Homotopy Perturbation Method Approximate Analytical Solution of Fuzzy Partial Differential Equation

Sarmad A. Altaie, Ali F. Jameel, Azizan Saaban

***Abstract*—This work employs the Homotopy Perturbation Method (HPM) to develop an approximate analytical solution for a Fuzzy Partial Differential Equations (FPDE). The method is applied to calculate the solution of fuzzy reaction-diffusion equation (FRDE) by using the properties of fuzzy set theory. Examples are given to verify results compared with the exact solution of the linear equation and with residual error of the nonlinear equation of the given problems and to illustrate the efficiency and the capability of the proposed method.**

***Index Terms*— Fuzzy Partial Differential Equations, Fuzzy Reaction-Diffusion equation, Approximate Analytical Solution, Homotopy Perturbation Method**

I. Introduction

Fuzzy differential equations (FDEs) are a significant part of the fuzzy analytic theory, and a valuable instrument to describe a dynamical phenomenon when the information about it is vague and its nature is under uncertainty [1,2]. They arise in the modeling of the real-life problems [3,4] when there is impreciseness, for example, population models [5,6], medicine [7] and physics [8] and control design [9].

The fuzzy partial differential equations (FPDEs) attracted a great deal of attention among scientists and engineers, because of its frequent involvement in the modeling of numerous industrialized applications, such as heat and mass transfer, electromagnetic fields, static and dynamic of structures, meteorology, biomechanics and many others. The numerical, and approximate analytical solution of FPDEs have been tackled by numerous authors like [10,11,12,13]. Yet the field still lacking for further accurate and capable solutions, since the exact solutions are rarely available especially for the nonlinear equations.

He [14] developed the homotopy perturbation method (HPM) and used the homotopy in topology for non-linear problems [15]. In HPM the approximate solution is obtained in the form of a series which converges rapidly to the exact solution. The main advantage of HPM is the flexibility to give approximate and exact solution to both linear and nonlinear problems without any need for discretization and linearization as in numerical methods [16]. In this work, we developed a method based on HPM for acquiring an approximate-analytical solution of the FRDE. As far as we know, obtaining a solution to a FRDE by means of HPM based method is the first to be developed.

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II. Development of HPM for solving FPDE

The HPM was applied to derive an approximate-analytical solution of linear and nonlinear time dependent partial differential equations [17,18], and these works motivated us to develop our proposed method. The methodology for the development of HPM for solving PDEs in fuzzy environment is given as follows. Let the succeeding FPDE,

$$\mathcal{L}\big(\tilde{u}(s;r)\big)+\mathcal{N}\big(\tilde{u}(s;r)\big)+\tilde{\Lambda}(s;r)=0\ ,s\in\Omega \tag{1}$$
$$\mathcal{B}\left(\tilde{u}(s;r),\frac{\partial\tilde{u}(s;r)}{\partial s}\right)=0 \qquad s\in\Gamma$$

where $\mathcal{L}$ is a linear operator, $\mathcal{N}$ is a nonlinear operator, $\tilde{\Lambda}(s;r)$ is a known fuzzy function, $\tilde{u}(s;r)$ is an unknown fuzzy function, and $\mathcal{B}$ is a boundary operator and Γ is the boundary of the domain Ω.

Now, a homotopy $\tilde{\omega}(s;r;p):\Omega\times[0,1]\to\mathbb{R}$ can be constructed using the homotopy technique, for an embedding parameter $p\in[0,1]$ that satisfies,

$$\mathcal{H}(\tilde{\omega},p)=(1-p)[\mathcal{L}(\tilde{\omega})-\mathcal{L}(\tilde{u}_a)]+p\big[\mathcal{L}(\tilde{\omega})+\mathcal{N}(\tilde{\omega})+\tilde{\Lambda}(s)\big]=0 \tag{2}$$
or
$$\mathcal{H}(\tilde{\omega},p)=\mathcal{L}(\tilde{\omega})-\mathcal{L}(\tilde{u}_a)+p\mathcal{L}(\tilde{u}_a)+p\big[\mathcal{N}(\tilde{\omega})-\tilde{\Lambda}(s)\big]=0 \tag{3}$$

where $\tilde{u}_a$ is an initial approximation of (1), which complies with the boundary conditions. Clearly, (2) and (3) will give,

$$\mathcal{H}(\tilde{\omega},0)=\mathcal{L}(\tilde{\omega})-\mathcal{L}(\tilde{u}_a)=0 \tag{4}$$
$$\mathcal{H}(\tilde{\omega},1)=\mathcal{L}(\tilde{\omega})+\mathcal{N}(\tilde{\omega})-\tilde{\Lambda}(s)=0 \tag{5}$$

In topology, the altering procedure of p from 0 to 1, is only the deformation of $\tilde{\omega}$ from the initial $\tilde{u}_a$ to the solution $\tilde{u}$. Furthermore, $\mathcal{L}(\tilde{\omega})-\mathcal{L}(\tilde{u}_a)$, $\mathcal{L}(\tilde{\omega})+\mathcal{N}(\tilde{\omega})-\tilde{\Lambda}(s)$ are called homotopic. Hence, the fundamental hypothesis is a solution for (2) and (3) can be expressed in

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power series of p,

$$\widetilde{\omega} = \sum_{i=0}^{\infty} p^i \widetilde{\omega}_i \tag{6}$$

Therefore, the approximate solution of (1) is obtained as,

$$\tilde{u} = \lim_{p\to 1} \widetilde{\omega} = \sum_{i=0}^{\infty} \widetilde{\omega}_i \tag{7}$$

III. Fuzzy Reaction-Diffusion Equation Analysis

According to [19,20], a general model for the FRDE will be specified using the properties of the fuzzy set theory. Suppose that $0 < x < l$, $0 < t \le T$, then

$$\frac{\partial}{\partial t}\tilde{u}(x,t) = \tilde{\mathrm{D}}(x)\frac{\partial^2}{\partial x^2}\tilde{u}(x,t) + \tilde{\mathrm{R}}\big(\tilde{u}(x,t)\big) + \tilde{\Lambda}(x,t) \tag{8}$$
$$\tilde{u}(x,0) = \tilde{\varphi}(x)$$

