On the Shephard Type Problems for General $L_p$-Projection Bodies

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Abstract—The notion of the $L_p$-projection body was introduced by Lutwak, Yang and Zhang. Whereafter, Ludwig proposed the asymmetric $L_p$-projection bodies, Haberl and Schuster introduced the general $L_p$-projection bodies. In this paper, associated with the $L_p$-geominimal surface area, we study the Shephard type problems for the general $L_p$-projection bodies.

Index Terms—Shephard type problem, general $L_p$-projection body, $L_p$-geominimal surface area.

I. INTRODUCTION

Let $K^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we write $K^n_o$ and $K^n_{os}$ respectively. Let $S^n_o$ denote the set of star bodies (about the origin) in $\mathbb{R}^n$. Let $S^{n-1}_o$ denote the unit sphere and $V(K)$ denote the $n$-dimensional volume of the body $K$. For the standard unit ball $B$, its volume is written as $V(B) = \omega_n$.

For $K \subseteq K^n_o$, its support function, $h(K, \cdot) : \mathbb{R}^n \to \mathbb{R}$, is defined by (see [3])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

(1.1)

where $x \cdot y$ denotes the standard inner product of $x$ and $y$.

The projection bodies were introduced by Minkowski at the previous century. For each $K \subseteq K^n_o$, the projection body, $\Pi K$, of $K$ is an origin-symmetric convex body whose support function is defined by (see [3])

$$h(\Pi K, u) = \frac{1}{2} \int_{S^{n-1}} |u \cdot v| dS(K, v),$$

for all $u \in S^{n-1}_o$. Here $S(K, \cdot)$ denotes the surface area measure of $K$.

Projection body is a central study object in the Brunn-Minkowski theory, a great deal of results are gathered in two good books (see [3], [16]). In 1964, Shephard [17] proposed the following problem about the projection bodies.

**Problem 1.1 (Shephard problem).** Suppose $K, L \subseteq K^n_o$. If $\Pi K \subseteq \Pi L$,

is it true that

$$V(K) \leq V(L)?$$

**Remark 1.1.** For centrally symmetric convex bodies $K$ and $L$, Problem 1.1 was solved independently by Petty [12] and Schneider [15], who showed that the answer is affirmative if $n \leq 2$ and negative if $n \geq 3$. They also proved that Problem 1.1 has an affirmative answer if $L$ is a projection body.

In 2000, Lutwak, Yang and Zhang [8] introduced the $L_p$-projection bodies as follows: For $K \subseteq K^n_o$ and $p \geq 1$, the $L_p$-projection body $\Pi_p K$ is an origin-symmetric convex body whose support function is given by

$$h^p(\Pi_p K, u) = \alpha_{n, p} \int_{S^{n-1}} |u \cdot v|^p dS_p(K, v),$$

(1.2)

for all $u \in S^{n-1}_o$, where $\alpha_{n, p} = 1/n\omega_n c_{n-2, p}$ with $c_{n, p} = \omega_{n+p}/\omega_{n}\omega_{n-1}p^{-1}$, and $S_p(K, \cdot)$ is the $L_p$-surface area measure of $K \subseteq K^n_o$ (see [6]). In particular, for $p = 1$, the convex body $\Pi_1 K$ is the projection body $\Pi K$ of $K$ under the normalization of definition (1.2).

As a fundamental notion of $L_p$-projection body in $L_p$-Brunn-Minkowski theory. In recent years, it has paid considerable attentions (see [9], [11], [14], [19], [20], [21], [22]).

For $p \geq 1$, Ludwig [5] introduced the asymmetric $L_p$-projection bodies: For $K \subseteq K^n_o$, $p \geq 1$, the asymmetric $L_p$-projection body, $\Pi_p^+ K$, of $K$ is defined by

$$h^p(\Pi_p^+ K, u) = 2\alpha_{n, p} \int_{S^{n-1}} (u \cdot v)_+^p dS_p(K, v),$$

(1.3)

where $(u, v)_+ = \max\{u, v, 0\}$. Afterwords, Haberl and Schuster [4] defined

$$\Pi_p^+ K = \Pi_p^+(−K),$$

(1.4)

