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Nabla Hukuhara Differentiability for Fuzzy Functions on Time Scales

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Abstract—**In this paper, we introduced a new class of derivative called nabla Hukuhara derivative for fuzzy functions on time scales under Hukuhara difference. We proved the uniqueness and existence of this derivative and obtain fundamental properties. The nabla Hukuhara derivative of scalar multiplication, sum and product of two fuzzy functions on time scales are established.**

Index Terms—**Fuzzy functions, Time scales, Hukuhara difference, Nabla Hukuhara derivative.**

I. Introduction

An exact description of any real world phenomenon is virtually impossible due to indeterminacy, which is caused by the inability to represent a real situation in precise form. To specify these vague or imprecise notions, Zadeh [25] introduced the theory of fuzzy sets. Hukuhara [9] introduced difference between two sets called Hukuhara difference and developed the theory of derivatives and integrals for set valued mappings. Later, Puri and Ralescu [19] studied Hukuhara derivative for fuzzy functions using Hukuhara difference. Kaleva [15], studied fuzzy differential equations under Hukuhara differentiability. Further, [24] studied dynamical behavior of first-order nonlinear fuzzy difference equations.

Time scales was initiated by the german mathematician Stefan Hilger [12]. The important features of time scales are extension, unification and generalization. The theory of time scale calculus is applicable to any field in which dynamic process is described by continuous or discrete time models. If we take time scales as real numbers, then the derivative of a function is equal to standard differentiation while, if we take time scales as integers then it turns to backward difference operator or forward difference operator. For basic results in time scale calculus and dynamic equations on time scales were found in [1], [5], [6]. In some recent studies and applications in economics [4], production, inventory models [3], adaptive control [14], neural networks [17] cellular neural networks [8], BAM neural networks with nabla derivative on time scales [10] suggested nabla derivative is more preferable than delta derivative on time scales.

Interval differential equations on time scales and Hukuhara differentiability of interval valued functions was studied by Lupulescu [18]. Recently, Fard and Bidgoli [7] introduced and studied delta Hukuhara derivative and Henstock-Kurzweil integrals of fuzzy valued functions on time scales, using generalized Hukuhara difference. Vasavi et. al. [20], [21], [22], [23] introduced Hukuhara delta derivative, second type Hukuhara delta derivative and generalized Hukuhara delta derivatives, using Hukuhara difference and studied fuzzy dynamic equations on time scales. M.Hu et. al. [11] studied the Positive Periodic Solutions in Shifts Delta(+/-) for a Neutral Dynamic Equation on Time Scales. Recently, we introduced and studied nabla integral for fuzzy functions on time scales [16]. With the importance and advantages of nabla derivatives and dynamic equations on time scales in recent applications, we proposed to develop the theory of fuzzy nabla dynamic equations on time scales. In this context, we introduced nabla Hukuhara derivative for fuzzy functions on time scales and study their properties. The rest of this paper is organized as follows. In section 2, we present some definitions, properties basic results relating to fuzzy sets, calculus of fuzzy functions and time scales calculus. Section 3 introduces nabla Hukuhara derivative of fuzzy functions on time scales and established uniqueness, existence of the derivative and also obtain some properties.

II. Preliminaries

It is important to recall some basic results and definitions related to fuzzy calculus. Let $\Re_k(\Re^n)$ be the family of all convex compact nonempty subsets of $\Re^n$. Denote the set addition and scalar multiplication in $\Re_k(\Re^n)$ as usual. Then $\Re_k(\Re^n)$ satisfies the properties of commutative semigroup [15] under addition with cancellation laws. Further, if $\alpha, \beta \in \Re$ and $S, T \in \Re_k(\Re^n)$, then

$$\alpha(S+T) = \alpha S + \alpha T, \;\; \alpha(\beta S) = (\alpha\beta)S, \;\; 1.S = S,$$

and if $\beta, \alpha \geq 0$ then $(\beta+\alpha)S = \beta S + \alpha S$. Let S and T be two bounded nonempty subsets of $\Re^n$. By using the Pampeiu-Hausdorff metric we defined the distance between S and T as follows

$$d_H(S,T) = \max\{\sup_{s\in S}\inf_{t\in T}\|s-t\|, \sup_{t\in T}\inf_{s\in S}\|s-t\|\}$$

here $||.||$ is the Euclidean norm in $\Re^n$. Then $(\Re_k(\Re^n), d_H)$ becomes a seperable and complete metric space [15].

Define

$$\mathbb{E}_n = \{\mu : \Re^n \to [0,1]/\mu \text{ satisfies(a)-(d) below}\}, \text{ where}$$

(a) If $\exists$ a $t \in \Re^n$ such that $\mu(t) = 1$ then μ is said to be normal,
(b) μ is fuzzy convex,
(c) μ is upper semicontinuous,
(d) the closure of $\{t \in \Re^n/\mu(t) > 0\}$ = $[\mu]^0$ is compact.

For $0 < \lambda \leq 1$, denote $[\mu]^\lambda = \{t \in \Re^n : \mu(t) \geq \lambda\}$, then from the above conditions we have that the λ-level set $[\mu]^\lambda \in \Re_k(\Re^n)$.

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We know that $[g(s,t)]^\lambda = g([s]^\lambda, [t]^\lambda)$, for all $s, t \in \mathbb{E}_n$ and g is continuous. The scalar multiplication $\odot$ and addition $\oplus$ of $s, t \in \mathbb{E}_n$ is defined as

$$[s \oplus t]^\lambda = [s]^\lambda + [t]^\lambda, [k \odot s]^\lambda = k[s]^\lambda,$$

where $s, t \in \mathbb{E}_n,\ k \in \Re,\ 0 \le \lambda \le 1$.

Theorem 2.1: [15] If $\mu \in \mathbb{E}_n$, then

(a) $[\mu]^\lambda \in \Re_k(\Re^n)$ for all $0 \le \lambda \le 1$,
(b) $[\mu]^{\lambda_2} \subset [\mu]^{\lambda_1}$ for all $0 \le \lambda_1 \le \lambda_2 \le 1$,
(c) If $\lambda_k \in [0,1]$ and $\{\lambda_k\}$ is a nondecreasing sequence converging to $\lambda > 0$, then

$$[\mu]^\lambda = \bigcap_{k \ge 1} [\mu]^{\lambda_k}.$$

Conversely, if $\{X^\lambda / 0 \le \lambda \le 1\}$ is a subsets of family of $\Re^n$ satisfying the above conditions from (a)-(c), then
$\exists$a $x \in \mathbb{E}_n \ni$

$$[\mu]^\lambda = X^\lambda, \ \forall \ \ \lambda \in (0,1] \text{ and}$$

$$[\mu]^0 = cl\left\{ \bigcup_{0<\lambda\le 1} X^\lambda \right\} \subset X^0,$$

here cl is the closure of the set

Theorem 2.2: [15] If sequence $\{X_n\}$ converging to X in $\Re_k(\Re^n)$ and
$d(X_n, X) \to 0$ as $n \to \infty$ then

$$X = \bigcap_{n \ge 1} cl\left\{ \bigcup_{m \ge n} X_m \right\}.$$

Define $D_H : \mathbb{E}_n \times \mathbb{E}_n \to [0, \infty)$ by

$$D_H(s,t) = \sup_{0 \le \lambda \le 1} d_H([s]^\lambda, [t]^\lambda),$$

here d_H is the Pampeiu Hausdorff metric defined in $\Re_k(\Re^n)$. Then $(\mathbb{E}_n, D_H)$ is a complete metric space [15].

The following theorem extend the properties of addition and scalar multiplication of fuzzy number valued functions ($\Re_F = \mathbb{E}_1$) to $\mathbb{E}_n$.