In (8), $\tilde{u}(x,t)$ represents the concentration variables, which is a crisp variables fuzzy function [21]. Furthermore, $\frac{\partial}{\partial t}\tilde{u}(x,t)$, $\frac{\partial^2}{\partial x^2}\tilde{u}(x,t)$ are fuzzy partial derivatives in the Hukuhara sense [1,22]. Also, $\widetilde{D}(x) = \tilde{\gamma}_1 D(x)$ is a fuzzy function of crisp variables represent the diffusion coefficient [21], $\tilde{\mathrm{R}}\big(\tilde{u}(x,t)\big)$ a nonlinear source term describes a local reaction kinetics, $\tilde{\Lambda}(x,t) = \tilde{\gamma}_2 \Lambda(x,t)$ is a fuzzy function of crisp variables as a nonhomogeneous term. Moreover, $\tilde{u}(x,0)$ is a fuzzy environment initial condition equals to a crisp variables fuzzy function $\tilde{\varphi}(x) = \tilde{\gamma}_3\varphi(x)$.

Finally, $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, $\tilde{\gamma}_3$ are convex fuzzy numbers [23,24], and $D(x)$, $\Lambda(x,t)$, $\varphi(x)$ are crisp functions. The defuzzification of this model for all the values of *r* between *0* and *1*, is acquired as the following,

$$[\tilde{u}(x,t)]_r = [\underline{u}(x,t;r), \overline{u}(x,t;r)],$$
$$\left[\frac{\partial}{\partial t}\tilde{u}(x,t)\right]_r = \left[\frac{\partial}{\partial t}\underline{u}(x,t;r), \frac{\partial}{\partial t}\overline{u}(x,t;r)\right],$$
$$\left[\frac{\partial^2}{\partial x^2}\tilde{u}(x,t)\right]_r = \left[\frac{\partial^2}{\partial x^2}\underline{u}(x,t;r), \frac{\partial^2}{\partial x^2}\overline{u}(x,t;r)\right],$$
$$\big[\widetilde{D}(x)\big]_r = [\underline{D}(x;r), \overline{D}(x;r)], \tilde{\gamma}_1 = \left[\underline{\gamma}_1(r), \overline{\gamma}_1(\alpha)\right],$$
$$\big[\tilde{\mathrm{R}}\big(\tilde{u}(x,t)\big)\big]_r = \left[\underline{R}\left(\underline{u}(x,t;r)\right), \overline{R}\big(\overline{u}(x,t;r)\big)\right],$$
$$\big[\tilde{\Lambda}(x,t)\big]_r = [\underline{\Lambda}(x,t;r), \overline{\Lambda}(x,t;r)], \tilde{\gamma}_2 = \left[\underline{\gamma}_2(r), \overline{\gamma}_2(r)\right],$$
$$[\tilde{u}(x,0)]_r = [\underline{u}(x,0;r), \overline{u}(x,0;r)],$$
$$[\tilde{\varphi}(x)]_r = \left[\underline{\varphi}(x;r), \overline{\varphi}(x;r)\right], \tilde{\gamma}_3 = \left[\underline{\gamma}_3(r), \overline{\gamma}_3(r)\right]$$

Now, by using the extension principle [25,26], the membership function of (8) is defined as follows,

$$\underline{u}(x,t;r) = min\{\tilde{u}\big(t,\tilde{\mu}(r)\big) | \tilde{\mu}(r) \in \tilde{u}(x,t;r)\}$$
$$\overline{u}(x,t;r) = max\{\tilde{u}\big(t,\tilde{\mu}(r)\big) | \tilde{\mu}(r) \in \tilde{u}(x,t;r)\}$$

Hence, for $0 < x < l$, $0 < t < T$ and all the values of *r* between *0* and *1*, (8) can be rewritten as,

$$\frac{\partial}{\partial t}\underline{u}(x,t;r) - \underline{D}(x;r)\frac{\partial^2}{\partial x^2}\underline{u}(x,t;r) - \underline{R}\left(\underline{u}(x,t;r)\right) - \underline{\Lambda}(x,t;r) = 0$$
$$\underline{u}(x,0;r) = \underline{\varphi}(x;r)$$
$$\frac{\partial}{\partial t}\overline{u}(x,t;r) - \overline{D}(x;r)\frac{\partial^2}{\partial x^2}\overline{u}(x,t;r) - \overline{R}\big(\overline{u}(x,t;r)\big) - \overline{\Lambda}(x,t;r) = 0$$
$$\overline{u}(x,0;r) = \overline{\varphi}(x;r)$$

hence,

$$\frac{\partial}{\partial t}\underline{u}(x,t;r) - \underline{\gamma}_1(r)D(x)\frac{\partial^2}{\partial x^2}\underline{u}(x,t;r) - \underline{R}\left(\underline{u}(x,t;r)\right) - \underline{\gamma}_2(r)\Lambda(x,t) = 0 \tag{9}$$
$$\underline{u}(x,0;r) = \underline{\gamma}_3(r)\varphi(x)$$
$$\frac{\partial}{\partial t}\overline{u}(x,t;r) - \overline{\gamma}_1(r)D(x)\frac{\partial^2}{\partial x^2}\overline{u}(x,t;r) - \overline{R}\big(\overline{u}(x,t;r)\big) - \overline{\gamma}_2(r)\Lambda(x,t) = 0 \tag{10}$$
$$\overline{u}(x,0;r) = \overline{\gamma}_3(r)\varphi(x)$$

