On the Shephard Type Problems for General L_p-Projection Bodies

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***Abstract*—The notion of the L_p-projection body was introduced by Lutwak, Yang and Zhang. Whereafter, Ludwig proposed the asymmetric L_p-projection bodies, Haberl and Schuster introduced the general L_p-projection bodies. In this paper, associated with the L_p-geominimal surface area, we study the Shephard type problems for the general L_p-projection bodies.**

***Index Terms*—Shephard type problem, general L_p-projection body, L_p-geominimal surface area.**

I. Introduction

LET $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbf{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies in $\mathbf{R}^n$, we write $\mathcal{K}_o^n$ and $\mathcal{K}_{os}^n$, respectively. Let $\mathcal{S}_o^n$ denote the set of star bodies (about the origin) in $\mathbf{R}^n$. Let S^{n-1} denote the unit sphere and $V(K)$ denote the n-dimensional volume of the body K. For the standard unit ball B, its volume is written as $V(B) = \omega_n$.

For $K \in \mathcal{K}^n$, its support function, $h(K, \cdot) : \mathbf{R}^n \to \mathbf{R}$, is defined by (see [3])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \qquad x \in \mathbf{R}^n, \tag{1.1}$$

where $x \cdot y$ denotes the standard inner product of x and y.

The projection bodies were introduced by Minkowski at the previous century. For each $K \in \mathcal{K}^n$, the projection body, ΠK, of K is an origin-symmetric convex body whose support function is defined by (see [3])

$$h(\Pi K, u) = \frac{1}{2} \int_{S^{n-1}} |u \cdot v| dS(K, v),$$

for all $u \in S^{n-1}$. Here $S(K, \cdot)$ denotes the surface area measure of K.

Projection body is a central study object in the Brunn-Minkowski theory, a great deal of results are gathered in two good books (see [3], [16]). In 1964, Shephard [17] proposed the following problem about the projection bodies.

Problem 1.1 (Shephard problem). *Suppose $K, L \in \mathcal{K}^n$. If*

$$\Pi K \subseteq \Pi L,$$

is it true that

$$V(K) \leq V(L)?$$

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Remark 1.1. For centrally symmetric convex bodies K and L, Problem 1.1 was solved independently by Petty [12] and Schneider [15], who showed that the answer is affirmative if $n \leq 2$ and negative if $n \geq 3$. They also proved that Problem 1.1 has an affirmative answer if L is a projection body.

In 2000, Lutwak, Yang and Zhang [8] introduced the L_p-projection bodies as follows: For $K \in \mathcal{K}_o^n$ and $p \geq 1$, the L_p-projection body, $\Pi_p K$, is an origin-symmetric convex body whose support function is given by

$$h^p(\Pi_p K, u) = \alpha_{n,p} \int_{S^{n-1}} |u \cdot v|^p dS_p(K, v), \tag{1.2}$$

for all $u \in S^{n-1}$, where $\alpha_{n,p} = 1/n\omega_n c_{n-2,p}$ with $c_{n,p} = \omega_{n+p}/\omega_2\omega_n\omega_{p-1}$, and $S_p(K, \cdot)$ is the L_p-surface area measure of $K \in \mathcal{K}_o^n$ (see [6]). In particular, for $p = 1$, the convex body $\Pi_1 K$ is the projection body ΠK of K under the normalization of definition (1.2).

As a fundamental notion of L_p-projection body in L_p-Brunn-Minkowski theory. In recent years, it has paid considerable attentions (see [9], [11], [14], [19], [20], [21], [22]).