Theorem 2.3: [2]

(a) If $\tilde{0}$ is the zero element in $\Re_F$, then $\hat{0} = (\tilde{0}, \tilde{0}, \ldots, \tilde{0})$ is the zero element in $\mathbb{E}_n$. i.e. $s \oplus \hat{0} = \hat{0} \oplus s = s \ \forall s \in \mathbb{E}_n$;
(b) For any $s \in \mathbb{E}_n$ has no inverse with respect to '$\oplus$';
(c) For any $\beta, \gamma \in \Re$ with $\beta, \gamma \ge 0$ or $\beta, \gamma \le 0$ and $s \in \mathbb{E}_n$, then $(\beta + \gamma) \odot s = (\beta \odot s) \oplus (\gamma \odot s)$;
(d) For any $\beta \in \Re$ and $s, t \in \mathbb{E}_n$, we have $\beta \odot (s \oplus t) = (\beta \odot s) \oplus (\beta \odot t)$;
(e) For any $\beta, \gamma \in \Re$ and $s \in \mathbb{E}_n$, we have $\beta \odot (\gamma \odot s) = (\beta\gamma) \odot s$.

Let $S, T \in \mathbb{E}_n$. If $\exists$ a $R \in \mathbb{E}_n$ such that $S = T \oplus R$ then we say that R is the H-difference of S and T and is denoted by $S \ominus_h T$. For any $S, T, R, U \in \mathbb{E}_n$ and $\alpha \in \Re$, the following holds

(a) $D_H(S,T) = 0 \Leftrightarrow S = T$;
(b) $D_H(\alpha S, \alpha T) = |\alpha| D_H(S,T)$;
(c) $D_H(S \oplus R, T \oplus R) = D_H(S,T)$;
(d) $D_H(S \ominus_h R, T \ominus_h R) = D_H(S,T)$;
(e) $D_H(S \oplus T, R \oplus U) \le D_H(S,R) + D_H(T,U)$;
(f) $D_H(S \ominus_h T, R \ominus_h U) \le D_H(S,R) + D_H(T,U)$.

provided the H-differences exists.

Now, we present some fundamental definitions and properties of Hukuhara derivative of fuzzy functions on the compact interval $I = [a,b]$, $a, b \in \Re$.

Definition 2.1: [15] A function $G : I \to \mathbb{E}_n$ is H-differentiable at $\theta \in I$, if $\exists$ a $G^{'}(\theta) \in \mathbb{E}_n$ such that the limits

$$\lim_{\hbar \to 0^+} \frac{G(\theta + \hbar) \ominus_h G(\theta)}{\hbar}, \ \lim_{\hbar \to 0^+} \frac{G(\theta) \ominus_h G(\theta - \hbar)}{\hbar}$$

exist in $\mathbb{E}_n$ and are equal to $G^{'}(\theta)$. Here we consider the limits in the metric space $(\mathbb{E}_n, D_H)$. At the end points of I, we will consider only the one-sided derivatives.

Remark 2.1: [15] A function G is said to be differentiable if the multivalued mapping G_λ is Hukuhara differentiable for all $\lambda \in [0,1]$ and

$$[G_\lambda(\theta)]^{'} = [G^{'}(\theta)]^\lambda,$$

where $[G_\lambda]^{'}$ is the H-derivative of G_λ.

Definition 2.2: [15] A mapping G is said to be strongly measurable if for each $\lambda \in [0,1]$, the fuzzy function $G : I \to \Re_k(\Re^n)$ defined by $G_\lambda(\theta) = [G(\theta)]^\lambda$ is measurable.

Remark 2.2: [15] If $\{\lambda_k\}$ is a nonincreasing sequence converging to 0 for all $x \in \mathbb{E}_n$, then

$$\lim_{k \to \infty} d_H([x]^0, [x]^{\lambda_k}) = 0.$$

Now, we present some fundamental definitions and results of time scales.

Definition 2.3: [5]

(a) Any nonempty closed subset of $\Re$ is defined as a time scale which is denoted by $\mathbb{T}$.
(b) $\rho : \mathbb{T} \to \mathbb{T}$ is the backward jump operator and $\nu : \mathbb{T} \to \mathbb{R}^+$, the graininess operator are defined by

$$\rho(\theta) = \sup\{\theta_0 \in \mathbb{T} : \theta_0 < \theta\}, \ \nu(\theta) = \theta - \rho(\theta) \text{ for } \theta \in \mathbb{T},$$

(c) The operator ρ is called left dense if $\rho(\theta) = \theta$, otherwise left scattered.
(d) $\mathbb{T}_k = \mathbb{T} - \{m\}$, if $\mathbb{T}$ has a right scattered minimum m. Otherwise $\mathbb{T}_k = \mathbb{T}$.
(e) A mapping $g^\rho : \mathbb{T} \to \Re$ defined by

$$g^\rho(\theta) = g(\rho(\theta)) \text{ for each } \theta \in \mathbb{T},$$

where $g : \mathbb{T} \to \Re$ is a function.
(f) The interval in time scale $\mathbb{T}$ is defined by

$$\mathbb{T}^{[a,b]} = \{\theta \in \mathbb{T} : a \le \theta \le b\} = [a,b] \cap \mathbb{T}$$

and

$$\mathbb{T}_k^{[a,b]} = \begin{cases} \mathbb{T}^{[a,b]}, & \text{if a is right dense} \\ \mathbb{T}^{[\sigma(a),b]}, & \text{if a is right scattered.} \end{cases}$$

Definition 2.4: [5] Let $g : \mathbb{T} \to \Re$ be a function and $\theta \in \mathbb{T}_k$. Then $g^\nabla(\theta)$ exists as a number provided for any given $\epsilon > 0$, $\exists$ a neighbourhood N_δ of θ (i.e., $N_\delta = (\theta - \delta, \theta + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|[g(\rho(\theta)) - g(\theta_0)] - g^\nabla(\theta)[\rho(\theta) - \theta_0]| \le \epsilon|\rho(\theta) - \theta_0|,$$

$\forall \ \theta_0 \in N_\delta$,

Here, $g^\nabla(\theta)$ is called the nabla derivative of g at θ. Moreover, g is said to be nabla (or Hilger) differentiable on $\mathbb{T}_k$, if $g^\nabla(\theta)$ exists $\forall\ \theta \in \mathbb{T}_k$. The function $g^\nabla : \mathbb{T}_k \to \Re$ is then called the nabla derivative of g on $\mathbb{T}_k$.

Definition 2.5: [5] A mapping $g : \mathbb{T} \to \Re$ is said to be regulated if its left sided limits exist and are finite at all ld-point (left dense points) in $\mathbb{T}$ and its right sided limits exist and are finite at all rd-points (right dense points) in $\mathbb{T}$

Definition 2.6: [5] Let $g : \mathbb{T} \to \Re$ be a function. g is said to be ld-continuous, if it is continuous at each ld-point in $\mathbb{T}$ and $\lim\limits_{\theta_0 \to \theta^+} g(\theta)$ exists as a finite number for all rd-points in $\mathbb{T}$.

III. Nabla Hukuhara Differentiability

In this section, first we introduced nabla Hukuhara derivative of fuzzy functions on time scales. Later, we established uniqueness and existence of this derivative and obtained some properties, results on nabla Hukuhara derivative.

Definition 3.1: [20] For any given $\epsilon > 0$ $\exists$ a $\delta > 0$, such that the fuzzy function $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ has a unique $\mathbb{T}$-limit $P \in \mathbb{E}_n$ at $\theta \in \mathbb{T}$ if $D_H(G(\theta) \ominus_h P, \hat{0}) \leq \epsilon$, for all $\theta \in N_{\mathbb{T}}(\theta, \delta)$ and it is denoted by $\mathbb{T} - \lim\limits_{\theta \to \theta_0} G(\theta)$.
Here $\mathbb{T}$-limit denotes the limit on time scale in the metric space $(\mathbb{E}_n, D_H)$.

Remark 3.1: From the above definition, we have

$$\mathbb{T} - \lim_{\theta \to \theta_0} G(\theta) = P \in \mathbb{E}_n \Longleftrightarrow \mathbb{T} - \lim_{\theta \to \theta_0} (G(\theta) \ominus_h P) = \hat{0},$$

where $\hat{0}$ is the zero element in $\mathbb{E}_n$.