IV. Application of Developed HPM to FRDE

Following the similar approaches as given in [17,18], we will discuss the application of the developed HPM in section 2 to FRDE. We use (9) and (10) from the analysis in section 3 similar to the work in [11] by constructing the family of equations,

$$(1-p)\left[\frac{\partial}{\partial t}\underline{\omega}(x,t;r) - \frac{\partial}{\partial t}\underline{u}_r(x,t;r)\right] + p\left[\frac{\partial}{\partial t}\underline{\omega}(x,t;r) - \underline{\gamma}_1(r)D(x)\frac{\partial^2}{\partial x^2}\underline{\omega}(x,t;r) - \underline{R}\left(\underline{\omega}(x,t;r)\right) - \underline{\gamma}_2(r)\Lambda(x,t)\right] = 0 \tag{11}$$
$$(1-p)\left[\frac{\partial}{\partial t}\overline{\omega}(x,t;r) - \frac{\partial}{\partial t}\overline{u}_r(x,t;r)\right] + p\left[\frac{\partial}{\partial t}\overline{\omega}(x,t;r) - \overline{\gamma}_1(r)D(x)\frac{\partial^2}{\partial x^2}\overline{\omega}(x,t;r) - \overline{R}\big(\overline{\omega}(x,t;r)\big) - \overline{\gamma}_2(r)\Lambda(x,t)\right] = 0 \tag{12}$$

The solution of (11) and (12) can be expressed as a power series in p, like the following,

$$\underline{\omega}(x,t;r) = \sum_{i=0}^{\infty} p^i \underline{\omega_i}(x,t;r) \tag{13}$$
$$\overline{\omega}(x,t;r) = \sum_{i=0}^{\infty} p^i \overline{\omega}_i(x,t;r) \tag{14}$$

The substitution of (13) and (14) into (11) and (12) yields,

$$\frac{\partial}{\partial t}\sum_{i=0}^{\infty} p^i \underline{\omega_i}(x,t;r) - \frac{\partial}{\partial t}\underline{u}_r(x,t;r) = p\Big[-\frac{\partial}{\partial t}\underline{u}_r(x,t;r) + \underline{\gamma}_1(r)D(x)\frac{\partial^2}{\partial x^2}\sum_{i=0}^{\infty} p^i \underline{\omega_i}(x,t;r) + \underline{R}\big(\sum_{i=0}^{\infty} p^i \underline{\omega_i}(x,t;r)\big) + \underline{\gamma}_2(r)\Lambda(x,t)\Big] \tag{15}$$
$$\frac{\partial}{\partial t}\sum_{i=0}^{\infty} p^i \overline{\omega}_i(x,t;r) - \frac{\partial}{\partial t}\overline{u}_a(x,t;r) = p\Big[-\frac{\partial}{\partial t}\overline{u}_r(x,t;r) + \overline{\gamma}_1(r)D(x)\frac{\partial^2}{\partial x^2}\sum_{i=0}^{\infty} p^i \overline{\omega}_i(x,t;r) + \overline{R}(\sum_{i=0}^{\infty} p^i \overline{\omega}_i(x,t;r)) + \overline{\gamma}_2(r)\Lambda(x,t)\Big] \tag{16}$$

The initial approximation of (15) and (16) that satisfies the initial conditions is given as,

$$\underline{u}_a(x,t;r) = \underline{\gamma}_3(r)\varphi(x) \tag{17}$$
$$\overline{u}_a(x,t;r) = \overline{\gamma}_3(r)\varphi(x) \tag{18}$$

Now, both sides with similar powers of p are compared to obtain the following for the lower band solution,

$$\frac{\partial}{\partial t}\underline{\omega}_0(x,t;r) = \frac{\partial}{\partial t}\underline{u}_a(x,t;r)$$
$$\frac{\partial}{\partial t}\underline{\omega}_1(x,t;r) = -\frac{\partial}{\partial t}\underline{u}_0(x,t;r) + \underline{\gamma}_1(r)D(x)\frac{\partial^2}{\partial x^2}\underline{\omega}_0(x,t;r) + \underline{R}\left(\underline{\omega}_0(x,t;r)\right) + \underline{\gamma}_2(\alpha)\Lambda(x,t)$$
$$\frac{\partial}{\partial t}\underline{\omega}_2(x,t;r) = \underline{\gamma}_1(r)D(x)\frac{\partial^2}{\partial x^2}\underline{\omega}_1(x,t;r) + \underline{R}\left(\underline{\omega}_1(x,t;r)\right)$$

$$\frac{\partial}{\partial t}\underline{\omega}_3(x,t;r) = \underline{\gamma}_1(r)D(x)\frac{\partial^2}{\partial x^2}\underline{\omega}_2(x,t;r) + \underline{R}\left(\underline{\omega}_2(x,t;r)\right)$$

and so on, and so forth. Similarly, for the upper bound solution,

$$\frac{\partial}{\partial t}\overline{\omega}_0(x,t;r) = \frac{\partial}{\partial t}\overline{u}_r(x,t;r)$$
$$\frac{\partial}{\partial t}\overline{\omega}_1(x,t;r) = -\frac{\partial}{\partial t}\overline{u}_r(x,t;r) +$$
$$\overline{\gamma}_1(r)D(x)\frac{\partial^2}{\partial x^2}\overline{\omega}_0(x,t;r) + \overline{R}\left(\overline{\omega}_0(x,t;r)\right) + \overline{\gamma}_2(\alpha)\Lambda(x,t)$$
$$\frac{\partial}{\partial t}\overline{\omega}_2(x,t;r) = \overline{\gamma}_1(r)D(x)\frac{\partial^2}{\partial x^2}\overline{\omega}_1(x,t;r) + \overline{R}\left(\overline{\omega}_1(x,t;r)\right)$$
$$\frac{\partial}{\partial t}\overline{\omega}_3(x,t;r) = \overline{\gamma}_1(r)D(x)\frac{\partial^2}{\partial x^2}\overline{\omega}_2(x,t;\alpha) + \overline{R}\left(\overline{\omega}_2(x,t;r)\right)$$

and so on, and so forth. For simplicity, $\widetilde{\omega}_0(x,t;r) = \widetilde{u}_0(x,t;r) = \widetilde{u}_0(x,0;r)$. thus, the following recurrent relation is obtained,