For $p \geq 1$, Ludwig [5] introduced the asymmetric L_p-projection bodies: For $K \in \mathcal{K}_o^n$, $p \geq 1$, the asymmetric L_p-projection body, $\Pi_p^+ K$, of K is defined by

$$h^p(\Pi_p^+ K, u) = 2\alpha_{n,p} \int_{S^{n-1}} (u \cdot v)_+^p dS_p(K, v), \tag{1.3}$$

where $(u, v)_+ = \max\{u \cdot v, 0\}$. Afterwords, Haberl and Schuster [4] defined

$$\Pi_p^- K = \Pi_p^+(-K). \tag{1.4}$$

Moreover, combined with function $\varphi_\tau : \mathbf{R} \to [0, +\infty)$ by $\varphi_\tau(t) = |t| + \tau t$ for $\tau \in [-1, 1]$, Ludwig [5], Haberl and Schuster [4] introduced general L_p-projection bodies as follows: For $K \in \mathcal{K}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$, the general L_p-projection body $\Pi_p^\tau K \in \mathcal{K}_o^n$ is defined by

$$h^p(\Pi_p^\tau K, u) = \alpha_{n,p}(\tau) \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p dS_p(K, v), \tag{1.5}$$

where

$$\alpha_{n,p}(\tau) = \frac{2\alpha_{n,p}}{(1+\tau)^p + (1-\tau)^p}.$$

The normalization is chosen such that $\Pi_p^\tau B = B$. Obviously, $\Pi_p^0 K = \Pi_p K$.

From (1.3), (1.4) and (1.5), Haberl and Schuster [4] deduced that for $K \in \mathcal{K}_o^n$, $p \geq 1$, $\tau \in [-1, 1]$ and all $u \in S^{n-1}$,

$$\begin{aligned} &h^p(\Pi_p^\tau K, u) \\ &= f_1(\tau) h^p(\Pi_p^+ K, u) + f_2(\tau) h^p(\Pi_p^- K, u), \end{aligned} \tag{1.6}$$

that is,

$$\Pi_p^\tau K = f_1(\tau)\cdot\Pi_p^+ K +_p f_2(\tau)\cdot\Pi_p^- K,$$

where $+_p$ denotes the L_p-Minkowski addition of convex bodies, and

$$f_1(\tau) = \frac{(1+\tau)^p}{(1+\tau)^p + (1-\tau)^p},$$

$$f_2(\tau) = \frac{(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p}.$$

From this, we easily know that

$$f_1(-\tau) = f_2(\tau), \quad f_2(-\tau) = f_1(\tau),$$
$$f_1(\tau) + f_2(\tau) = 1. \tag{1.7}$$

The general L_p-projection bodies belong to asymmetric L_p-Brunn-Minkowski theory. More results, also see [23], [24], [25]. In particular, Wang and Wan [23] researched the Shephard type problems of general L_p-projection bodies for volumes and L_p-affine surface areas, respectively.

Theorem 1.A. *Let $K \in \mathcal{K}_o^n$, $p \geq 1$ and $\tau \in [-1,1]$. If $L \in \mathcal{P}_p^{\tau,n}$ and $\Pi_p^\tau K \subseteq \Pi_p^\tau L$, then for $n > p \geq 1$,*

$$V(K) \leq V(L);$$

for $n < p$,

$$V(K) \geq V(L).$$

In each case, equality holds for $p = 1$ if and only if K is a translation of L, and for $p > 1$ if and only if $K = L$. Here $\mathcal{P}_p^{\tau,n}$ denotes the set of general L_p-projection bodies with a parameter τ.

Theorem 1.B. *Let $K \in \mathcal{F}_o^n$, $p \geq 1$ and $\tau \in [-1,1]$. If $L \in \mathcal{W}_p^{\tau,n}$ and $\Pi_p^\tau K \subseteq \Pi_p^\tau L$, then*

$$\Omega_p(K) \leq \Omega_p(L),$$

with equality for $p = 1$ if and only if K is a translation of L, and for $p > 1$ if and only if $K = L$. Here $\mathcal{W}_p^{\tau,n} = \{Q \in \mathcal{F}_o^n : \textit{there exists}\ \ Z \in \mathcal{P}_p^{\tau,n}\ \textit{with}\ f_p(Q,\cdot) = h(Z,\cdot)^{-(n+p)}\}$, where $f_p(Q,\cdot)$ is the L_p-curvature function of Q and $\mathcal{F}_o^n$ denotes the set of convex bodies in $\mathcal{K}_o^n$ with positive continuous L_p-curvature function.