Definition 3.2: A fuzzy mapping $G : \mathbb{T} \to \mathbb{E}_n$ is continuous at $\theta_0 \in \mathbb{T}$, if $\mathbb{T} - \lim\limits_{\theta \to \theta_0} G(\theta) \in \mathbb{E}_n$ exists and $\mathbb{T} - \lim\limits_{\theta \to \theta_0} G(\theta) = G(\theta_0)$, i.e.

$$\mathbb{T} - \lim_{\theta \to \theta_0} (G(\theta) \ominus_h G(\theta_0)) = \hat{0}.$$

Remark 3.2: If $G : \mathbb{T} \to \mathbb{E}_n$ is continuous at $\theta \in \mathbb{T}$, then for every $\epsilon > 0$, $\exists$ a $\delta > 0$, such that

$$D_H(G(\theta) \ominus_h G(\theta_0), \hat{0}) \leq \epsilon, \text{ for all } \theta \in N_{\mathbb{T}}.$$

Remark 3.3: Let $G : \mathbb{T} \to \mathbb{E}_n$ and $\theta_0 \in \mathbb{T}$.

(a) If $\mathbb{T} - \lim_{\theta \to \theta_0^+} G(\theta) = G(\theta_0)$, then G is said to be right continuous at θ_0.
(b) If $\mathbb{T} - \lim_{\theta \to \theta_0^-} G(\theta) = G(\theta_0)$, then G is said to be left continuous at θ_0.
(c) If $\mathbb{T} - \lim_{\theta \to \theta_0^+} G(\theta) = G(\theta_0) = \mathbb{T} - \lim_{\theta \to \theta_0^-} G(\theta)$, then G is continuous at θ_0.

Definition 3.3: Suppose $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is a fuzzy function and $\theta \in \mathbb{T}_k^{[a,b]}$. Let $G^{\nabla_h}(\theta)$ be an element of $\mathbb{E}_n$ exists provided for any given $\epsilon > 0$, $\exists$ a neighbourhood $N_{\mathbb{T}^{[a,b]}}$ of θ and for some $\delta > 0$ such that

$$\begin{aligned} &D_H[(G(\theta+\hbar) \ominus_h G(\rho(\theta)), (\hbar + \nu(\theta)) \odot G^{\nabla_h}(\theta)] \\ &\qquad \leq \epsilon|\hbar + \nu(\theta)|, \\ &D_H[(G(\rho(\theta)) \ominus_h G(\theta - \hbar), (\hbar - \nu(\theta)) \odot G^{\nabla_h}(\theta)] \\ &\qquad \leq \epsilon|\hbar - \nu(\theta)|, \end{aligned} \tag{1}$$

for all $\theta - \hbar, \theta + \hbar \in N_{\mathbb{T}^{[a,b]}}$ with $0 < h < \delta$ where $\nu(\theta) = \theta - \rho(\theta)$. Then G is called nabla Hukuhara (nabla-h) differentiable at θ and is denoted by $G^{\nabla_h}(\theta)$.

or

A fuzzy function $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is sasid to be nabla-h differentiable at $\theta \in \mathbb{T}_k^{[a,b]}$ if $\exists$ a $G^{\nabla_h}(\theta) \in \mathbb{E}_n$ such that the limits

$$\mathbb{T} - \lim_{\hbar \to 0^+} \frac{G(\theta + \hbar) \ominus_h G(\rho(\theta))}{\hbar + \nu(\theta)}$$

and

$$\mathbb{T} - \lim_{\hbar \to 0^+} \frac{G(\rho(\theta)) \ominus_h G(\theta - \hbar)}{\hbar - \nu(\theta)}$$

exists and are equal to $G^{\nabla_h}(\theta)$.

Moreover, if nabla-h derivative exists for each $\theta \in \mathbb{T}_k^{[a,b]}$, then G is nabla-h differentiable on $\mathbb{T}_k^{[a,b]}$. We consider only right limit at left scattered points and one-sided limit at the end points of $\mathbb{T}_k^{[a,b]}$.
Note. If both $\mathbb{T}$-limits exists at left scattered point, then the nabla-h derivative is in $\Re^n$ (crisp). It will restrict the nabla-h differentiability of fuzzy functions on time scales having left scattered points. To avoid this, we considered only right limit at left scattered points.

Lemma 3.1: If G is nabla-h differentiable at θ then nabla-h derivative exists and it is unique.

Proof: Suppose that $G^{\nabla_{h1}}(\theta)$ and $G^{\nabla_{h2}}(\theta)$ are ∇_h-derivatives of G at θ. Then

$$D_H[(\hbar+\nu(\theta)) \odot G^{\nabla_{h1}}(\theta), G(\theta+\hbar) \ominus_h G(\rho(\theta))] \leq \frac{\epsilon}{2}|\hbar+\nu(\theta)|,$$

$$D_H[(\hbar+\nu(\theta)) \odot G^{\nabla_{h2}}(\theta), G(\theta+\hbar) \ominus_h G(\rho(\theta))] \leq \frac{\epsilon}{2}|\hbar+\nu(\theta)|.$$

Consider

$$\begin{aligned} &D_H[G^{\nabla_{h1}}(\theta), G^{\nabla_{h2}}(\theta)] \\ &= \frac{D_H[(\hbar+\nu(\theta)) \odot G^{\nabla_{h1}}(\theta), (\hbar+\nu(\theta)) \odot G^{\nabla_{h2}}(\theta)]}{|\hbar+\nu(\theta)|} \\ &\leq \frac{D_H\left[(\hbar+\nu(\theta)) \odot G^{\nabla_{h1}}(\theta), G(\theta+\hbar) \ominus_h G(\rho(\theta))\right]}{|\hbar+\nu(\theta)|} \\ &+ \frac{D_H\left[G(\theta+\hbar) \ominus_h G(\rho(\theta)), (\hbar+\nu(\theta)) \odot G^{\nabla_{h2}}(\theta)\right]}{|\hbar+\nu(\theta)|} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \ \forall\ |\hbar+\nu(\theta)| \neq 0. \end{aligned}$$

Since $\epsilon > 0$, then $D_H[G^{\nabla_{h1}}(\theta), G^{\nabla_{h2}}(\theta)] = 0$. Therefore, $G^{\nabla_{h1}}(\theta) = G^{\nabla_{h2}}(\theta)$. Hence nabla-h derivative exists and is unique. ∎

Theorem 3.1: Let $\theta \in \mathbb{T}_k^{[a,b]}$ and $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ be a fuzzy function. Then we have

(a) If $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is nabla-h differentiable at θ, then G is continuous when θ is left dense and right continuous when θ is left scattered.
(b) If $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is right continuous at θ and θ is left scattered, then $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is nabla-h differentiable at θ with

$$G^{\nabla_h}(\theta) = \frac{G(\theta) \ominus_h G(\rho(\theta))}{\nu(\theta)},$$

provided $G(\theta) \ominus_h G(\rho(\theta))$ exists.
(c) If θ is left dense, then $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is nabla-h differentiable at θ iff the limits

$$\lim_{\hbar \to 0^+} \frac{G(\theta) \ominus_h G(\theta - \hbar)}{\hbar}, \quad \lim_{\hbar \to 0^+} \frac{G(\theta+\hbar) \ominus_h G(\theta)}{\hbar}$$

exist as a finite number and

$$\lim_{\hbar \to 0^+} \frac{G(\theta) \ominus_h G(\theta - \hbar)}{\hbar} = \lim_{\hbar \to 0^+} \frac{G(\theta+\hbar) \ominus_h G(\theta)}{\hbar}$$

$$= G^{\nabla_h}(\theta).$$

(d) If $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is nabla-h differentiable at θ, then

$$G(\rho(\theta)) = G(\theta) \ominus_h (\nu(\theta) \odot G^{\nabla_h}(\theta)).$$

Proof: **(a)** Suppose G is nabla-h differentiable at θ. Let $\epsilon \in (0,1)$. Choose $\epsilon^1 = \epsilon[1 + 2\nu(\theta) + \|G^{\nabla_h}(\theta)\|]^{-1}$. Clearly, $\epsilon^1 \in (0,1)$. Let $\hat{0}$ be the zero element in $\mathbb{E}_n$ and from the definition of D_H, we have

$$D_H[\tilde{p}, \hat{0}] = \|\tilde{p}\|.$$

For any $s, t \in \mathbb{E}_n, D_H[s,t] \leq \|s \ominus_h t\|$. Since G is nabla-h differentiable, $\exists\ N_{\mathbb{T}^{[a,b]}}$ a neighbourhood of θ such that

$$D_H[(G(\theta+\hbar) \ominus_h G(\rho(\theta)), (\hbar + \nu(\theta)) \odot G^{\nabla_h}(\theta)]$$