Moreover, combined with function $\varphi_\tau : \mathbb{R} \to [0, +\infty)$ by $\varphi_\tau(t) = |t| + \tau t$ for $\tau \in [-1, 1]$, Ludwig [5], Haberl and Schuster [4] introduced general $L_p$-projection bodies as follows: For $K \subseteq K^n_o$, $p \geq 1$ and $\tau \in [-1, 1]$, the general $L_p$-projection body $\Pi_p K \subseteq K^n_o$ is defined by

$$h^p(\Pi_p K, u) = \alpha_{n, p}(\tau) \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p dS_p(K, v),$$

(1.5)

where

$$\alpha_{n, p}(\tau) = \frac{2\alpha_{n, p}}{(1 + \tau)^p + (1 - \tau)^p}.$$

The normalization is chosen such that $\Pi_p^+ B = B$. Obviously, $\Pi^o_p K = \Pi_p K$.

From (1.3), (1.4) and (1.5), Haberl and Schuster [4] deduced that for $K \subseteq K^n_o$, $p \geq 1$, $\tau \in [-1, 1]$ and all $u \in S^{n-1}_o$,

$$h^p(\Pi_p^+ K, u) = f_1(\tau) h^p(\Pi_p K, u) + f_2(\tau) h^p(\Pi_p K, u),$$

(1.6)
K was introduced by Lutwak [7]. For surface areas. The notion of the general Shephard type problem for general $L_p$-projection bodies. More researches about the Shephard type problems of general $L_p$-projection bodies for volumes and $L_p$-affine surface areas, respectively.

**Theorem 1.1.** Let $K \in \mathcal{K}_p^n$, $p \geq 1$ and $\tau \in [-1, 1]$. If $L \in \mathcal{P}^\tau_p$ and $\Pi^\tau_p K \subseteq \Pi^\tau_p L$, then for $n > p \geq 1$,

$$V(K) \leq V(L);$$

for $n < p$,

$$V(K) \geq V(L).$$

In each case, equality holds for $p = 1$ if and only if $K$ is a translation of $L$, and for $p > 1$ if and only if $K = L$. Here $\mathcal{P}^\tau_p$ denotes the set of general $L_p$-projection bodies with a parameter $\tau$.

**Theorem 1.2.** Let $K \in \mathcal{F}^\alpha_p$, $p \geq 1$ and $\tau \in [-1, 1]$. If $L \in \mathcal{W}^\tau_p$ and $\Pi^\tau_p K \subseteq \Pi^\tau_p L$, then

$$\Omega_p(K) \leq \Omega_p(L),$$

with equality for $p = 1$ if and only if $K$ is a translation of $L$, and for $p > 1$ if and only if $K = L$. Here $\mathcal{W}^\tau_p = \{Q \in \mathcal{F}^\alpha_p : f_p(Q) = h(Z, \cdot)^{(1-p)}\}$, where $f_p(Q, \cdot)$ is the $L_p$-curvature function of $Q$ and $\mathcal{F}^\alpha_p$ denotes the set of convex bodies in $\mathcal{K}_p^n$ with positive continuous $L_p$-curvature function.

In this article, we will research the Shephard type problems of the general $L_p$-projection bodies for $L_p$-geominimal surface areas. The notion of $L_p$-geominimal surface areas was introduced by Lutwak [7]. For $K \in \mathcal{K}_p^n$, $p \geq 1$, the $L_p$-geominimal surface area, $G_p(K)$, of $K$ is defined by

$$\omega_n^p G_p(K) = \inf \{n V_p(K, Q) V(Q)^{\frac{p}{n}} : Q \in \mathcal{F}_p^{\tau,n}\},$$

where $V_p(M, N)$ denotes the $L_p$-mixed volume of $M, N \in \mathcal{K}_p^n$. More researches about $L_p$-geominimal surface areas, also see [10], [26], [27], [28], [29], [30].

In (1.8), if $Q \in \mathcal{P}^\tau_p$, then we define $G_p^\tau(K)$ by

$$\omega_n^p G_p^\tau(K) = \inf \{n V_p(K, Q) V(Q)^{\frac{p}{n}} : Q \in \mathcal{P}^\tau_p\}.$$  

Combining with (1.9), we first give an affirmative answer of the Shephard type problem for general $L_p$-projection bodies.