$$\widetilde{\omega}_1(x,t;r) = \int_0^T\Big[-\frac{\partial}{\partial t}\widetilde{u}_0(x,t;r) +$$
$$\widetilde{\gamma}_1(r)D(x)\frac{\partial^2}{\partial x^2}\widetilde{\omega}_0(x,t;r) + \underline{R}\left(\widetilde{\omega}_0(x,t;r)\right) +$$
$$\widetilde{\gamma}_2(r)\Lambda(x,t)\Big]\,dt$$
$$\widetilde{\omega}_2(x,t;r) = \int_0^T\Big[-\frac{\partial}{\partial t}\widetilde{u}_1(x,t;r) +$$
$$\widetilde{\gamma}_1(r)D(x)\frac{\partial^2}{\partial x^2}\widetilde{\omega}_1(x,t;r) + \underline{R}\left(\widetilde{\omega}_1(x,t;r)\right)\Big]\,dt$$
$$\widetilde{\omega}_3(x,t;r) = \int_0^T\Big[-\frac{\partial}{\partial t}\widetilde{u}_2(x,t;r) +$$
$$\widetilde{\gamma}_1(r)D(x)\frac{\partial^2}{\partial x^2}\widetilde{\omega}_2(x,t;r) + \underline{R}\left(\widetilde{\omega}_2(x,t;r)\right)\Big]\,dt$$
$$\widetilde{\omega}_n(x,t;r) = \int_0^T\Big[-\frac{\partial}{\partial t}\widetilde{u}_{n-1}(x,t;r) +$$
$$\widetilde{\gamma}_1(r)D(x)\frac{\partial^2}{\partial x^2}\widetilde{\omega}_{n-1}(x,t;r) + \underline{R}\left(\widetilde{\omega}_{n-1}(x,t;r)\right)\Big]\,dt,$$

where $n \geq 2$. The approximate solution of (8) can be obtained as,

$$\widetilde{u} = \lim_{n\to\infty}\widetilde{\omega}_n(x,t;r) \tag{19}$$

V. ILLUSTRATION

Case 1. Consider the linear Cauchy FRDE, where $0 < x < 0.4, 0 < t < 0.6$,

$$\frac{\partial \widetilde{u}(t,x)}{\partial t} = \frac{\partial^2 \widetilde{u}(t,x)}{\partial x^2} + \widetilde{u}(t,x) \tag{20}$$
$$\widetilde{u}(0,x) = [r-1, 1-r]x^3.$$

The exact solution of (20) has been obtained by help of Wolfram Mathematica 10 as,

$$\widetilde{u}(t,x;r) = e^t[r-1, 1-r]x(6t+x^2) \tag{21}$$

The initial approximation of (20) are specified by

$$\begin{cases} \underline{U}_0(t,x;r) = (r-1)x^3 \\ \overline{U}_0(t,x;r) = (1-r)x^3 \end{cases} \tag{22}$$

According to HPM as in section 4 we have

$$\begin{cases} \underline{U}_1(x,t;r) = \int_0^t\left[\frac{\partial^2 \underline{U}_0(t,x;r)}{\partial x^2} + \underline{U}_0(t,x;r)\right] \\ \underline{U}_2(x,t;r) = \int_0^t\left[\frac{\partial^2 \underline{U}_1(t,x;r)}{\partial x^2} + \underline{U}_1(t,x;r)\right] \\ \quad\vdots \\ \underline{U}_k(x,t;r) = \int_0^t\left[\frac{\partial^2 \underline{U}_{k-1}(t,x;r)}{\partial x^2} + \underline{U}_{k-1}(t,x;r)\right] \end{cases} \tag{23}$$

$$\begin{cases} \overline{U}_1(x,t;r) = \int_0^t\left[\frac{\partial^2 \overline{U}_0(t,x;r)}{\partial x^2} + \overline{U}_0(t,x;r)\right] \\ \overline{U}_2(x,t;r) = \int_0^t\left[\frac{\partial^2 \overline{U}_1(t,x;r)}{\partial x^2} + \overline{U}_1(t,x;r)\right] \\ \quad\vdots \\ \overline{U}_k(x,t;r) = \int_0^t\left[\frac{\partial^2 \overline{U}_{k-1}(t,x;r)}{\partial x^2} + \overline{U}_{k-1}(t,x;r)\right] \end{cases} \tag{24}$$

Additionally, absolute error of the approximate-analytical solution of (20) is given by,

$$[\widetilde{E}]_r = \left|\widetilde{U}(t,x;r) - \widetilde{u}(t,x;r)\right| \tag{25}$$

TABLE I
EQUATION (20) 10TH-ORDER HPM LOWER SOLUTION FOR $0 \leq r \leq 1$, $x = 0.4$, AND $t = 0.6$

r	$\underline{U}$ HPM	$\underline{u}$ Exact	$\underline{E}$
0	2.74046667	2.74046668	2.54331×10^{-9}
0.2	2.19237334	2.19237334	2.03464×10^{-9}
0.4	1.644280003	1.644280005	1.52598×10^{-9}
0.6	1.09618667	1.09618667	1.01732×10^{-9}
0.8	0.54809333	0.54809334	5.08662×10^{-10}
1	2.7767×10^{-16}	0	2.77664×10^{-16}

TABLE II
EQUATION (20) 10TH-ORDER HPM UPPER SOLUTION FOR $0 \leq r \leq 1$, $x = 0.4$, AND $t = 0.6$

r	$\overline{U}$ HPM	$\overline{u}$ Exact	$\overline{E}$
0	-2.74046667	-2.74046667	2.54331×10^{-9}
0.2	-2.19237334	-2.19237334	2.03465×10^{-9}
0.4	-1.64428	-1.64428	1.52599×10^{-9}
0.6	-1.09618667	-1.09618667	1.01732×10^{-9}
0.8	-0.54809333	-0.54809333	5.08662×10^{-10}
1	2.77665×10^{-16}	0	2.77665×10^{-16}