$\leq \epsilon^1|\hbar + \nu(\theta)|$, for all $0 < \hbar < \delta$ with $\theta - \hbar, \theta + \hbar \in N_{\mathbb{T}^{[a,b]}}$. Therefore, for all $\theta - \hbar, \theta + \hbar \in N_{\mathbb{T}^{[a,b]}} \cap (\theta - \epsilon^1, \theta + \epsilon^1)$ and $0 < \hbar < \epsilon^1$, we have

$$D_H[G(\theta+\hbar), G(\theta)] \leq \|G(\theta+\hbar) \ominus_h G(\theta)\|$$

Now,

$$\begin{aligned}
&D_H[G(\theta+\hbar) \ominus_h G(\theta), \hat{0}] \\
&\leq D_H[(G(\theta+\hbar) \ominus_h G(\rho(\theta))) \ominus_h (G(\theta) \ominus_h G(\rho(\theta)))], \\
&\qquad (\hbar + \nu(\theta)) \odot G^{\nabla_h}(\theta) \ominus_h (\hbar + \nu(\theta)) \odot G^{\nabla_h}(\theta)] \\
&\leq D_H[(G(\theta+\hbar) \ominus_h G(\rho(\theta)), (\hbar + \nu(\theta)) \odot G^{\nabla_h}(\theta)] \\
&\qquad + D_H[G(\theta) \ominus_h G(\rho(\theta)), (\hbar + \nu(\theta)) \odot G^{\nabla_h}(\theta)] \\
&\leq \epsilon^1|\hbar + \nu(\theta)| + D_H[G(\theta) \ominus_h G(\rho(\theta)) \oplus \hat{0}, \\
&\qquad \nu(\theta) \odot G^{\nabla_h}(\theta) \oplus \hbar \odot G^{\nabla_h}(\theta)] \\
&\leq \epsilon^1|\hbar + \nu(\theta)| + D_H[G(\theta) \ominus_h G(\rho(\theta)), \nu(\theta) \odot G^{\nabla_h}(\theta)] \\
&\qquad + D_H[\hat{0}, \hbar \odot G^{\nabla_h}(\theta)] \\
&\leq \epsilon^1(\hbar + \nu(\theta)) + \epsilon^1\nu(\theta) + +\hbar D_H[\hat{0}, G^{\nabla_h}(\theta)] \\
&< \epsilon^1(1 + 2\nu(\theta) + \|G^{\nabla_h}(\theta)\|) = \epsilon.
\end{aligned}$$

Therefore, for θ being left dense or left scattered

$$\mathbb{T} - \lim_{\hbar \to 0} G(\theta + \hbar) = G(\theta).$$

For left dense point θ, it is easy to prove that

$$\mathbb{T} - \lim_{\hbar \to 0} G(\theta - \hbar) = G(\theta).$$

Hence G is continuous at left dense points and right continuous at left scattered points in $\mathbb{T}_k^{[a,b]}$.

(b) Suppose that θ is left scattered. Consider

$$G^{\nabla_h}(\theta) = \mathbb{T} - \lim_{\hbar \to 0^+} \frac{G(\theta+\hbar) \ominus_h G(\rho(\theta))}{\hbar + \nu(\theta)}.$$

Since G is right continuous, then

$$G^{\nabla_h}(\theta) = \frac{G(\theta) \ominus_h G(\rho(\theta))}{\nu(\theta)}.$$

Hence G is nabla-h differentiable at θ.

(c) Suppose that G is nabla-h differentiable at θ and θ is ld-point. Since G is nabla-h differentiable at θ, for any given $\epsilon > 0$, $\exists\ N_{\mathbb{T}^{[a,b]}}$ a neighbourhood of $\theta\ \ni$

$$D_H[(G(\rho(\theta)) \ominus_h G(\theta-\hbar), (\hbar - \nu(\theta)) \odot G^{\nabla_h}(\theta)] \leq \epsilon|\hbar - \nu(\theta)|,$$

$$D_H[G(\theta+\hbar) \ominus_h G(\rho(\theta)), (\hbar + \nu(\theta)) \odot G^{\nabla_h}(\theta)] \leq \epsilon|\hbar + \nu(\theta)|,$$

for all $0 < \hbar < \delta$ with $\theta - \hbar, \theta + \hbar \in N_{\mathbb{T}^{[a,b]}}$. Since $\rho(\theta) = \theta$, i.e. $\nu(\theta) = 0$, we have

$$D_H[(G(\theta) \ominus_h G(\theta - \hbar), \hbar \odot G^{\nabla_h}(\theta)] \leq \epsilon\hbar,$$

$$D_H[(G(\theta+\hbar) \ominus_h G(\theta), \hbar \odot G^{\nabla_h}(\theta)] \leq \epsilon\hbar,$$

for all $0 < \hbar < \delta$ with $\theta - \hbar, \theta + \hbar \in N_{\mathbb{T}^{[a,b]}}$. This implies that

$$D_H\left[\frac{G(\theta) \ominus_h G(\theta - \hbar)}{\hbar}, G^{\nabla_h}(\theta)\right] \leq \epsilon,$$

$$D_H\left[\frac{G(\theta+\hbar) \ominus_h G(\theta)}{\hbar}, G^{\nabla_h}(\theta)\right] \leq \epsilon,$$

for all $0 < \hbar < \delta$ with $\theta - \hbar, \theta + \hbar \in N_{\mathbb{T}^{[a,b]}}$. since ϵ is arbitrary, we have

$$\begin{aligned}
G^{\nabla_h}(\theta) &= \lim_{\hbar \to 0} \frac{G(\theta) \ominus_h G(\theta - \hbar)}{\hbar} \\
&= \lim_{\hbar \to 0} \frac{G(\theta + \hbar) \ominus_h G(\theta)}{\hbar}.
\end{aligned}$$

Conversely, suppose that for all $0 < \hbar < \delta$ with $\theta - \hbar, \theta + \hbar \in N_{\mathbb{T}}$, $\exists$ a neighbourhood $N_{\mathbb{T}}$ of θ and θ is left dense such that

$$D_H\left[\frac{G(\theta) \ominus_h G(\theta - \hbar)}{\hbar}, G^{\nabla_h}(\theta)\right] \leq \epsilon,$$

$$D_H\left[\frac{G(\theta+\hbar) \ominus_h G(\theta)}{\hbar}, G^{\nabla_h}(\theta)\right] \leq \epsilon.$$

From the above inequalities, G is nabla-h differentiable at θ.

(d) If $\rho(\theta) = \theta$, then $\nu(\theta) = 0$ and we have,

$$G(\rho(\theta)) = G(\theta) = G(\theta) \ominus_h (\nu(\theta) \odot G^{\nabla_h}(\theta)).$$

If $\rho(\theta) < \theta$, then by (b)

$$\nu(\theta) \odot G^{\nabla_h}(\theta) = G(\theta) \ominus_h G(\rho(\theta))$$

Therefore, $G(\rho(\theta)) = G(\theta) \ominus_h (\nu(\theta) \odot G^{\nabla_h}(\theta))$. ■

Example 3.1: Let us consider $\mathbb{T} = \Re$ or $\mathbb{T} = t\mathbb{Z} = \{tk : k \in \mathbb{Z}\}$.