**Theorem 1.3.** Let $K, L \in \mathcal{K}_p^n$, $1 \leq p < n$ and $\tau \in [-1, 1]$. If $\Pi^\tau_p K \subseteq \Pi^\tau_p L$, then

$$G^\tau_p(K) \leq G^\tau_p(L),$$

with equality when $\Pi_p^\tau K = \Pi_p^\tau L$.

Let $G_p^\tau_r$ denotes the set of all general $L_p$-centroid bodies (see [2]), thus $G_p^\tau_r \subseteq \mathcal{K}_p^n$. If $Q \in G_p^\tau_r$ in (1.8), then we write $G_p^\tau_r(K)$ by

$$\omega_n^p G_p^\tau_r(K) = \inf \{n V_p(K, Q) V(Q)^{\frac{p}{n}} : Q \in G_p^\tau_r\}.  \ (1.10)$$

Based on (1.10), we give the other affirmative form of the Shephard type problems for the general $L_p$-projection bodies.

**Theorem 1.4.** Let $K, L \in \mathcal{K}_p^n$, $1 \leq p < n$ and $\tau \in [-1, 1]$. If $\Pi_p^\tau K \subseteq \Pi_p^\tau L$, then

$$G_p^\tau(K) \leq G_p^\tau(L),$$

with equality when $\Pi_p^\tau K = \Pi_p^\tau L$.

In particular, if $\tau = 0$ in Theorem 1.3, the following result is obvious.

**Corollary 1.1.** Let $L \in \mathcal{K}_p^n$, $1 \leq p < n$. If $L$ is not an origin-symmetric convex body, then there exists $K \in \mathcal{K}_p^n \setminus \mathcal{K}_p^o$, such that

$$\Pi_p K \subseteq \Pi_p L,$$

but

$$G_p(K) \geq G_p(L).$$

Corollary 1.1 shows the symmetric negative solutions of the Shephard type problem of $L_p$-projection bodies for the $L_p$-geominimal surface areas. Actually, by the general $L_p$-Blaschke bodies, we find the asymmetric negative solutions in Corollary 1.1, i.e., we generalize the scope of negative solutions in Corollary 1.1 from $\mathcal{K}_p^n$ to $\mathcal{K}_p^n$.

**Theorem 1.5.** Let $L \in \mathcal{K}_p^n$ and $1 \leq p < n$. If $L$ is not an origin-symmetric convex body, then there exists $K \in \mathcal{K}_p^n \setminus \mathcal{K}_p^o$, such that

$$\Pi_p K \subseteq \Pi_p L,$$

but

$$G_p(K) > G_p(L).$$
From (1.1) and (2.1), it follows that if $K \in \mathcal{K}^n_o$, then
\[ h(K^*, \cdot) = \frac{1}{\mu(K, \cdot)}, \quad \rho(K^*, \cdot) = \frac{1}{h(K, \cdot)}. \] (2.2)

B. $L_p$-Mixed Volume and $L_p$-Dual Mixed Volume

For $K, L \in \mathcal{K}^n_o$, $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the $L_p$-Minkowski combination, $\lambda \cdot K +_p \mu \cdot L$, of $K$ and $L$ is defined by (see [6])
\[ h(\lambda \cdot K +_p \mu \cdot L, \cdot) = \lambda h(K, \cdot)^p + \mu h(L, \cdot)^p, \]
where $+_p$ denotes the $L_p$-Minkowski addition, $\lambda \cdot K$ denotes the $L_p$-Minkowski scalar multiplication.

Together with $L_p$-Minkowski combination, Lutwak [6] introduced $L_p$ mixed volume as follows: For $K, L \in \mathcal{K}^n_o$, $p > 0$ and $p \geq 1$, the $L_p$ mixed volume $V_p(K, L)$ is defined by
\[ \frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V_p(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}. \]

Besides, Lutwak [6] also gave its integral formula:
\[ V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(l(u), u)^p dS_p(K, u). \] (2.3)

Here $S_p(K, \cdot)$ is the $L_p$-surface area measure of $K$. It turns out that the measure $S_p(K, \cdot)$ is absolutely continuous with respect to $S(K, \cdot)$, and has Radon-Nikodym derivative (see [7])
\[ \frac{dS_p(K, \cdot)}{dS(K, \cdot)} = h(K, \cdot)^{1-p}. \] (2.4)