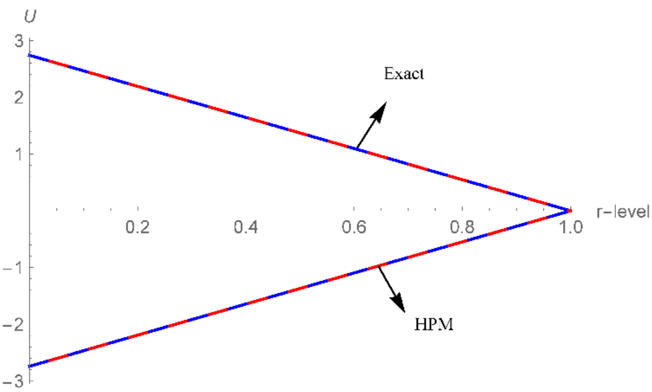

Fig. 1. Equation (20) 10th-order HPM solution at $x = 0.4$, $t = 0.6$, and $0 \leq r \leq 1$

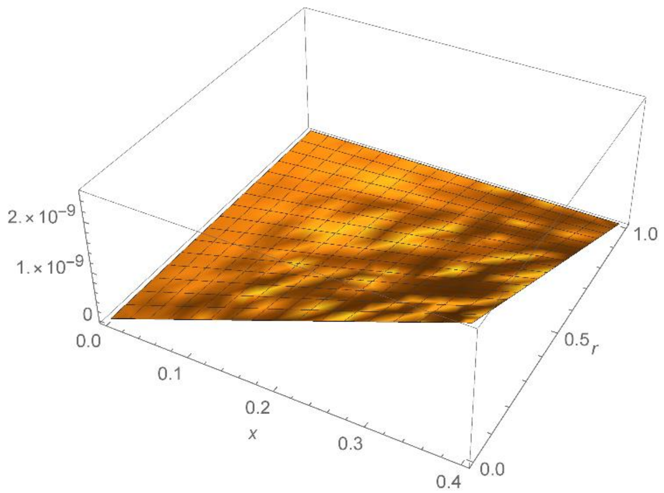

Fig. 2. 10th-order HPM solution of (20) with lower bound accuracy at $t=0.6$, $0 \le x \le 0.4$, and $0 \le r \le 1$

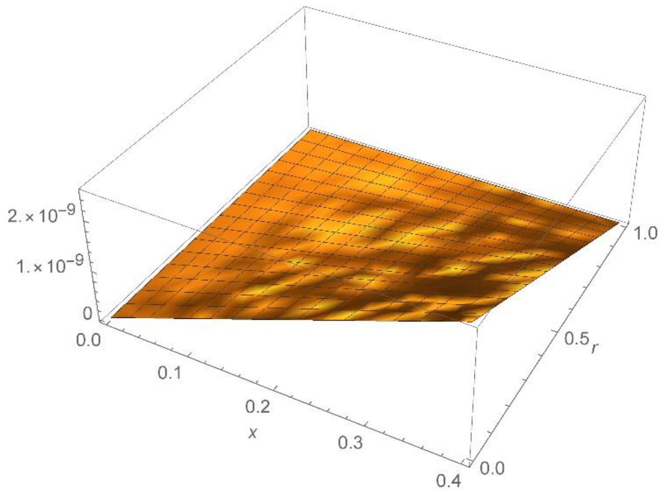

Fig. 3. 10th-order HPM solution of (20) with upper bound accuracy at $t=0.6$, $0 \le x \le 0.4$, and $0 \le r \le 1$

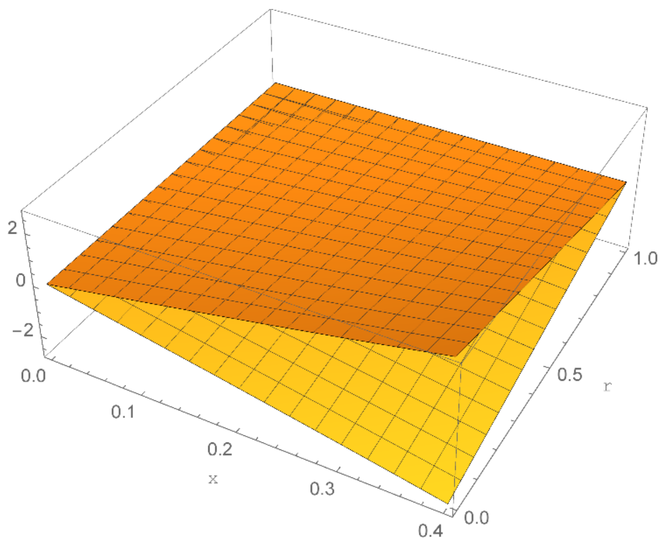

Fig. 4. 10-order HPM solution of (20) at $t=0.6$, $0 \le x \le 0.4$, and $0 \le r \le 1$

From tables I, II and Fig. 1 to 3 one can conclude that the 10-order HPM solution of (20) satisfies the convex triangular fuzzy number properties [24,27] for the values of $0 \le t \le 1$ and $0 \le r \le 1$.

Case 2. Consider the nonlinear Cauchy FRDE, where $x > 0, t > 0$,

$$\frac{\partial \tilde{u}(t,x)}{\partial t} = \frac{\partial^2 \tilde{u}(t,x)}{\partial x^2} + \tilde{u}(t,x)(1-\tilde{u}(t,x)) \tag{26}$$
$$\tilde{u}(0,x) = [r-1, 1-r]x^2.$$

The initial approximation of (26) are specified by

$$\begin{cases} \underline{U}_0(t,x;r) = (r-1)x^2 \\ \overline{U}_0(t,x;r) = (1-r)x^2 \end{cases} \tag{27}$$