In this article, we will research the Shephard type problems of the general L_p-projection bodies for L_p-geominimal surface areas. The notion of L_p-geominimal surface areas was introduced by Lutwak [7]. For $K \in \mathcal{K}_o^n$, $p \geq 1$, the L_p-geominimal surface area, $G_p(K)$, of K is defined by

$$\omega_n^{\frac{p}{n}} G_p(K) = \inf\{nV_p(K,Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\}, \tag{1.8}$$

where $V_p(M,N)$ denotes the L_p-mixed volume of $M, N \in \mathcal{K}_o^n$. More researches about L_p-geominimal surface areas, also see [10], [26], [27], [28], [29], [30].

In (1.8), if $Q \in \mathcal{P}_p^{\tau,n}$, then we define $G_p^o(K)$ by

$$\omega_n^{\frac{p}{n}} G_p^o(K) = \inf\{nV_p(K,Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{P}_p^{\tau,n}\}. \tag{1.9}$$

Combining with (1.9), we first give an affirmative answer of the Shephard type problem for general L_p-projection bodies.

Theorem 1.1. *Let $K, L \in \mathcal{K}_o^n$, $1 \leq p < n$ and $\tau \in [-1,1]$. If $\Pi_p^\tau K \subseteq \Pi_p^\tau L$, then*

$$G_p^o(K) \leq G_p^o(L),$$

with equality when $\Pi_p^\tau K = \Pi_p^\tau L$.

Let $\mathcal{C}_p^{\tau,n}$ denotes the set of all general L_p-centroid bodies (see [2]), thus $\mathcal{C}_p^{\tau,n} \subseteq \mathcal{K}_o^n$. If $Q \in \mathcal{C}_p^{\tau,n}$ in (1.8), then we write $G_p^\star(K)$ by

$$\omega_n^{\frac{p}{n}} G_p^\star(K) = \inf\{nV_p(K,Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{C}_p^{\tau,n}\}. \tag{1.10}$$

Based on (1.10), we give the other affirmative form of the Shephard type problems for the general L_p-projection bodies.

Theorem 1.2. *Let $K, L \in \mathcal{K}_o^n$, $1 \leq p < n$ and $\tau \in [-1,1]$. If $\Pi_p^\tau K \subseteq \Pi_p^\tau L$, then*

$$G_p^\star(K) \leq G_p^\star(L),$$

with equality when $\Pi_p^\tau K = \Pi_p^\tau L$.

Further, we also give a negative answer as follows:

Theorem 1.3. *Let $L \in \mathcal{K}_o^n$, $1 \leq p < n$ and $\tau \in (-1,1)$. If L is not origin-symmetric convex body, then there exists $K \in \mathcal{K}_o^n$ ($\tau = 0$, $K \in \mathcal{K}_{os}^n$), such that*

$$\Pi_p^\tau K \subset \Pi_p^\tau L,$$

but

$$G_p(K) > G_p(L).$$

In particular, if $\tau = 0$ in Theorem 1.3, the following result is obvious.

Corollary 1.1. *Let $L \in \mathcal{K}_o^n$, $1 \leq p < n$. If L is not a origin-symmetric convex body, then there exists $K \in \mathcal{K}_{os}^n$, such that*

$$\Pi_p K \subset \Pi_p L,$$

but

$$G_p(K) > G_p(L).$$

Corollary 1.1 shows the symmetric negative solutions of the Shephard type problem of L_p-projection bodies for the L_p-geominimal surface areas. Actually, by the general L_p-Blaschke bodies, we find the asymmetric negative solutions in Corollary 1.1, i.e., we generalize the scope of negative solutions in Corollary 1.1 from $\mathcal{K}_{os}^n$ to $\mathcal{K}_o^n$.

