, 10.4194)$ of System (3) is Locally Asymptotically Stable.

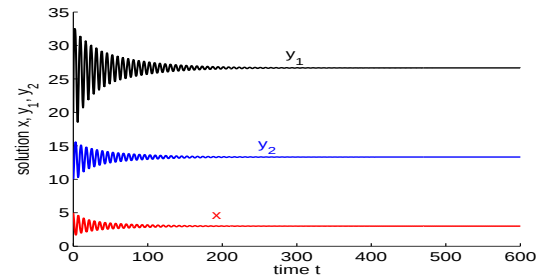


Fig. 5. $\tau = 1.8 < 2.0387 = \tau_0, a = 21, E^* = (3, 26.6452, 13.3226)$ of System (3) is Locally Asymptotically Stable, System (3) Undergoes a Hopf Bifurcation at E^* When τ_0 .

positive equilibrium of system (3) when the delay crosses the critical value τ_0 , further we show periodic solutions. When $\tau = 0$, then system (3) is reduced to nondelay system, and we give the stability of nondelay system in the Table V.

TABLE V
EQUILIBRIA $E_i, i = 0, 1, 2$ AND E^* OF SYSTEM (3) WITH $\tau = 0$.

$0 < a < a_1$		E_0 GAS; $E_{1,2}, E^*$ inexistent
$A > \frac{b(x^*)^2}{d}$	$a_1 < a < a_2$	$E_{0,1}$ LAS; E_2 unstable; E^* inexistent
	$a > a_2$	E_0 LAS; $E_{1,2}, E^*$ unstable
$A = \frac{b(x^*)^2}{d}$	$a > a_1$	E_0 LAS; $E_{1,2}, E^*$ unstable
$A < \frac{b(x^*)^2}{d},$ $p_2(p_1 + q_1) > p_0 + q_0$	$a_1 < a < a_2$	$E_{0,1}$ LAS; E_2 unstable; E^* inexistent
	$a_2 < a < a_3$	E_0 LAS; $E_{1,2}$ unstable; E^* LAS
	$a > a_3$	E_0 LAS; $E_{1,2}, E^*$ unstable

Table VI presents that the results of Theorems 3.1-3.5. When the initial density of prey and predators are not low, we show the results of Table VI in Fig. 7, Fig. 8 and Fig. 9. with the same initial values. With the increasing of birth rate, the stability of equilibriums of system (3) will be changed. For example, if the birth rate is large enough, the stability of prey and predators will be destroyed, and the system becomes oscillation.

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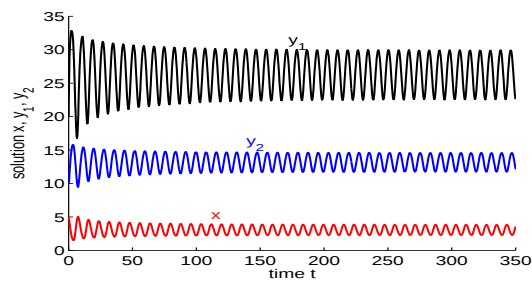


Fig. 6. $\tau = 2.2 > 2.0387 = \tau_0$, $a = 21$, $E^* = (3, 26.6452, 13.3226)$ of System (3) is Unstable, System (3) Undergoes a Hopf Bifurcation at E^* When τ_0 .

TABLE VI
SYSTEM (3) WITH $p_2(p_1 + q_1) > p_0 + q_0$ AND $A < \frac{b(x^*)^2}{d}$.

Birth rate a	Equilibria
$0 < a < a_1$	E_0 GAS; $E_{1,2}$, E^* does not exist
$a_1 < a < a_2$	$E_{0,1}$ LAS; E_2 unstable; E^* does not exist
$a_2 < a \leq a_4$	E_0 LAS; $E_{1,2}$ unstable; E^* LAS
$a_4 < a < a_3$	E_0 LAS; $E_{1,2}$ unstable; E^* exists a Hopf bifurcation

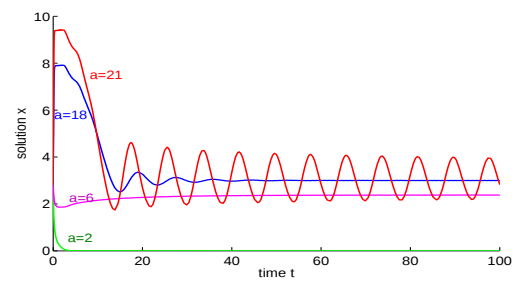


Fig. 7. Dynamic Behavior of the Prey of System (3) with $A = 0.1$, $b = 2$, $\beta = 0.5$, $d = m = n = \delta = e = 1$ and $\tau = 2.2$, Respectively.

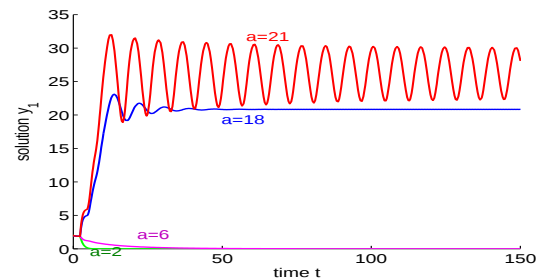


Fig. 8. Dynamic Behavior of the Immature Predator of System (3) with $A = 0.1$, $b = 2$, $\beta = 0.5$, $d = m = n = \delta = e = 1$ and $\tau = 2.2$, Respectively.

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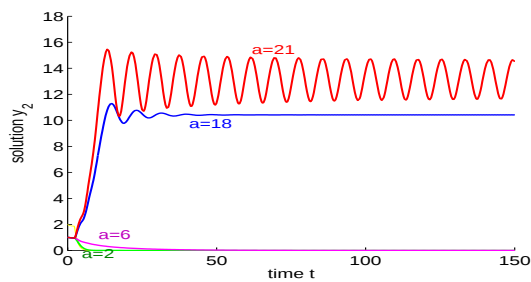


Fig. 9. Dynamic Behavior of the Mature Predator of System (3) with $A = 0.1$, $b = 2$, $\beta = 0.5$, $d = m = n = \delta = e = 1$ and $\tau = 2.2$, Respectively.

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