(a) If $\mathbb{T} = \Re$, then from Theorem 3.1 (c) $G : \Re \to \mathbb{E}_n$ is nabla-h differentiable at $\theta \in \Re$ iff

$$\begin{aligned}
G^{\nabla_h}(\theta) &= \lim_{\hbar \to 0} \frac{G(\theta) \ominus_h G(\theta - \hbar)}{\hbar} \\
&= \lim_{\hbar \to 0} \frac{G(\theta + \hbar) \ominus_h G(\theta)}{\hbar} \\
&= G'(\theta).
\end{aligned}$$

(b) If $\mathbb{T} = t\mathbb{Z}$, then every point $\theta \in \mathbb{T}$ is isolated and

$$\rho(\theta) = \sup\{\theta - nt : n \in \mathbb{N}\} = \theta - t,$$

$$\nu(\theta) = \theta - \rho(\theta) = \theta - (\theta - t) = t.$$

From Theorem 3.1(b) $G : t\mathbb{Z} \to \mathbb{E}_n$ is ∇_h-differentiable at $\theta \in t\mathbb{Z}$ and

$$\begin{aligned}
G^{\nabla_h}(\theta) &= \frac{G(\theta) \ominus_h G(\rho(\theta))}{\nu(\theta)} \\
&= \frac{G(\theta) \ominus_h G(\theta - t)}{t} \\
&= \frac{1}{t} \odot \nabla G(\theta),
\end{aligned}$$

where ∇ is the backward Hukuhara difference operator.

Theorem 3.2: Denote $[G(\theta)]^\lambda = G_\lambda(\theta)$ for each $\lambda \in [0,1]$, where $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ be the fuzzy function and if G is nabla-h differentiable, then G_λ is also nabla-h differentiable and

$$[G^{\nabla_h}(\theta)]^\lambda = G_\lambda^{\nabla_h}(\theta).$$

Proof: If θ is left scaterred and G is nabla-h differentiable at $\theta \in \mathbb{T}_k^{[a,b]}$, then for each $\lambda \in [0,1]$, from theorem 3.1 (b) we get

$$[G^{\nabla_h}(\theta)]^\lambda = \frac{[G(\theta)]^\lambda \ominus_h [G(\rho(\theta)]^\lambda}{\nu(\theta)} = \frac{G_\lambda(\theta) \ominus_h G_\lambda(\rho(\theta))}{\nu(\theta)} = G_\lambda^{\nabla_h}(\theta).$$

If G is nabla-h differentiable at $\theta \in \mathbb{T}_k^{[a,b]}$ and θ is left dense, then for $\lambda \in [0,1]$, we get

$$[G(\theta) \ominus_h G(\theta - \hbar)]^\lambda = [G_\lambda(\theta) \ominus_h G_\lambda(\theta - \hbar)]$$

dividing by $\hbar > 0$ and taking the limit $\hbar \to 0^+$, we have

$$\lim_{\hbar \to 0^+} \frac{1}{\hbar}[G_\lambda(\theta) \ominus_h G_\lambda(\theta - \hbar)] = G_\lambda^{\nabla_h}(\theta).$$

Similarly,

$$\lim_{\hbar \to 0^+} \frac{1}{\hbar}\big[G_\lambda(\theta + \hbar) \ominus_h G_\lambda(\theta)\big] = G_\lambda^{\nabla_h}(\theta).$$

■

Remark 3.4: The above Theorem 3.2, states that if G is nabla-h differentiable then the multivalued mapping G_λ is nabla-h differentiable for all $\lambda \in [0,1]$, but the converse of the theorem need not be true. That is the existence of H-differences of λ-level sets $[p]^\lambda \ominus_h [q]^\lambda$ does not imply the existence of H-difference of $p \ominus_h q$.
However, for the converse of the theorem we have the following:

Theorem 3.3: Suppose that $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ satisfy the following conditions:

(1) For each $\theta \in \mathbb{T}^{[a,b]}$ and θ is left dense
  - (a) $\exists$ a $\beta > 0$, $\ni$ the Hukuhara differences $G(\theta) \ominus_h G(\theta - \hbar)$ and $G(\theta + \hbar) \ominus_h G(\theta)$ exists, for all $0 < \hbar < \beta$ and $\theta - \hbar, \theta + \hbar \in N_{\mathbb{T}^{[a,b]}}$;
  - (b) the fuzzy mappings G_λ, $\lambda \in [0,1]$, are uniformly nabla-h differentiable with derivative $G_\lambda^{\nabla_h}$, i.e., to each $\theta \in \mathbb{T}^{[a,b]}$ and $\epsilon > 0$ $\exists$ a $\delta > 0$ such that

$$D_H\left\{\frac{G_\lambda(\rho(\theta)) \ominus_h G_\lambda(\theta - \hbar)}{\hbar - \nu(\theta)}, G_\lambda^{\nabla_h}(\theta)\right\} < \epsilon,$$
$$D_H\left\{\frac{G_\lambda(\theta + \hbar) \ominus_h G_\lambda(\rho(\theta))}{\hbar + \nu(\theta)}, G_\lambda^{\nabla_h}(\theta)\right\} < \epsilon,$$

  for all $0 < \hbar < \delta$, $\theta - \hbar, \theta + \hbar \in N_T^{[a,b]}$, $\lambda \in [0,1]$.

(2) for each $\theta \in \mathbb{T}^{[a,b]}$ and θ is left scattered
  - (a) there exists a $\beta > 0$, $\ni$ the Hukuhara differences $G(\theta) \ominus_h G(\rho(\theta))$ exists and;
  - (b) the fuzzy mappings G_λ, $\lambda \in [0,1]$, are uniformly nabla-h differentiable with derivative $G_\lambda^{\nabla_h}$, i.e., to each $\theta \in \mathbb{T}^{[a,b]}$ and $\epsilon > 0$ $\exists$ a $\delta > 0$ such that

$$D_H\left\{\frac{G_\lambda(\theta) \ominus_h G_\lambda(\rho(\theta))}{\nu(\theta)}, G_\lambda^{\nabla_h}(\theta)\right\} < \epsilon. \quad (2)$$

Then G is nabla-h differentiable and its derivative is given by $G_\lambda^{\nabla_h}(\theta) = [G^{\nabla_h}(\theta)]^\lambda$.

Proof: **Case(1):** For left dense points and $\theta \in \mathbb{T}^{[a,b]}$ then the proof is obvious from Theorem 5.1 [15].

Case(2): For left scattered points and $\theta \in \mathbb{T}^{[a,b]}$ Consider, $\{G_\lambda^{\nabla_h}(\theta), \lambda \in [0,1]\}$, where $G_\lambda^{\nabla_h}(\theta)$ is convex, compact and nonempty subset of $\Re^n$. If $\lambda_1 \le \lambda_2$ then by our supposition (a), we have

$$G_{\lambda_1}(\theta) \ominus_h G_{\lambda_1}(\rho(\theta)) \supset G_{\lambda_2}(\theta) \ominus_h G_{\lambda_2}(\rho(\theta)) \quad (3)$$

For $0 < \hbar < \beta$, we have $G_{\lambda_1}^{\nabla_h}(\theta) \supset G_{\lambda_2}^{\nabla_h}(\theta)$.
Let $\{\lambda_n\}$ be a nondecreasing sequence coverges to $\lambda > 0$.
For $\epsilon > 0$ choose $\hbar > 0$ $\ni$ the equation (2) holds.
Now, let us consider

$$\begin{aligned}
&D_H(G_\lambda^{\nabla_h}(\theta), G_{\lambda_n}^{\nabla_h}(\theta)) \\
&\le D_H\left(G_\lambda^{\nabla_h}(\theta), \frac{G_\lambda(\theta) \ominus_h G_\lambda(\rho(\theta))}{\nu(\theta)}\right) \\
&\quad + D_H\left(\frac{G_\lambda(\theta) \ominus_h G_\lambda(\rho(\theta))}{\nu(\theta)}, G_{\lambda_n}^{\nabla_h}(\theta)\right) \\
&< \epsilon + \frac{D_H[G_\lambda(\theta) \ominus_h G_\lambda(\rho(\theta)), G_{\lambda_n}(\theta) \ominus_h G_{\lambda_n}(\rho(\theta))]}{\nu(\theta)} \\
&\quad + \frac{D_H[G_{\lambda_n}(\theta) \ominus_h G_{\lambda_n}(\rho(\theta)), \nu(\theta) G_{\lambda_n}^{\nabla_h}(\theta)]}{\nu(\theta)} \\
&< 2\epsilon + \frac{1}{\nu(\theta)} D_H[G_\lambda(\theta) \ominus_h G_\lambda(\rho(\theta)), \\
&\quad G_{\lambda_n}(\theta) \ominus_h G_{\lambda_n}(\rho(\theta))].
\end{aligned}$$