According to HPM section IV we have

$$\begin{cases} \underline{U}_1(x,t;r) = \int_0^t \begin{bmatrix} \frac{\partial^2 \underline{U}_0(t,x;r)}{\partial x^2} + \underline{U}_0(t,x;r) \\ -\underline{U}_0(t,x;r)^2 \end{bmatrix} \\ \underline{U}_2(x,t;r) = \int_0^t \begin{bmatrix} \frac{\partial^2 \underline{U}_1(t,x;r)}{\partial x^2} + \underline{U}_1(t,x;r) \\ -2\underline{U}_1(t,x;r)\underline{U}_0(t,x;r) \end{bmatrix} \\ \vdots \\ \underline{U}_k(x,t;r) = \int_0^t \begin{bmatrix} \frac{\partial^2 \underline{U}_{k-1}(t,x;r)}{\partial x^2} + \underline{U}_{k-1}(t,x;r) \\ -\sum_{k-1=0}^{n-1} \underline{U}_{k-1}(t,x;r)\, \underline{U}_{n-k-2}(t,x;r) \end{bmatrix} \end{cases} \tag{28}$$

$$\begin{cases} \overline{U}_1(x,t;r) = \int_0^t \begin{bmatrix} \frac{\partial^2 \overline{U}_0(t,x;r)}{\partial x^2} + \overline{U}_0(t,x;r) \\ -\overline{U}_0(t,x;r)^2 \end{bmatrix} \\ \overline{U}_2(x,t;r) = \int_0^t \begin{bmatrix} \frac{\partial^2 \overline{U}_1(t,x;r)}{\partial x^2} + \overline{U}_1(t,x;r) \\ -2\overline{U}_1(t,x;r)\overline{U}_0(t,x;r) \end{bmatrix} \\ \vdots \\ \overline{U}_k(x,t;r) = \int_0^t \begin{bmatrix} \frac{\partial^2 \overline{U}_{k-1}(t,x;r)}{\partial x^2} + \overline{U}_{k-1}(t,x;r) \\ -\sum_{k-1=0}^{n-1} \overline{U}_{k-1}(t,x;r)\, \overline{U}_{n-k-2}(t,x;r) \end{bmatrix} \end{cases} \tag{29}$$

Since the exact solution cannot be found from (26) [28], we define the residual error [29,30] to analyze the accuracy of the approximate solution approximate-analytical such that

$$\tilde{E}(x,t;r) = \\ = \left| \frac{\partial \tilde{U}_k(t,x;r)}{\partial t} - \frac{\partial^2 \tilde{U}_k(t,x;r)}{\partial x^2} - \tilde{U}_k(t,x;r) + \left[\tilde{U}_k(t,x;r)\right]^2 \right| \tag{30}$$

TABLE III
EQUATION (20) 15TH-ORDER HPM OF (26) WITH LOWER SOLUTION FOR $0 \le r \le 1, x = 0.1$, AND $t = 0.1$

r	$\underline{U}$ HPM	$\underline{E}$
0	-0.2411520	$6.514161121629058 \times 10^{-8}$
0.2	-0.1914090	$1.077814432148827 \times 10^{-8}$
0.4	-0.1424460	$1.084306144871760 \times 10^{-9}$
0.6	-0.0942395	$4.46025438805000 \times 10^{-11}$
0.8	-0.0467649	$2.20198859146592 \times 10^{-13}$
1	-7.56609×10^{-19}	$1.11093988383018 \times 10^{-19}$

TABLE IV
15TH-ORDER HPM OF (26) WITH LOWER SOLUTION FOR $0 \le r \le 1, x = 0.1$, AND $t = 0.1$

r	$\overline{U}$ HPM	$\underline{E}$
0	0.22388700	$2.686822725417315 \times 10^{-8}$
0.2	0.18037200	$4.694926836190660 \times 10^{-9}$
0.4	0.13624300	$4.738556769190438 \times 10^{-10}$
0.6	0.09148440	$1.726832565829283 \times 10^{-11}$
0.8	-0.0467649	$4.450606549966096 \times 10^{-14}$
1	-7.56609×10^{-16}	$1.110939883830187 \times 10^{-19}$

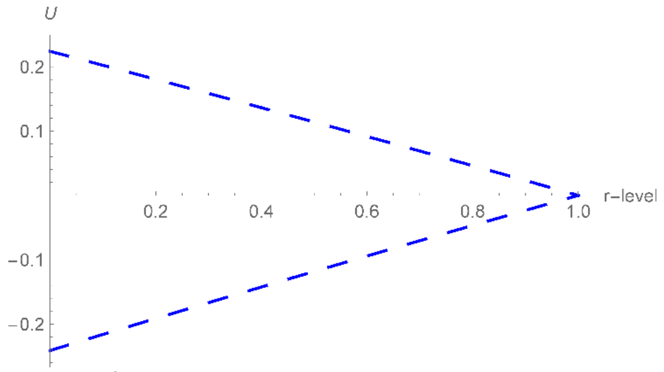

Fig. 5. 15th-order HPM solution of (26) at $0 \le r \le 1$, $t = 0.1$, and $x = 0.1$

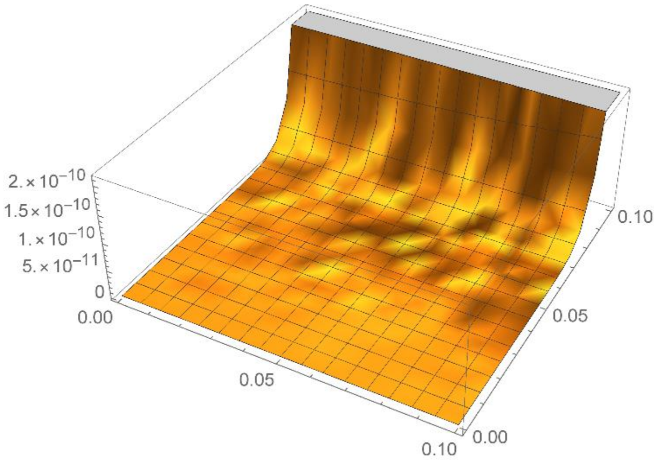

Fig. 6. 15th-order HPM solution of (26) with Lower bound accuracy $\forall\, t, x \in [0,0.1]$ and $r = 0.4$