Theorem 1.4. *Let $L \in \mathcal{K}_o^n$ and $1 \leq p < n$. If L is not origin-symmetric convex body, then there exists $K \in \mathcal{K}_o^n$, such that*

$$\Pi_p K \subset \Pi_p L,$$

but

$$G_p(K) > G_p(L).$$

For more investigations of the Shephard type problems, we also see articles [1], [11], [13], [18], [23].

II. Preliminaries

A. Radial Function and Polar Body

If K is a compact star-shaped (about the origin) in $\mathbf{R}^n$, then its radial function, $\rho_K = \rho(K,\cdot) : \mathbf{R}^n \backslash \{0\} \to [0,\infty)$, is defined by (see [3], [16])

$$\rho(K,x) = \max\{\lambda \geq 0 : \lambda\cdot x \in K\}, \quad x \in \mathbf{R}^n\backslash\{0\}. \tag{2.1}$$

If $K \in \mathcal{K}_o^n$, the polar body, K^*, of K is defined by (see [3], [16])

$$K^* = \{x \in \mathbf{R}^n : x\cdot y \leq 1, y \in K\}.$$

From (1.1) and (2.1), it follows that if $K \in \mathcal{K}_o^n$, then

$$h(K^*, \cdot) = \frac{1}{\rho(K, \cdot)}, \quad \rho(K^*, \cdot) = \frac{1}{h(K, \cdot)}. \tag{2.2}$$

B. L_p-Mixed Volume and L_p-Dual Mixed Volume

For $K, L \in \mathcal{K}_o^n$, $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the L_p-Minkowski combination, $\lambda \cdot K +_p \mu \cdot L$, of K and L is defined by (see [6])

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K, \cdot)^p + \mu h(L, \cdot)^p,$$

where $+_p$ denotes the L_p-Minkowski addition, $\lambda \cdot K$ denotes the L_p-Minkowski scalar multiplication.

Together with L_p-Minkowski combination, Lutwak [6] introduced L_p mixed volume as follows: For $K, L \in \mathcal{K}_o^n$, $\varepsilon > 0$ and $p \geq 1$, the L_p mixed volume $V_p(K, L)$ is defined by

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V_p(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

Besides, Lutwak [6] also gave its integral formula:

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p dS_p(K, u). \tag{2.3}$$

Here $S_p(K, \cdot)$ is the L_p-surface area measure of K. It turns out that the measure $S_p(K, \cdot)$ is absolutely continuous with respect to $S(K, \cdot)$, and has Radon-Nikodym derivative (see [7])

$$\frac{dS_p(K, \cdot)}{dS(K, \cdot)} = h(K, \cdot)^{1-p}. \tag{2.4}$$

If $c > 0$, $n \neq p$, according to (2.4), we have

$$S_p(cK, \cdot) = c^{n-p} S_p(K, \cdot). \tag{2.5}$$

The L_p-dual mixed volume was introduced by Lutwak (see [7]). For $K, L \in \mathcal{S}_o^n$ and $p \geq 1$, the L_p-dual mixed volume, $\widetilde{V}_{-p}(K, L)$, of K and L is defined by (see [7])

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p}(u) \rho_L^{-p}(u) dS(u). \tag{2.6}$$

C. General L_p-Blaschke Bodies

According to the existence's theorem of L_p-Minkowski problem (see Theorem 9.2.3 in [16]), the L_p-Blaschke combinations of convex bodies was stated as follows: For $K, L \in \mathcal{K}_o^n$, $1 \leq p \neq n$, $\lambda, \mu \geq 0$ (not both zero), the L_p-Blaschke combination $\lambda \odot K \mp_p \mu \odot L \in \mathcal{K}_o^n$ of K, L is defined by

$$S_p(\lambda \odot K \mp_p \mu \odot L, \cdot) = \lambda S_p(K, \cdot) + \mu S_p(L, \cdot). \tag{2.7}$$

where $\mp_p$ denotes the L_p-Blaschke addition, and $\lambda \odot K$ denotes the L_p-Blaschke scalar multiplication.