By our supposition (a), the rightmost term converges to zero as $n \to \infty$ and hence

$$\lim_{n \to \infty} D_H(G_\lambda^{\nabla_h}(\theta), G_{\lambda_n}^{\nabla_h}(\theta)) = 0.$$

From Theorem 2.1 and (3) we have

$$G_\lambda^{\nabla_h}(\theta) = \bigcap_{n \ge 1} cl\left\{\bigcup_{m \ge n} G_{\lambda_m}^{\nabla_h}(\theta)\right\}.$$

If $\lambda = 0$, we can write it as

$$\lim_{n \to \infty} D_H(G_0^{\nabla_h}(\theta), \nabla_h G_{\lambda_n}^{\nabla_h}(\theta)) = 0,$$

where the nondecreasing sequence $\{\lambda_n\}$ tends to zero , and as a result of this

$$G_0^{\nabla_h}(\theta) = cl\left(\bigcup_{n \ge 1} G_{\lambda_n}^{\nabla_h}(\theta)\right).$$

Then from Theorem 2.1, $\exists$ an element $\tilde{u} \in \mathbb{E}_n$ such that

$$[\tilde{u}]^\lambda = G_\lambda^{\nabla_h}(\theta), \quad \lambda \in [0,1].$$

Let $\theta \in \mathbb{T}^{[a,b]}$, $\epsilon > 0$, $\delta > 0$ and $\theta - \hbar, \theta + \hbar \in N_T$ be as in supposition (b) then, we have

$$\begin{aligned}
&D_H\left(\frac{G_\lambda(\theta) \ominus_h G_\lambda(\rho(\theta))}{\nu(\theta)}, \tilde{u}^\lambda\right) \\
&\quad = D_H\left(\frac{G_\lambda(\theta) \ominus_h G_\lambda(\rho(\theta))}{\nu(\theta)}, G_\lambda^{\nabla_h}(\theta)\right) < \epsilon
\end{aligned}$$

Thus G is nabla-h differentiable. ■

Theorem 3.4: Let $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ defined by $G(\theta) = g(\theta) \odot u$ for all $\theta \in \mathbb{T}^{[a,b]}$, where $u \in \mathbb{E}_n$ and $g : \mathbb{T}^{[a,b]} \to \mathbb{T}_+$ is nabla differentiable at $\theta_0 \in \mathbb{T}^{[a,b]}$. If $g^\nabla(\theta_0) > 0$, then G is nabla-h differentiable at θ_0 with $G^{\nabla_h}(\theta_0) = g^\nabla(\theta_0) \odot u$.

Proof: Since g is nabla differentiable at θ_0, then g is continuous at θ_0.

Case(i): If θ_0 is left scattered then we have

$$g^{\nabla}(\theta_0) = \frac{g(\theta_0) - g(\rho(\theta_0))}{\nu(\theta_0)}.$$

Since $g^{\nabla}(\theta_0) > 0$, then

$$g(\theta_0) - g(\rho(\theta_0)) = g^{\nabla}(\theta_0)\nu(\theta_0) > 0.$$

It implies that

$$g(\theta_0) = g(\rho(\theta_0)) + g^{\nabla}(\theta_0)\nu(\theta_0).$$

Since $g(\theta_0) > 0, g(\rho(\theta_0)) > 0, g^{\nabla}(\theta_0)\nu(\theta_0) > 0$ and from Theorem 2.3(c), by multiplying the above equation with $u \in \mathbb{E}_n$ on both sides, we get

$$g(\theta_0) \odot u = [g(\rho(\theta_0)) \odot u] \oplus \left[g^{\nabla}(\theta_0)\nu(\theta_0) \odot u\right].$$

It implies that

$$[g(\theta_0) \odot u] \ominus_h [g(\rho(\theta_0)) \odot u] = [g^{\nabla}(\theta_0)\nu(\theta_0)] \odot u$$

and then

$$G(\theta_0) \ominus_h G(\rho(\theta_0)) = [g^{\nabla}(\theta_0)\nu(\theta_0)] \odot u.$$

Multiplying by $\dfrac{1}{\nu(\theta_0)}$ and using Theorem 2.3 (e), we get

$$\frac{G(\theta_0) \ominus_h G(\rho(\theta_0))}{\nu(\theta_0)} = [g^{\nabla}(\theta_0)] \odot u$$

and hence

$$G^{\nabla_h}(\theta_0) = g^{\nabla}(\theta_0) \odot u.$$

Case(ii): If θ_0 is left dense, then

$$g^{\nabla}(\theta_0) = g^{'}(\theta_0) > 0$$

and

$$g^{'}(\theta_0) = \lim_{\hbar \to 0} \frac{g(\theta_0) - g(\theta_0 - \hbar)}{\hbar}.$$

It follows that for $h > 0$ sufficiently small, we have

$$g(\theta_0) - g(\theta_0 - \hbar) = \phi(\theta_0, \hbar) > 0,$$

it implies

$$g(\theta_0) = g(\theta_0 - \hbar) + \phi(\theta_0, \hbar).$$

Since $g(\theta_0) > 0, g(\theta_0 - \hbar) > 0, \phi(\theta_0, \hbar) > 0$ and from Theorem 2.3(c), by multiplying the above equation with $u \in \mathbb{E}_n$ on both sides, we get

$$g(\theta_0) \odot u = [g(\theta_0 - \hbar) \odot u] \oplus [\phi(\theta_0, \hbar) \odot u].$$

It implies

$$G(\theta_0) = G(\theta_0 - \hbar) \oplus [(\phi(\theta_0, \hbar) \odot u].$$

Therefore, $G(\theta_0) \ominus_h G(\theta_0 - h)$ exists and hence G is left nabla-h differentiable at θ_0. Similarly, we can prove G is right nabla-h differentiable at θ_0.
It follows that, G is nabla-h differentiable at θ_0 with

$$G^{\nabla_h}(\theta_0) = g^{\nabla}(\theta_0) \odot u.$$

■

Example 3.2: Let us define $G(\theta) = \theta^2 \odot u, \forall\, \theta \in \mathbb{T}^{[1,5]}$, $G : \mathbb{T}^{[1,5]} \to \mathbb{E}_1$ is a fuzzy function and $u = (2, 3, 4)$ is the triangular fuzzy number. From Theorem 3.4, since $g(\theta) = \theta^2$ and $g^{\nabla}(\theta) = \theta + \rho(\theta) > 0, \forall\, \theta \in \mathbb{T}^{[1,5]}$, then $G(\theta)$ is nabla-h differentiable and $G^{\nabla}(\theta) = (\theta + \rho(\theta)) \odot u, \forall\, \theta \in \mathbb{T}^{[1,5]}$.

Theorem 3.5: Let $[G(\theta)]^{\lambda} = [g_{\lambda}(\theta), h_{\lambda}(\theta)], \lambda \in [0, 1]$ and $G : \mathbb{T}^{[a,b]} \to \mathbb{E}^1$ be nabla-h differentiable on $\mathbb{T}^{[a,b]}$. Then g_{λ} and h_{λ} are nabla differentiable on $\mathbb{T}^{[a,b]}$ and

$$[G^{\nabla_h}(\theta)]^{\lambda} = [g^{\nabla}_{\lambda}(\theta), h^{\nabla}_{\lambda}(\theta)].$$

Proof: If G is nabla-h differentiable at $\theta \in \mathbb{T}^{[a,b]}_k$ and θ is left scaterred, then for any $\lambda \in [0, 1]$,

$$[G(\theta) \ominus_h G(\rho(\theta))]^{\lambda} = [g_{\lambda}(\theta) - g_{\lambda}(\rho(\theta)), h_{\lambda}(\theta) - h_{\lambda}(\rho(\theta))].$$