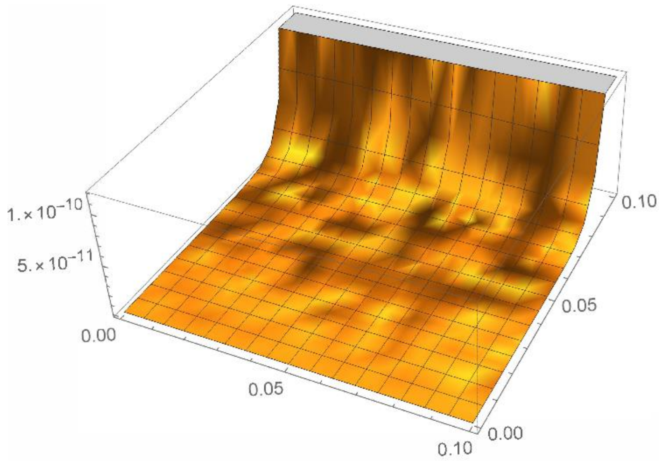

Fig. 7. 15th-order HPM solution of (26) with Upper bound accuracy $\forall\, t, x \in [0,0.1]$ and $r = 0.4$

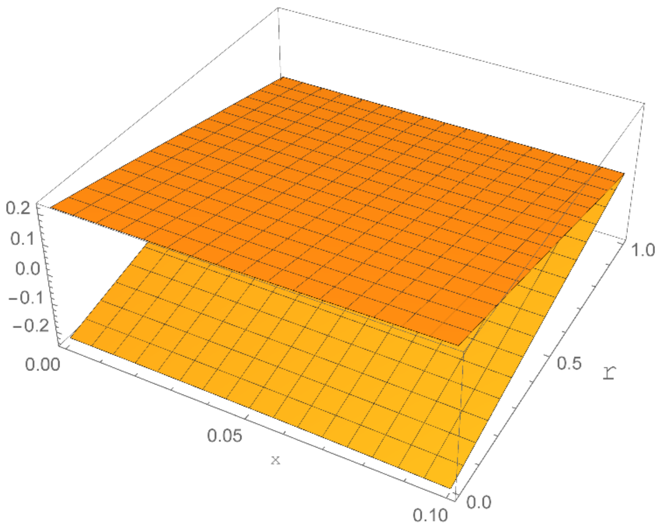

Fig. 8. 15th-order HPM solution of (26) at $0 \le r \le 1$, $x \in [0,0.1]$ and $t = 0.1$.

from Tables III, IV and Fig. 5 to 8 one can conclude that the 15-order HPM solution of (26) satisfies the convex triangular fuzzy number [24,27] for the values of $0 \le r \le 1$.

Case 3. Consider the nonlinear nonhomogeneous Cauchy FRDE, where $x \ge 0, t \ge 0$, $\tilde{\alpha} = [0.9 + 0.1r, 1.1 - 0.1r]$

$$\frac{\partial \tilde{u}(t,x)}{\partial t} = \frac{\partial^2 \tilde{u}(t,x)}{\partial x^2} - [\tilde{u}(t,x)]^2 + \tilde{\alpha} x^2 t^2 \tag{31}$$
$$\tilde{u}(0,x) = 0, \frac{\partial}{\partial x}\tilde{u}(0,x) = \tilde{\alpha} x,$$

The initial approximation of (31) are specified by

$$\begin{cases} \underline{U}_0(t,x;r) = (0.9 + 0.1r)x \\ \overline{U}_0(t,x;r) = (1.1 - 0.1r)x \end{cases} \tag{32}$$

According to HPM section 4 we have

$$\begin{cases} \underline{U}_1(x,t;r) = \int_0^t \begin{bmatrix} \frac{\partial^2 \underline{U}_0(t,x;r)}{\partial x^2} \\ -\underline{U}_0(t,x;r)^2 + \underline{\alpha} x^2 t^2 \end{bmatrix} \\ \underline{U}_2(x,t;r) = \int_0^t \begin{bmatrix} \frac{\partial^2 \underline{U}_1(t,x;r)}{\partial x^2} \\ -2\underline{U}_1(t,x;r)\underline{U}_0(t,x;r) \end{bmatrix} \\ \vdots \\ \underline{U}_k(x,t;r) = \int_0^t \begin{bmatrix} \frac{\partial^2 \underline{U}_{k-1}(t,x;r)}{\partial x^2} \\ -\sum_{k-1=0}^{n-1} \underline{U}_{k-1}(t,x;r)\, \underline{U}_{n-k-2}(t,x;r) \end{bmatrix} \end{cases} \tag{33}$$

$$\begin{cases} \overline{U}_1(x,t;r) = \int_0^t \begin{bmatrix} \frac{\partial^2 \overline{U}_0(t,x;r)}{\partial x^2} + \\ -\overline{U}_0(t,x;r)^2 + \overline{\alpha} x^2 t^2 \end{bmatrix} \\ \overline{U}_2(x,t;r) = \int_0^t \begin{bmatrix} \frac{\partial^2 \overline{U}_1(t,x;r)}{\partial x^2} \\ -2\overline{U}_1(t,x;r)\overline{U}_0(t,x;r) \end{bmatrix} \\ \vdots \\ \overline{U}_k(x,t;r) = \int_0^t \begin{bmatrix} \frac{\partial^2 \overline{U}_{k-1}(t,x;r)}{\partial x^2} \\ -\sum_{k-1=0}^{n-1} \overline{U}_{k-1}(t,x;r)\, \overline{U}_{n-k-2}(t,x;r) \end{bmatrix} \end{cases} \tag{34}$$

Since the exact solution cannot be found from (31) [28], we define the residual error as in case 2 to analyze the accuracy of the approximate solution approximate-analytical such that

$$\tilde{E}(x,t;r) = \\ = \left|\frac{\partial \tilde{U}_k(t,x;r)}{\partial t} - \frac{\partial^2 \tilde{U}_k(t,x;r)}{\partial x^2} + \left[\tilde{U}_k(t,x;r)\right]^2 - \tilde{\alpha} x^2 t^2\right| \qquad (35)$$

TABLE V
12TH-ORDER HPM OF (31) WITH LOWER SOLUTION FOR $0 \le r \le 1, x = 0.3$, AND $t = 0.3$

r	$\underline{U}$ HPM	$\underline{E}$
0	0.18977833277389260	0.00017066078329187884
0.2	0.19233109735968892	0.00017066078329187884
0.4	0.19481676927664257	0.00017066078329187884
0.6	0.19723547686377807	0.00017066078329187884
0.8	0.19958733383886260	0.00017066078329187884
1	0.20187243969820795	0.00017066078329187884