If $K, L \in \mathcal{K}_{os}^n$, then definition (2.7) is owe to Lutwak [6].

Let $\lambda = f_1(\tau)$, $\mu = f_2(\tau)$ and $L = -K$ in (2.7), where $f_1(\tau)$ and $f_2(\tau)$ satisfy (1.7). We define the general L_p-Blaschke body, $\nabla_p^\tau K$, of $K \in \mathcal{K}_o^n$ by

$$\nabla_p^\tau K = f_1(\tau) \odot K \mp_p f_2(\tau) \odot (-K). \tag{2.8}$$

Obviously, by (1.7) and (2.8) we see that if $\tau = \pm 1$, then $\nabla_p^\tau K = \nabla_p^\pm K = \pm K$.

D. General L_p-Centroid Bodies

In 2015, Feng et al. [2] introduced the general L_p-centroid body as follows: For $K \in \mathcal{S}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$, the general L_p-centroid body, $\Gamma_p^\tau K$, of K is a convex body whose support function is defined by

$$\begin{aligned} &h^p_{\Gamma_p^\tau K}(u) \\ &= \frac{2}{c_{n,p}(\tau)(n+p)V(K)} \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p \rho_K(v)^{n+p} dv, \end{aligned}$$

where

$$c_{n,p}(\tau) = c_{n,p}[(1+\tau)^p + (1-\tau)^p].$$

III. Results and Proofs

In this part, we will give the proofs of Theorems 1.1-1.4. First, in order to prove theorem 1.1, the following lemma is required.

Lemma 3.1 ([24]). *If $K, L \in \mathcal{K}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$, then*

$$V_p(K, \Pi_p^\tau L) = V_p(L, \Pi_p^\tau K). \tag{3.1}$$

Proof of Theorem 1.1. Since $K, L \in \mathcal{K}_o^n$, $1 \leq p < n$, if $\Pi_p^\tau K \subseteq \Pi_p^\tau L$, then for all $u \in S^{n-1}$,

$$h(\Pi_p^\tau K, u) \leq h(\Pi_p^\tau L, u). \tag{3.2}$$

From (2.3), (3.1) and (3.2), we have for any $M \in \mathcal{K}_o^n$,

$$\begin{aligned} &V_p(K, \Pi_p^\tau M) \\ &= V_p(M, \Pi_p^\tau K) \\ &\leq V_p(M, \Pi_p^\tau L) \\ &= V_p(L, \Pi_p^\tau M). \end{aligned} \tag{3.3}$$

Since $\Pi_p^\tau M \in \mathcal{P}_p^{\tau,n}$, thus by (1.9) and (3.3), we get

$$\begin{aligned} &\omega_n^{\frac{p}{n}} G_p^o(K) \\ &= \inf\{nV_p(K, \Pi_p^\tau M) V(\Pi_p^{\tau,*} M)^{\frac{p}{n}} : \Pi_p^\tau M \in \mathcal{P}_p^{\tau,n}\} \\ &\leq \inf\{nV_p(L, \Pi_p^\tau M) V(\Pi_p^{\tau,*} M)^{\frac{p}{n}} : \Pi_p^\tau M \in \mathcal{P}_p^{\tau,n}\} \\ &= \omega_n^{\frac{p}{n}} G_p^o(L), \end{aligned}$$

i.e.,

$$G_p^o(K) \leq G_p^o(L).$$

Equality holds when $\Pi_p^\tau K = \Pi_p^\tau L$.