Multiplying by $\dfrac{1}{\nu(\theta)}$, we get

$$\begin{aligned} &\frac{1}{\nu(\theta)} \odot [G(\theta) \ominus_h G(\rho(\theta))]^{\lambda} \\ &= \frac{1}{\nu(\theta)} \odot ([g_{\lambda}(\theta)), h_{\lambda}(\theta)] \ominus [g_{\lambda}(\rho(\theta)), h_{\lambda}(\rho(\theta))]) \\ &= \left[\frac{g_{\lambda}(\theta) - g_{\lambda}(\rho(\theta))}{\nu(\theta)}, \frac{h_{\lambda}(\theta) - h_{\lambda}(\rho(\theta))}{\nu(\theta)}\right] \\ &= \left[g^{\nabla}_{\lambda}(\theta), h^{\nabla}_{\lambda}(\theta)\right]. \end{aligned}$$

If G is nabla-h differentiable at $\theta \in \mathbb{T}^{[a,b]}_k$ and θ is ld-point, then for any $\lambda \in [0, 1]$.

$$[G(\rho(\theta)) \ominus_h G(\theta - \hbar)]^{\lambda} = [g_{\lambda}(\theta) - g_{\lambda}(\theta - \hbar), h_{\lambda}(\theta) - h_{\lambda}(\theta - \hbar)].$$

Dividing by $\hbar > 0$ and taking limit as $\hbar \to 0^+$, we get

$$\begin{aligned} &\lim_{\hbar \to 0^+} \frac{1}{\hbar}[G(\theta) \ominus_h G(\theta - \hbar)]^{\lambda} \\ &= \left[\lim_{\hbar \to 0^+} \frac{g_{\lambda}(\theta) - g_{\lambda}(\theta - \hbar)}{\hbar}, \lim_{\hbar \to 0^+} \frac{h_{\lambda}(\theta) - h_{\lambda}(\theta - \hbar)}{\hbar}\right] \\ &= [g^{\nabla_h}_{\lambda}(\theta), h^{\nabla_h}_{\lambda}(\theta)]. \end{aligned}$$

Similarly, we can prove

$$\begin{aligned} &\lim_{\hbar \to 0^+} \frac{1}{\hbar}[G(\theta + \hbar) \ominus_h G(\rho(\theta)]^{\lambda} \\ &= [g^{\nabla_h}_{\lambda}(\theta), h^{\nabla_h}_{\lambda}(\theta)]. \end{aligned}$$

Thus, g_{λ} and h_{λ} are nabla differentiable on $\mathbb{T}^{[a,b]}$ and

$$[G^{\nabla_h}(\theta)]^{\lambda} = [g^{\nabla}_{\lambda}(\theta), h^{\nabla}_{\lambda}(\theta)].$$

■

Example 3.3: Consider the fuzzy function $G(\theta)$ as in Example 3.2. Then $u^{\lambda} = [2 + \lambda,\ 4 - \lambda]$ is λ-level set of u and

$$\begin{aligned} [G(\theta)]^{\lambda} &= [g_{\lambda}(\theta), h_{\lambda}(\theta)] \\ &= \theta^2 \odot [2 + \lambda,\ 4 - \lambda] \\ &= [\theta^2(2 + \lambda),\ \theta^2(4 - \lambda)]. \end{aligned}$$

From Example 3.2, $G(\theta)$ is nabla-h differentiable and $G^{\nabla_h}(\theta) = (\theta + \rho(\theta)) \odot u$. Clearly, $g^{\nabla}_{\lambda}(\theta), h^{\nabla}_{\lambda}(\theta)$ are nabla differentiable and $g^{\nabla}_{\lambda}(\theta) = (\theta + \rho(\theta))(2 + \lambda),\ h^{\nabla}_{\lambda}(\theta) = (\theta + \rho(\theta))(4 - \lambda)$. From Theorem 3.5, we have

$$\begin{aligned} [G^{\nabla_h}(\theta)]^{\lambda} &= \left[g^{\nabla}_{\lambda}(\theta), h^{\nabla}_{\lambda}(\theta)\right] \\ &= [(\theta + \rho(\theta))(2 + \lambda),\ (\theta + \rho(\theta))(4 - \lambda)] \\ &= (\theta + \rho(\theta)) \odot [2 + \lambda,\ 4 - \lambda] \\ &= (\theta + \rho(\theta)) \odot u^{\lambda}. \end{aligned}$$

If $\mathbb{T}=\mathbb{R}$, then $\rho(\theta)=\theta$ and

$$G^{\nabla_h(\theta)}=(\theta+\rho(\theta))\odot u^{\lambda}=2\theta\odot u^{\lambda}.$$

If $\mathbb{T}=q^{\mathbb{N}}$, then $\rho(\theta)=\frac{\theta}{q}$ and

$$\begin{aligned}G^{\nabla_h(\theta)}&=(\theta+\rho(\theta))\odot u^{\lambda}\\&=(\theta+\frac{\theta}{q})\odot u^{\lambda}.\end{aligned}$$

Now, we obtain the nabla-h derivatives of addition, scalar multiplication and product of fuzzy nabla Hukuhara differentiable (nabla-h differentiable) functions on time scales.

Theorem 3.6: Let $G,H:\mathbb{T}^{[a,b]}\to\mathbb{E}_n$ be nabla-h differentiable at $\theta\in\mathbb{T}_k^{[a,b]}$. Then

(a) If $G\oplus H:\mathbb{T}^{[a,b]}\to\mathbb{E}_n$ is nabla-h differentiable at $\theta\in\mathbb{T}_k^{[a,b]}$, then

$$(G\oplus H)^{\nabla_h}(\theta)=G^{\nabla_h}(\theta)\oplus H^{\nabla_h}(\theta).$$

(b) For any scalar $\gamma\geq 0$, $\gamma\odot G:\mathbb{T}^{[a,b]}\to\mathbb{E}_n$ is nabla-h differentiable at θ with

$$(\gamma\odot G)^{\nabla_h}(\theta)=\gamma\odot G^{\nabla_h}(\theta).$$

(c) If $GH:\mathbb{T}^{[a,b]}\to\mathbb{E}_n$ is nabla-h differentiable at $\theta\in\mathbb{T}_k^{[a,b]}$, then

$$\begin{aligned}(GH)^{\nabla_h}(\theta)&=G(\rho(\theta))H^{\nabla_h}(\theta)\oplus H(\theta)G^{\nabla_h}(\theta)\\&=G(\theta)H^{\nabla_h}(\theta)\oplus H(\rho(\theta))G^{\nabla_h}(\theta).\end{aligned}$$

Proof: Let G and H be nabla-h differentiable at $\theta\in\mathbb{T}_k^{[a,b]}$.

(a) If θ is left dense and G,H are nabla-h differentiable at θ then from Theorem 3.1(c), we have

$$\begin{aligned}\lim_{\hbar\to 0}\frac{G(\theta+\hbar)\ominus_h G(\theta)}{\hbar}&=\lim_{\hbar\to 0}\frac{G(\theta)\ominus_h G(\theta-\hbar)}{\hbar}\\&=G^{\nabla_h}(\theta)\end{aligned}$$

and

$$\begin{aligned}\lim_{\hbar\to 0}\frac{H(\theta+\hbar)\ominus_h H(\theta)}{\hbar}&=\lim_{\hbar\to 0}\frac{H(\theta)\ominus_h H(\theta-\hbar)}{\hbar}\\&=H^{\nabla_h}(\theta).\end{aligned}$$

Now, we consider

$$\begin{aligned}&\lim_{\hbar\to 0}\frac{(G\oplus H)(\theta+\hbar)\ominus_h(G\oplus H)(\theta)}{\hbar}\\&=\lim_{\hbar\to 0}\frac{1}{\hbar}\odot[(G(\theta+\hbar)\ominus_h G(\theta))\\&\qquad\oplus(H(\theta+\hbar)\ominus_h H(\theta))]\\&=\lim_{\hbar\to 0}\frac{G(\theta+\hbar)\ominus_h G(\theta)}{\hbar}\oplus\lim_{\hbar\to 0}\frac{H(\theta+\hbar)\ominus_h H(\theta)}{\hbar}\\&=G^{\nabla_h}(\theta)\oplus H^{\nabla_h}(\theta).\end{aligned}$$

