Dual Quermassintegral Quotient Functions Inequalities of Radial Blaschke-Minkowski Homomorphisms

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***Abstract*—In this paper, we establish some Brunn-Minkowski inequalities for the dual quermassintegral quotient functions of the radial Blaschke Minkowski homomorphisms of star bodies.**

***Index Terms*—star bodies, dual quermassintegral quotient functions, radial Blaschke Minkowski homomorphisms, intersection body operators.**

I. Introduction

In[7], Schuster first introduced the radial Blaschke Minkowski homomorphisms as follows: A map $\Psi:\mathcal{S}^n\rightarrow\mathcal{S}^n$ is called a radial Blaschke Minkowski homomorphism if it satisfies the following conditions:

(1) Ψ is continuous.

(2) $\Psi(K)$ is radial Blaschke Minkowski additive, i.e., $\Psi(K\check{+}_{n-1}L)=\Psi K\tilde{+}\Psi L$ for all $K,L\in\mathcal{S}^n$.

(3) Ψ intertwines rotations, i.e. $\Psi(\vartheta K)=\vartheta\Psi K$ for all $K\in\mathcal{S}^n$ and all $\vartheta\in SO(n)$.

Here, $\mathcal{S}^n$ denotes the set of star bodies, $SO(n)$ denotes the group of special origin-rotation transformation, $K\check{+}_{n-1}L$ denotes the radial Blaschke sum of the K and L and $\Psi K\tilde{+}\Psi L$ denotes the radial Minkowski sum of ΨK and ΨL.

In[7], Schuster established the following Brunn-Minkowski inequality for radial Blaschke Minkowski homomorphism of star bodies.

Theorem A [7] *If $K_1,K_2\in\mathcal{S}^n$, then*

$$V(\Psi(K_1\tilde{+}K_2))^{\frac{1}{n(n-1)}}\leq V(\Psi K_1)^{\frac{1}{n(n-1)}}+V(\Psi K_2)^{\frac{1}{n(n-1)}},$$

with equality if and only if K_1 and K_2 are dilates.

Theorem B [7] *There is a continuous operator*

$$\Psi:\underbrace{\mathcal{S}^n\times\cdots\times\mathcal{S}^n}_{n-1}\rightarrow\mathcal{S}^n$$

symmetric in its arguments such that, for $K_1,\cdots,K_m\in\mathcal{S}^n$ and $\lambda_1,\cdots,\lambda_m\geq0$,

$$\Psi(\lambda_1K_1\tilde{+}\cdots\tilde{+}\lambda_mK_m)$$
$$=\tilde{\sum}_{i_1,\cdots,i_{n-1}}\lambda_{i_1}\cdots\lambda_{i_{n-1}}\Psi(K_{i_1},\cdots,K_{i_{n-1}}),\qquad(1.1)$$

where the sum is with respect to radial Minkowski addition.

Clearly, (1.1) generalizes the notion of radial Blaschke Minkowski homomorphism and we will call the map

$$\Psi:\mathcal{S}^n\times\cdots\times\mathcal{S}^n\rightarrow\mathcal{S}^n$$

mixed radial Blaschke Minkowski homomorphism induced by Ψ. For $K,L\in\mathcal{S}^n$, let $\Psi_i(K,L)$ denote the mixed radial Blaschke Minkowski homomorphism $\Psi_i(K,\cdots,K,L,\cdots,L)$, with i copies of L and $n-i-1$ copies of K. We write $\Psi_i(K)$ for $\Psi_i(K,\cdots,K,B,\cdots,B)$ and call $\Psi_i(K)$ the radial Blaschke Minkowski homomorphism of order i.

And in fact, Schuster established a more general version of Brunn-Minkowski inequality for radial Blaschke Minkowski homomorphism: If $K_1,K_2\in\mathcal{S}^n$, integers i,j satisfy $0\leq i\leq n-1$, $0\leq j\leq n-2$, then

$$\widetilde{W}_i(\Psi_j(K_1\tilde{+}K_2))^{\frac{1}{(n-i)(n-j-1)}}$$
$$\leq\widetilde{W}_i(\Psi_jK_1)^{\frac{1}{(n-i)(n-j-1)}}+\widetilde{W}_i(\Psi_jK_2)^{\frac{1}{(n-i)(n-j-1)}},$$

with equality if and only if K_1 and K_2 are dilates.

And for $K,L\in\mathcal{S}^n$, if $0\leq i,j\leq n-2$, then[7]

$$\widetilde{W}_i(K,\Psi_jL)=\widetilde{W}_j(L,\Psi_iK).\qquad(1.2)$$

In[11], Zhao defined the dual quermassintegral difference function of Ψ by

$$D_{\widetilde{W}_i}(\Psi K,\Psi D)=\widetilde{W}_i(\Psi K)-\widetilde{W}_i(\Psi D),D\subseteq K,0\leq i\leq n.$$

And in[11], Minkowski and Brunn-Minkowski type inequalities for volume difference function of radial Blaschke Minkowski homomorphism were established.