Lemma 3.2 ([8]). *If $M \in \mathcal{K}_o^n$, $N \in \mathcal{S}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$, then*

$$V_p(M, \Gamma_p^\tau N) = \frac{\omega_n}{V(N)} \widetilde{V}_{-p}(N, \Pi_p^{\tau,*} M). \tag{3.4}$$

Proof of Theorem 1.2. Since $K, L \in \mathcal{K}_o^n$, $1 \leq p < n$, if $\Pi_p^\tau K \subseteq \Pi_p^\tau L$, then $\Pi_p^{\tau,*} K \supseteq \Pi_p^{\tau,*} L$. From (2.2), (2.6) and (3.4), for any $N \in \mathcal{S}_o^n$, we obtain

$$\begin{aligned} &V_p(K, \Gamma_p^\tau N) \\ &= \frac{\omega_n}{V(N)} \widetilde{V}_{-p}(N, \Pi_p^{\tau,*} K) \\ &\leq \frac{\omega_n}{V(N)} \widetilde{V}_{-p}(N, \Pi_p^{\tau,*} L) \\ &= V_p(L, \Gamma_p^\tau N). \end{aligned} \tag{3.5}$$

Taking $Q = \Gamma_p^\tau N$, $N \in S_o^n$, thus by (1.10) and (3.5), we have

$$G_p^\star(K) \leq G_p^\star(L).$$

Equality holds when $\Pi_p^\tau K = \Pi_p^\tau L$.

Lemma 3.3. *If $K \in \mathcal{K}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$, then*

$$G_p(\nabla_p^\tau K) \geq G_p(K), \tag{3.6}$$

with equality for $\tau \in (-1, 1)$ if and only if K is origin-symmetric. For $\tau = \pm 1$, (3.6) becomes an equality.

Proof. By (1.8), (2.8), (2.3), (2.7) and (1.7), we have

$$\begin{aligned}
&\omega_n^{\frac{p}{n}} G_p(\nabla_p^\tau K)\\
&= \inf\{nV_p(\nabla_p^\tau K, Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\}\\
&= \inf\{nV_p(f_1(\tau) \odot K \mp_p f_2(\tau) \odot (-K), Q)V(Q^*)^{\frac{p}{n}} :\\
&\quad Q \in \mathcal{K}_o^n\}\\
&= \inf\{n(f_1(\tau)V_p(K, Q) + f_2(\tau)V_p(-K, Q))V(Q^*)^{\frac{p}{n}} :\\
&\quad Q \in \mathcal{K}_o^n\}\\
&\geq \inf\{nf_1(\tau)V_p(K, Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\}\\
&\quad + \inf\{nf_2(\tau)V_p(-K, Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\}\\
&= f_1(\tau) \inf\{nV_p(K, Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\}\\
&\quad + f_2(\tau) \inf\{nV_p(K, -Q)V((-Q)^*)^{\frac{p}{n}} : -Q \in \mathcal{K}_o^n\}\\
&= \omega_n^{\frac{p}{n}} (f_1(\tau)G_p(K) + f_2(\tau)G_p(K))\\
&= \omega_n^{\frac{p}{n}} G_p(K).
\end{aligned}$$

For any $Q \in \mathcal{K}_o^n$ and $\tau \in (-1, 1)$, with equality if and only if $f_1(\tau)V_p(K, Q)$ and $f_1(\tau)V_p(-K, Q)$ are proportional, i.e., $f_1(\tau)S_p(K, .)$ and $f_2(\tau)S_p(-K, .)$ are proportional. This together with Lutwak's result (see [6]) imples that equality holds in (3.6) if and only if K and $-K$ are dilates, namely, K is origin-symmetric.

Obviously, by $\nabla_p^{\pm 1} K = \pm K$ we see that if $\tau = \pm 1$, then (3.6) is an equality.