Similarly, we can prove

$$\begin{aligned}&\lim_{\hbar\to 0}\frac{(G\oplus H)(\theta)\ominus_h(G\oplus H)(\theta-\hbar)}{\hbar}\\&\qquad=G^{\nabla_h}(\theta)\oplus H^{\nabla_h}(\theta).\end{aligned}$$

Therefore, from Theorem 3.1(c) $G\oplus H$ is nabla-h differentiable at θ and

$$(G\oplus H)^{\nabla_h}(\theta)=G^{\nabla_h}(\theta)\oplus H^{\nabla_h}(\theta).$$

If θ is left scattered and G,H are nabla-h differentiable at θ, then from theorem 3.1(a) & (b), we have

$$\frac{G(\theta)\ominus_h G(\rho(\theta))}{\nu(\theta)}=G^{\nabla_h}(\theta)$$

and

$$\frac{H(\theta)\ominus_h H(\rho(\theta))}{\nu(\theta)}=H^{\nabla_h}(\theta).$$

Clearly, $G\oplus H$ is right continuous at θ and from Theorem 3.1(b), we have

$$\begin{aligned}(G\oplus H)^{\nabla_h}(\theta)&=\frac{(G\oplus H)(\theta)\ominus_h(G\oplus H)(\rho(\theta))}{\nu(\theta)}\\&=\frac{(G(\theta)\ominus_h G(\rho(\theta))\oplus(H(\theta)\ominus_h H(\rho(\theta)))}{\nu(\theta)}\\&=\frac{G(\theta)\ominus_h G(\rho(\theta))}{\nu(\theta)}\oplus\frac{H(\theta)\ominus_h H(\rho(\theta))}{\nu(\theta)}\\&=G^{\nabla_h}(\theta)\oplus H^{\nabla_h}(\theta).\end{aligned}$$

(b) For $\gamma=0$, the result is obvious. Now let us assume that $\gamma>0$, since G is nabla-h differentiable at $\theta\in\mathbb{T}_k^{[a,b]}$ and θ is left dense then from Theorem 3.1(c),

$$\begin{aligned}&\lim_{\hbar\to 0}\frac{\gamma\odot G(\theta+\hbar)\ominus_h\gamma\odot G(\theta)}{\hbar}\\&\qquad=\gamma\odot\lim_{\hbar\to 0}\frac{G(\theta+\hbar)\ominus_h G(\theta)}{\hbar}\\&\qquad=\gamma\odot G^{\nabla_h}(\theta).\end{aligned}$$

Similarly, we can prove

$$\lim_{\hbar\to 0}\frac{\gamma\odot G(\theta)\ominus_h\gamma\odot G(\theta-\hbar)}{\hbar}=\gamma\odot G^{\nabla_h}(\theta).$$

If θ is left scattered, then from Theorem 3.1(a) & (b), $\gamma\odot G$ is right continuous at θ and

$$\begin{aligned}(\gamma\odot G)^{\nabla_h}(\theta)&=\frac{(\gamma\odot G)(\theta)\ominus_h(\gamma\odot G)(\rho(\theta))}{\nu(\theta)}\\&=\frac{\gamma\odot G(\theta)\ominus_h\gamma\odot G(\rho(\theta))}{\nu(\theta)}\\&=\gamma\odot\frac{G(\theta)\ominus_h G(\rho(\theta))}{\nu(\theta)}\\&=\gamma\odot G^{\nabla_h}(\theta).\end{aligned}$$

(c) Since G,H are nabla-h differentiable and if θ is left dense, then from Theorem 3.1(c), we have

$$\begin{aligned}&\lim_{\hbar\to 0}\frac{GH(\theta)\ominus_h GH(\theta-\hbar)}{\hbar}\\&=\lim_{\hbar\to 0}\frac{1}{\hbar}\odot[G(\theta)[(H(\theta)\ominus_h H(\theta-\hbar)]\\&\qquad\oplus[G(\theta)\ominus_h G(\theta-\hbar)]H(\theta)]\\&=G(\theta)\lim_{\hbar\to 0}\frac{H(\theta)\ominus_h H(\theta-\hbar)}{\hbar}\\&\qquad\oplus H(\theta)\lim_{\hbar\to 0}\frac{G(\theta)\ominus_h G(\theta-\hbar)}{\hbar}\\&=G(\theta)H^{\nabla_h}(\theta)\oplus G^{\nabla_h}(\theta)H(\theta).\end{aligned}$$

Similarly, we get

$$\begin{aligned}&\lim_{\hbar\to 0}\frac{(GH)(\theta+\hbar)\ominus_h(GH)(\theta)}{\hbar}\\&\qquad=G(\theta)H^{\nabla_h}(\theta)\oplus G^{\nabla_h}(\theta)H(\theta).\end{aligned}$$

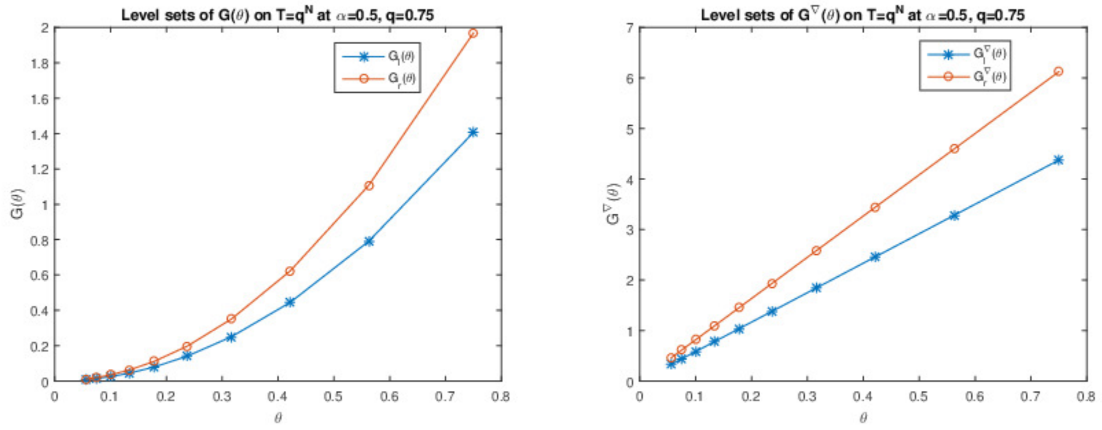

Fig. 1. Graphical representation of level sets $G(\theta)$ and $G^{\nabla}(\theta)$ for Example 3.3

If θ is left scattered then from Theorem 3.1(a) & (b), GH is right continuous at θ and

$$
\begin{aligned}
(GH)^{\nabla_h}(\theta) &= \frac{(GH)(\theta) \ominus_h (GH)(\rho(\theta))}{\nu(\theta)} \\
&= \frac{G(\theta)H(\theta) \ominus_h G(\rho(\theta))H(\rho(\theta))}{\nu(\theta)} \\
&\qquad \oplus \frac{G(\theta)H(\rho(\theta)) \ominus_h G(\theta)H(\rho(\theta))}{\nu(\theta)} \\
&= G(\theta)\left[\frac{H(\theta) \ominus_h H(\rho(\theta))}{\nu(\theta)}\right] \\
&\qquad \oplus H(\rho(\theta))\left[\frac{G(\theta) \ominus_h G(\rho(\theta))}{\nu(\theta)}\right] \\
&= G(\theta)H^{\nabla_h}(\theta) \oplus H(\rho(\theta))G^{\nabla_h}(\theta).
\end{aligned}
$$

Thus $(GH)^{\nabla_h}(\theta) = G(\theta)H^{\nabla_h}(\theta) \oplus H(\rho(\theta))G^{\nabla_h}(\theta)$ holds at θ. Similarly, we get the another product rule in (c) by interchanging G and H and which follows from the last equation.

■

IV. Conclusions

In this paper, we developed nabla Hukuhara derivative for fuzzy functions on time scales using Hukuhara difference and studied its properties. We propose to study generalizations of nabla Hukuhara differentials and integrals for fuzzy functions on time scales in our future work. Further, these concepts can be applied to study the fuzzy nabla dynamic equations on time scales.

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