If $c > 0$, $n \neq p$, according to (2.4), we have
\[ S_p(cK, \cdot) = c^{n-p} S_p(K, \cdot). \] (2.5)

The $L_p$-dual mixed volume was introduced by Lutwak (see [7]). For $K, L \in \mathcal{S}^n_o$ and $p \geq 1$, the $L_p$-dual mixed volume, $\tilde{V}_p(K, L)$, of $K$ and $L$ is defined by (see [7])
\[ \tilde{V}_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^{n+p} (u) \rho_L^p (u) dS(u). \] (2.6)

C. General $L_p$-Blaschke Bodies

According to the existence’s theorem of $L_p$-Minkowski problem (see Theorem 9.2.3 in [16]), the $L_p$-Blaschke combinations of convex bodies was stated as follows: For $K, L \in \mathcal{K}^n_o$, $1 \leq p \neq n$, $\lambda, \mu \geq 0$ (not both zero), the $L_p$-Blaschke combination $\lambda \odot K +_p \mu \odot L \in \mathcal{K}^n_o$ of $K$, $L$ is defined by
\[ S_p(\lambda \odot K +_p \mu \odot L, \cdot) = \lambda S_p(K, \cdot) + \mu S_p(L, \cdot). \] (2.7)

where $+_p$ denotes the $L_p$-Blaschke addition, and $\lambda \odot K$ denotes the $L_p$-Blaschke scalar multiplication.

If $K, L \in \mathcal{K}^n_o$, then definition (2.7) is owe to Lutwak [6].

Let $\lambda = f_1(\tau), \mu = f_2(\tau)$ and $L = -K$ in (2.7), where $f_1(\tau)$ and $f_2(\tau)$ satisfy (1.7). We define the general $L_p$-Blaschke body, $\nabla_p^\tau K$, of $K \in \mathcal{K}^n_o$ by
\[ \nabla_p^\tau K = f_1(\tau) \odot K +_p f_2(\tau) \odot (-K). \] (2.8)

Obviously, by (1.7) and (2.8) we see that if $\tau = \pm 1$, then
\[ \nabla_p^\tau K = \nabla_p^\pm K = \pm K. \]

D. General $L_p$-Centroid Bodies

In 2015, Feng et al. [2] introduced the general $L_p$-centroid body as follows: For $K \in S^m_p$, $p \geq 1$ and $\tau \in [-1, 1]$, the general $L_p$-centroid body, $\Gamma^\tau_p K$, of $K$ is a convex body whose support function is defined by
\[ h^\tau_p K(u) = \frac{2}{c_{n,p}(\tau)(n+p)V(K)} \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p \rho_K(v)^{n+p} dv, \]
where
\[ c_{n,p}(\tau) = c_{n,p}((1+\tau)^p + (1-\tau)^p). \]

III. RESULTS AND PROOFS

In this part, we will give the proofs of Theorems 1.1-1.4. First, in order to prove theorem 1.1, the following lemma is required.

Lemma 3.1 (24)]. If $K, L \in \mathcal{K}^n_o$, $p \geq 1$ and $\tau \in [-1, 1]$, then
\[ V_p(K, \Pi^p_n L) = V_p(L, \Pi^p_n K). \] (3.1)

Proof of Theorem 1.1. Since $K, L \in \mathcal{K}^n_o$, $1 \leq p < n$, if $\Pi^p_n K \subseteq \Pi^p_n L$, then for all $u \in S^{n-1}$
\[ h(\Pi^p_n K, u) \leq h(\Pi^p_n L, u). \] (3.2)

From (2.3), (3.1) and (3.2), we have for any $M \in \mathcal{K}^n_o$,
\[ V_p(K, \Pi^p_n M) = V_p(M, \Pi^p_n K) \leq V_p(M, \Pi^p_n L) = V_p(L, \Pi^p_n M). \] (3.3)

Since $\Pi^p_n M \in \mathcal{P}^p_n$, by (1.9) and (3.3), we get
\[ \omega^n_p G^p_o(K) = \inf \{nV_p(K, \Pi^p_n M) V(\Pi^p_n M)^{\frac{p}{n}} : \Pi^p_n M \in \mathcal{P}^p_n\} \leq \inf \{nV_p(L, \Pi^p_n M) V(\Pi^p_n M)^{\frac{p}{n}} : \Pi^p_n M \in \mathcal{P}^p_n\} = \omega^n_p G^p_o(L), \]
i.e.,
\[ G^n_o(K) \leq G^n_o(L). \]

Equality holds when $\Pi^p_n K = \Pi^p_n L$.