Lemma 3.4. *If $K \in \mathcal{K}_o^n$, $p \geq 1$ and $\tau \in (-1, 1)$, then*

$$\Pi_p^+ \nabla_p^\tau K = \Pi_p^\tau K, \tag{3.7}$$

and

$$\Pi_p^- \nabla_p^\tau K = \Pi_p^{-\tau} K. \tag{3.8}$$

Proof. By (1.3), (2.8), (2.7), (1.4) and (1.6), we get for all $u \in S^{n-1}$,

$$\begin{aligned}
&h^p(\Pi_p^+ \nabla_p^\tau K, u)\\
&= 2\alpha_{n,p} \int_{S^{n-1}} (u \cdot v)_+^p dS_p(\nabla_p^\tau K, v)\\
&= 2\alpha_{n,p} \int_{S^{n-1}} (u \cdot v)_+^p dS_p(f_1(\tau) \odot K \mp_p f_2(\tau) \odot (-K), v)\\
&= 2\alpha_{n,p} \int_{S^{n-1}} (u \cdot v)_+^p d[f_1(\tau)S_p(K, v) + f_2(\tau)S_p((-K), v)]\\
&= f_1(\tau)h^p(\Pi_p^+ K, u) + f_2(\tau)h^p(\Pi_p^+(-K), u)\\
&= f_1(\tau)h^p(\Pi_p^+ K, u) + f_2(\tau)h^p(\Pi_p^- K, u)\\
&= h^p(\Pi_p^\tau K, u).
\end{aligned}$$

This immediately gives (3.7).

Similarly, we have for all $u \in S^{n-1}$,

$$h^p(\Pi_p^- \nabla_p^\tau K, u) = h^p(\Pi_p^{-\tau} K, u).$$

This yields (3.8).

Lemma 3.5. *Let $L \in \mathcal{K}_o^n$, $1 \leq p < n$ and $\tau \in (-1, 1)$. If L is not origin-symmetric convex body, then there exists $K \in \mathcal{K}_o^n$ ($\tau = 0$, $K \in \mathcal{K}_{os}^n$), such that*

$$\Pi_p^+ K \subset \Pi_p^\tau L, \quad \Pi_p^- K \subset \Pi_p^{-\tau} L,$$

but

$$G_p(K) > G_p(L).$$

Proof. Since L is not origin-symmetric and $\tau \in (-1, 1)$, thus by Lemma 3.3, we know $G_p(\nabla_p^\tau L) > G_p(L)$. Choose $\varepsilon > 0$, such that $1 - \varepsilon > 0$, and $K = (1 - \varepsilon)\nabla_p^\tau L \in \mathcal{K}_o^n$ satisfies

$$G_p(K) = G_p((1 - \varepsilon)\nabla_p^\tau L) > G_p(L).$$

But by (1.5) and (2.5), we have

$$\Pi_p^\tau cK = c^{n-p} \Pi_p^\tau K, \quad (c > 0). \tag{3.9}$$

Therefore, for $n > p$, by (3.7), (3.8) and (3.9), we respectively have

$$\begin{aligned}
\Pi_p^+ K &= \Pi_p^+[(1 - \varepsilon)\nabla_p^\tau L] = (1 - \varepsilon)^{n-p} \Pi_p^+ \nabla_p^\tau L\\
&= (1 - \varepsilon)^{n-p} \Pi_p^\tau L \subset \Pi_p^\tau L,
\end{aligned}$$

and

$$\begin{aligned}
\Pi_p^- K &= \Pi_p^-[(1 - \varepsilon)\nabla_p^\tau L] = (1 - \varepsilon)^{n-p} \Pi_p^- \nabla_p^\tau L\\
&= (1 - \varepsilon)^{n-p} \Pi_p^{-\tau} L \subset \Pi_p^{-\tau} L.
\end{aligned}$$

This obtains the desired result.