TABLE VI
12TH-ORDER HPM OF (31) WITH LOWER SOLUTION FOR $0 \le r \le 1, x = 03$, AND $t = 0.3$

r	$\overline{U}$ HPM	$\overline{E}$
0	0.21229924682509757	0.0008404323866298136
0.2	0.21034686942973005	0.0008404323866298136
0.4	0.20832804156767476	0.0008404323866298136
0.6	0.20624272758025700	0.0008404323866298136
0.8	0.20409088014939222	0.0008404323866298136
1	0.20187243969820792	0.0008404323866298136

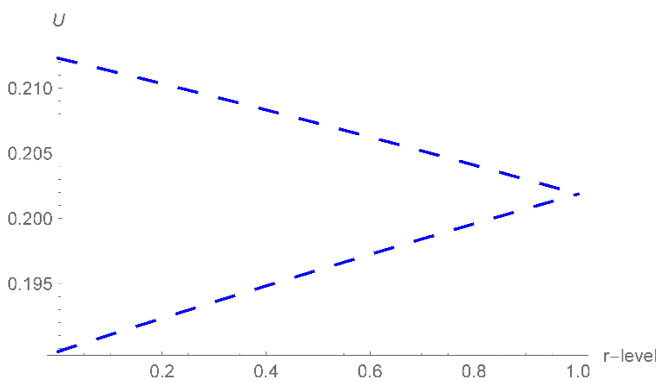

Fig. 9. 12th-order HPM solution of (31) at $0 \le r \le 1$, $t = 0.3$, and $x = 0.3$

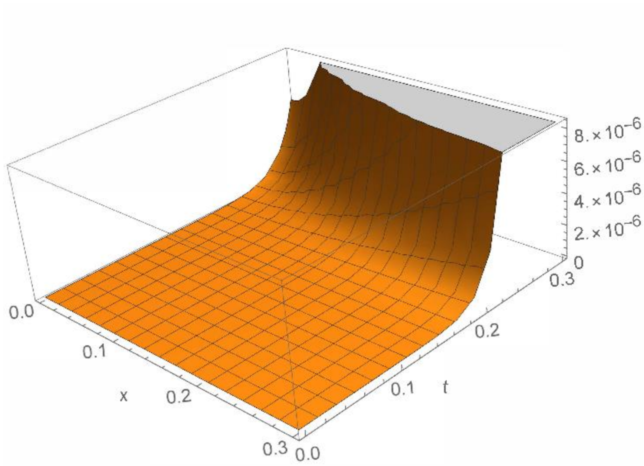

Fig. 10. 12th-order HPM solution of equation (31) with Lower bound accuracy $\forall\, t, x \in [0,0.3]$ and $r = 0.2$

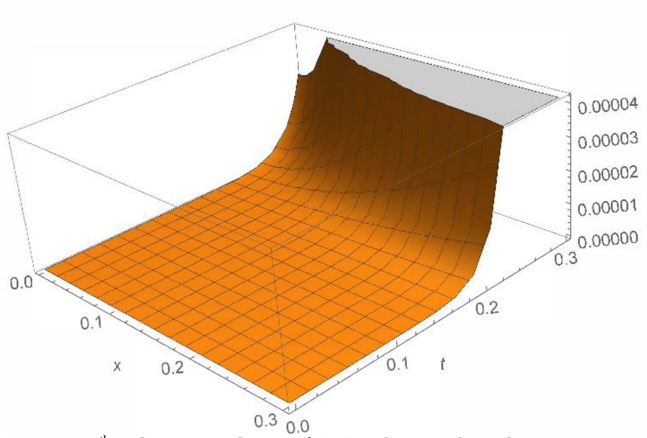

Fig. 11. 12th-order HPM solution of (31) with Upper bound accuracy $\forall\, t, x \in [0,0.3]$ and $r = 0.2$

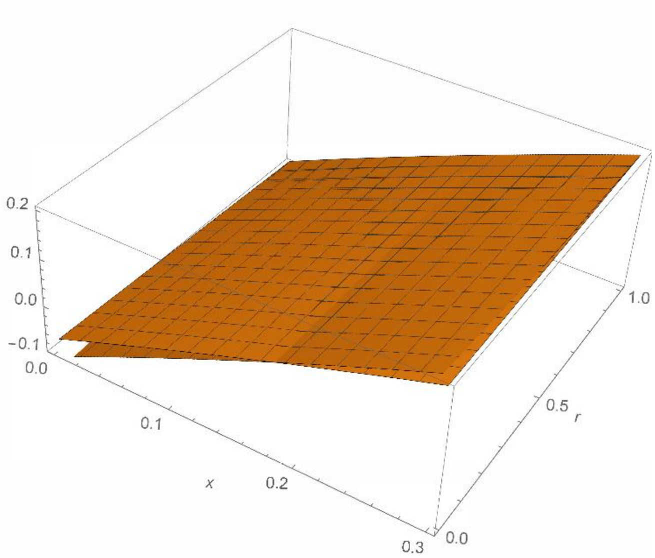

Fig. 12. 12th-order HPM solution of (31) at $0 \le r \le 1$, $x \in [0,0.3]$ and $t = 0.3$.

from Tables V, VI and Fig. 9 to 12 one can conclude that the 12th-order HPM solution of (31) satisfies the convex triangular fuzzy number [24,27] for the values of $0 \le r \le 1$.

VI. CONCLUSION

The main objective of this research with regard to approximate-analytical solution for the FRDE has been presented. We have achieved this aim by formulating and applying HPM befitting from fuzzy set theory properties. The solution provided by this method has useful feature of fast converging power series with the elegantly computable convergence of for the nonlinear problem without need to compare with exact solution. As far as we know, this is the earliest attempt to solve FRDE with HPM. Three test cases shows that the HPM is a capable and accurate method for obtaining approximate-analytical solution of FPDEs. In addition, the acquired solution demonstrates that HPM results are satisfying the properties of triangular shape fuzzy numbers.

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