Lemma 3.2 (8)]. If $\hat{M} \in \mathcal{K}^n_o$, $N \in \mathcal{S}^n_o$, $p \geq 1$ and $\tau \in [-1, 1]$, then
\[ V_p(M, \Gamma^\tau_p N) = \frac{\omega^n_p}{V(N)} \tilde{V}_p(N, \Pi^p_n M). \] (3.4)

Proof of Theorem 1.2. Since $K, L \in \mathcal{K}^n_o$, $1 \leq p < n$, if $\Pi^p_n K \subseteq \Pi^p_n L$, then $\Pi^p_n K \supseteq \Pi^p_n L$. From (2.2), (2.6) and (3.4), for any $N \in \mathcal{S}^n_o$, we obtain
\[ V_p(K, \Gamma^\tau_p N) = \frac{\omega^n_p}{V(N)} \tilde{V}_p(N, \Pi^p_n K) \leq \frac{\omega^n_p}{V(N)} \tilde{V}_p(N, \Pi^p_n L) = V_p(L, \Gamma^\tau_p N). \] (3.5)
Taking $Q = \Gamma_p^\tau N$, $N \in S^n_0$, thus by (1.10) and (3.5), we have
\[ G_p^{\tau}(K) \leq G_p^n(L). \]
Equality holds when $\Pi_{p}^{\tau}K = \Pi_{p}^{\tau}L$.

**Lemma 3.3.** If $K \in K^n_0$, $p \geq 1$ and $\tau \in [-1, 1]$, then
\[ G_p^{\tau}(\nabla_p^\tau K) \geq G_p(K), \tag{3.6} \]
with equality for $\tau \in (-1, 1)$ if and only if $K$ is origin-symmetric. For $\tau = \pm 1$, (3.6) becomes an equality.

**Proof.** By (1.8), (2.8), (2.3), (2.7) and (1.7), we have
\[ \omega_p^{\tau} G_p^{\tau}(\nabla_p^\tau K) \]
\[ \begin{aligned} &= \inf \{ nV_p^p(\nabla_p^\tau K, Q)V(Q^*) \cdot \bar{z} : Q \in K_0^n \} \\ &= \inf \{ nV_p^p(f_1(\tau) + f_2(\tau))V(Q^*) \cdot \bar{z} : Q \in K_0^n \} \\ &= \inf \{ n(f_1(\tau)V_p^p(K, Q) + f_2(\tau)V_p^p(-K, Q))V(Q^*) \cdot \bar{z} : Q \in K_0^n \} \end{aligned} \]
\[ \geq \inf \{ n(f_1(\tau)V_p^p(K, Q) + f_2(\tau)V_p^p(-K, Q))V(Q^*) \cdot \bar{z} : Q \in K_0^n \} \]
\[ \begin{aligned} &= f_1(\tau) \inf \{ nV_p^p(K, Q)V(Q^*) \cdot \bar{z} : Q \in K_0^n \} \\ &+ f_2(\tau) \inf \{ nV_p^p(-K, Q)V(Q^*) \cdot \bar{z} : Q \in K_0^n \} \end{aligned} \]
\[ = \omega_p^{\tau} G_p(K). \]

For any $Q \in K_0^n$ and $\tau \in (-1, 1)$, with equality if and only if $f_1(\tau)V_p^p(K, Q)$ and $f_2(\tau)V_p^p(-K, Q)$ are proportional, i.e., $f_1(\tau)S_p^p(K, \cdot)$ and $f_2(\tau)S_p^p(-K, \cdot)$ are proportional. This together with Lutwak’s result (see [6]) implies that equality holds in (3.6) if and only if $K$ and $-K$ are dilates, namely, $K$ is origin-symmetric.

Obviously, by $\nabla_p^\tau K = \pm K$ we see that if $\tau = \pm 1$, then (3.6) is an equality.