Proof of Theorem 1.3. Since L is not origin-symmetric and $\tau \in (-1, 1)$, thus by Lemma 3.5, there exists $K \in \mathcal{K}_o^n$, such that

$$\Pi_p^+ K \subset \Pi_p^\tau L, \quad \Pi_p^- K \subset \Pi_p^{-\tau} L,$$

but

$$G_p(K) > G_p(L).$$

Because $\tau \in (-1, 1)$ is equivalent to $-\tau \in (-1, 1)$, we have $\Pi_p^+ K \subset \Pi_p^\tau L$, $\Pi_p^- K \subset \Pi_p^{-\tau} L$, these imply

$$\Pi_p^+ K \subset \Pi_p^\tau L, \quad \Pi_p^- K \subset \Pi_p^\tau L.$$

From these and together with (1.6) and (1.7), we obtain for any $u \in S^{n-1}$,

$$\begin{aligned}
&h(\Pi_p^\tau K, u)^p\\
&= f_1(\tau)h(\Pi_p^+ K, u)^p + f_2(\tau)h(\Pi_p^- K, u)^p\\
&< f_1(\tau)h(\Pi_p^\tau L, u)^p + f_2(\tau)h(\Pi_p^\tau L, u)^p\\
&= h(\Pi_p^\tau L, u)^p,
\end{aligned}$$

i.e.,

$$\Pi_p^\tau K \subset \Pi_p^\tau L.$$

This yields desired result.

Lemma 3.6. *Let $K, L \in \mathcal{K}_o^n$, $p \geq 1$ and $\tau \in [-1, 1]$, then*

$$\Pi_p(\nabla_p^\tau K) = \Pi_p K. \tag{3.10}$$

Proof. By (1.2), (2.8) and (2.7), we obtain for any $u \in S^{n-1}$,

$$\begin{aligned}
&h^p(\Pi_p(\nabla_p^\tau K), u)\\
&= \alpha_{n,p} \int_{S^{n-1}} |u{\cdot}v|^p dS_p(\nabla_p^\tau K, v)\\
&= \alpha_{n,p} \int_{S^{n-1}} |u \cdot v|^p dS_p(f_1(\tau) \odot K \mp_p f_2(\tau) \odot (-K), v)\\
&= \alpha_{n,p} \int_{S^{n-1}} |u{\cdot}v|^p d[f_1(\tau) S_p(K, v) + f_2(\tau) S_p((-K), v)]\\
&= f_1(\tau) h^p(\Pi_p K, u) + f_2(\tau) h^p(\Pi_p(-K), u).
\end{aligned}$$

Note that $\Pi_p K = \Pi_p(-K)$, thus

$$h^p(\Pi_p(\nabla_p^\tau K), u) = h^p(\Pi_p K, u),$$

i.e.,

$$\Pi_p(\nabla_p^\tau K) = \Pi_p K.$$

This yields (3.10). *Proof of Theorem 1.4.* Since L is not origin-symmetric, from Lemma 3.3, we know for $\tau \in (-1, 1)$,

$$G_p(\nabla_p^\tau L) > G_p(L).$$

Choose $0 < \varepsilon < 1$, such that

$$G_p((1-\varepsilon)\nabla_p^\tau L) > G_p(L).$$

Let $K = (1-\varepsilon)\nabla_p^\tau L$, then $K \in \mathcal{K}_o^n$ (for $\tau \neq 0$, $K \in \mathcal{K}_o^n \backslash \mathcal{K}_{os}^n$; for $\tau = 0$, $K \in \mathcal{K}_{os}^n$) and

$$G_p(K) > G_p(L).$$

But by (1.2) and (2.5), we have

$$\Pi_p cK = c^{n-p} \Pi_p K, \quad (c > 0). \tag{3.11}$$

Hence, for $n > p$, (3.10) and (3.11) mean that

$$\begin{aligned}
\Pi_p K = \Pi_p((1-\varepsilon)\nabla_p^\tau L) &= (1-\varepsilon)^{n-p} \Pi_p \nabla_p^\tau L\\
&= (1-\varepsilon)^{n-p} \Pi_p L \subset \Pi_p L.
\end{aligned}$$

This obtains the desired result.

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