**Lemma 3.4.** If $K \in K^n_0$, $p \geq 1$ and $\tau \in (-1, 1)$, then
\[ \Pi_{p}^{\tau} \nabla_p^\tau K = \Pi_{p}^{\tau} K, \tag{3.7} \]
and
\[ \Pi_{p}^{\tau} \nabla_p K = \Pi_{p}^{\tau} K. \tag{3.8} \]

**Proof.** By (1.3), (2.8), (2.7), (1.4) and (1.6), we get for all $u \in S^{n-1}$,
\[ h_p^p(\Pi_{p}^{\tau} \nabla_p^\tau K, u) \]
\[ = 2\alpha_{n,p} \int_{S^{n-1}} (u \cdot v)_p^a dS_p(\nabla_p^\tau K, v) \]
\[ = 2\alpha_{n,p} \int_{S^{n-1}} (u \cdot v)_p^a dS_p(f_1(\tau) \circ K + f_2(\tau) \circ (-K), v) \]
\[ = 2\alpha_{n,p} \int_{S^{n-1}} (u \cdot v)_p^a d[f_1(\tau)S_p^p(K, \cdot) + f_2(\tau)S_p^p((-K), \cdot)] \]
\[ = f_1(\tau)h_p^p(\Pi_{p}^{\tau} K, u) + f_2(\tau)h_p^p(\Pi_{p}^{\tau}(-K), u) \]
\[ = f_1(\tau)h_p^p(\Pi_{p}^{\tau} K, u) + f_2(\tau)h_p^p(\Pi_{p}^{\tau} K, u) \]
\[ = h_p^p(\Pi_{p}^{\tau} K, u). \]

This immediately gives (3.7).

Similarly, we have for all $u \in S^{n-1}$,
\[ h_p^p(\Pi_{p}^{\tau} \nabla_p^\tau K, u) = h_p^p(\Pi_{p}^{\tau}(-K), u). \]

This yields (3.8).

**Lemma 3.5.** Let $L \in K^n_0$, $1 \leq p < n$ and $\tau \in (-1, 1)$. If $L$ is not origin-symmetric convex body, then there exists $K \in K^n_0(\tau = 0, K \in K^n_0)$, such that
\[ \Pi_{p}^{\tau} K \subset \Pi_{p}^{\tau} L, \quad \Pi_{p}^{\tau} L \subset \Pi_{p}^{\tau} K, \]
but
\[ G_p^{\tau}(K) > G_p^{\tau}(L). \]

**Proof.** Since $L$ is not origin-symmetric and $\tau \in (-1, 1)$, thus by Lemma 3.3, we know $G_p^{\tau}(\nabla_p^\tau L) > G_p^{\tau}(L)$. Choose $\varepsilon > 0$, such that $1 - \varepsilon > 0$, and $K = (1 - \varepsilon)V_p^p L \in K^n_0$ satisfies
\[ G_p(K) = G_p((1 - \varepsilon)V_p^p L) > G_p(L). \]

But by (1.5) and (2.5), we have
\[ \Pi_{p}^{\tau} eK = e^{n-p}\Pi_{p}^{\tau} K, \quad (\varepsilon > 0). \tag{3.9} \]

Therefore, for $n > p$, by (3.7), (3.8) and (3.9), we respectively have
\[ \Pi_{p}^{\tau} K = \Pi_{p}^{\tau}((1 - \varepsilon)V_p^p L)] = (1 - \varepsilon)^{n-p}\Pi_{p}^{\tau} V_p^p L \]
\[ = (1 - \varepsilon)^{n-p}\Pi_{p}^{\tau} L \subset \Pi_{p}^{\tau} K, \]
and
\[ \Pi_{p}^{\tau} K = \Pi_{p}^{\tau}((1 - \varepsilon)V_p^p L) = (1 - \varepsilon)^{n-p}\Pi_{p}^{\tau} V_p^p L \]
\[ = (1 - \varepsilon)^{n-p}\Pi_{p}^{\tau} L \subset \Pi_{p}^{\tau} L. \]

This obtains the desired result.

**Proof of Theorem 1.3.** Since $L$ is not origin-symmetric and $\tau \in (-1, 1)$, thus by Lemma 3.5, there exists $K \in K^n_0$, such that
\[ \Pi_{p}^{\tau} K \subset \Pi_{p}^{\tau} L, \quad \Pi_{p}^{\tau} L \subset \Pi_{p}^{\tau} K, \]
but
\[ G_p^{\tau}(K) > G_p^{\tau}(L). \]

Because $\tau \in (-1, 1)$ is equivalent to $-\tau \in (-1, 1)$, we have $\Pi_{p}^{\tau} K \subset \Pi_{p}^{\tau} L$, $\Pi_{p}^{\tau} L \subset \Pi_{p}^{\tau} K$, these imply
\[ \Pi_{p}^{\tau} K \subset \Pi_{p}^{\tau} L, \quad \Pi_{p}^{\tau} K \subset \Pi_{p}^{\tau} L. \]

From these and together with (1.6) and (1.7), we obtain for any $u \in S^{n-1}$,
\[ h_p^p(\Pi_{p}^{\tau} K, u) \]
\[ = f_1(\tau)h_p^p(\Pi_{p}^{\tau} K, u) + f_2(\tau)h_p^p(\Pi_{p}^{\tau}(-K), u) \]
\[ < f_1(\tau)h_p^p(\Pi_{p}^{\tau} L, u) + f_2(\tau)h_p^p(\Pi_{p}^{\tau}L, u) \]
\[ = h_p^p(\Pi_{p}^{\tau} L, u), \]
i.e.,
\[ \Pi_{p}^{\tau} K \subset \Pi_{p}^{\tau} L. \]

This yields desired result.

**Lemma 3.6.** Let $K, L \in K^n_0$, $p \geq 1$ and $\tau \in [-1, 1]$, then
\[ \Pi_{p}^{\tau} \nabla_p^\tau K = \Pi_{p}^{\tau} K. \tag{3.10} \]
**Proof.** By (1.2), (2.8) and (2.7), we obtain for any \( u \in S^{n-1}, \)
\[
\begin{align*}
\Delta^p(\Pi_p(\nabla_\tau^p K), u) \\
= \alpha_{n,p} \int_{S^{n-1}} |v|^p dS_p(\nabla_\tau^p K, v) \\
= \alpha_{n,p} \int_{S^{n-1}} |v|^p dS_p(f_1(\tau) \circ K + \tau, f_2(\tau) \circ (-K), v) \\
= \alpha_{n,p} \int_{S^{n-1}} |v|^p dS_p(f_1(\tau) S_p(K, v) + f_2(\tau) S_p((-K), v)) \\
= f_1(\tau) h^p(\Pi_p(\nabla_\tau^p K), u) + f_2(\tau) h^p(\Pi_p(-K), u).
\end{align*}
\]
Note that \( \Pi_pK = \Pi_p(-K), \) thus
\[
\begin{align*}
\Delta^p(\Pi_p(\nabla_\tau^p K), u) &= h^p(\Pi_p(\nabla_\tau^p K), u) \\
i.e.,
\Delta^p(\Pi_p(\nabla_\tau^p K)) &= \Pi_p K.
\end{align*}
\]
This yields (3.10). **Proof of Theorem 1.4.** Since \( L \) is not origin-symmetric, from Lemma 3.3, we know for \( \tau \in (-1, 1), \)
\[
G_p(\nabla_\tau^p L) > G_p(L).
\]
Choose \( 0 < \epsilon < 1 \), such that
\[
G_p((1 - \epsilon) \nabla_\tau^p L) > G_p(L).
\]
Let \( K = (1 - \epsilon) \nabla_\tau^p L \), then \( K \in \mathcal{K}_0^m \) (for \( \tau \neq 0, K \in \mathcal{K}_0^{m} \backslash \mathcal{K}_0^{m+} \) for \( \tau = 0, K \in \mathcal{K}_0^{m} \)) and
\[
G_p(K) > G_p(L).
\]
But by (1.2) and (2.5), we have
\[
\Pi_p\epsilon K = \epsilon^{n-p} \Pi_p K, \quad (\epsilon > 0).
\]
Hence, for \( n > p, (3.10) \) and (3.11) mean that
\[
\Pi_p K = \Pi_p((1 - \epsilon) \nabla_\tau^p L) = (1 - \epsilon)^{n-p} \Pi_p \nabla_\tau^p L
\]
\[
= (1 - \epsilon)^{n-p} \Pi_p L \subset \Pi_p L,
\]
This obtains the desired result.

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**REFERENCES**