Motivated by the work of Zhao, we give the dual quermassintegral quotient function of Ψ by

$$Q_{\widetilde{W}_{i,j}}(\Psi K,\Psi D)=\frac{\widetilde{W}_i(\Psi K)}{\widetilde{W}_j(\Psi D)}.\qquad(1.3)$$

The goal of this paper is to establish the Brunn-Minkowski inequalities for the quermassintegral quotient function of Ψ. Associated with L_p radial combination "$\tilde{+}_p$", we obtain

Theorem 1.1 *Let $\Psi:\mathcal{S}^n\rightarrow\mathcal{S}^n$ be a radial Blaschke Minkowski homomorphism. For $K_1,K_2,D_1,D_2\in\mathcal{S}^n$, and*

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let K_1 be a dilated copy of K_2, if i, j satisfy $0 \le i, j \le n-2$ and $\frac{n-1}{p} > 1$, $\frac{(n-j)(n-1)}{p} \ge 1 \ge \frac{(n-i)(n-1)}{p} \ge 0$, then

$$\left(\frac{\widetilde{W}_j(\Psi(D_1\tilde{+}_p D_2))}{\widetilde{W}_i(\Psi(K_1\tilde{+}_p K_2))}\right)^{\frac{p}{(i-j)(n-1)}} \le \left(\frac{\widetilde{W}_j(\Psi D_1)}{\widetilde{W}_i(\Psi K_1)}\right)^{\frac{p}{(i-j)(n-1)}} + \left(\frac{\widetilde{W}_j(\Psi D_2)}{\widetilde{W}_i(\Psi K_2)}\right)^{\frac{p}{(i-j)(n-1)}}, \tag{1.4}$$

with equality if and only if D_1 and D_2 are dilates, and $\frac{\widetilde{W}_j(\Psi D_1)}{\widetilde{W}_j(\Psi D_2)} = \frac{\widetilde{W}_i(\Psi K_1)}{\widetilde{W}_i(\Psi K_2)}$.

Furthermore, we establish the following Brunn-Minkowski inequality for the dual quermassintegral quotient function of Ψ about the L_p harmonic radial combination " $\hat{+}_{-p}$".

Theorem 1.2 *Let $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ be a radial Blaschke Minkowski homomorphism. For $K_1, K_2, D_1, D_2 \in \mathcal{S}^n$, and let K_1 be a dilated copy of K_2, if i, j satisfy $0 \le i, j \le n-2$, $p \ge 1$ and $\frac{(n-i)(n-1)}{p} \ge 1 \ge \frac{(n-j)(n-1)}{p} \ge 0$, then*

$$\left(\frac{\widetilde{W}_j(\Psi(D_1\hat{+}_{-p} D_2))}{\widetilde{W}_i(\Psi(K_1\hat{+}_{-p} K_2))}\right)^{\frac{p}{(j-i)(n-1)}} \le \left(\frac{\widetilde{W}_j(\Psi D_1)}{\widetilde{W}_i(\Psi K_1)}\right)^{\frac{p}{(j-i)(n-1)}} + \left(\frac{\widetilde{W}_j(\Psi D_2)}{\widetilde{W}_i(\Psi K_2)}\right)^{\frac{p}{(j-i)(n-1)}}, \tag{1.5}$$

with equality if and only if D_1 and D_2 are dilates, and $\frac{\widetilde{W}_j(\Psi D_1)}{\widetilde{W}_j(\Psi D_2)} = \frac{\widetilde{W}_i(\Psi K_1)}{\widetilde{W}_i(\Psi K_2)}$.

Finally, we establish the Brunn-Minkowski inequality for the dual quermassintegral quotient function of Ψ about the L_p radial Blaschke combination "$\check{+}_{n-p}$ " as follows:

Theorem 1.3 *Let $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ be a radial Blaschke Minkowski homomorphism. For $K_1, K_2, D_1, D_2 \in \mathcal{S}^n$, let K_1 be a dilated copy of K_2, and if i, j satisfy $0 \le i, j \le n-2$, $j+1 \le p < n$, $i+1 \le p < n$, and $\frac{(n-i)(n-1)}{n-p} \ge 1 \ge \frac{(n-j)(n-1)}{n-p} \ge 0$, then*

$$\left(\frac{\widetilde{W}_i(\Psi(D_1\check{+}_{n-p} D_2))}{\widetilde{W}_j(\Psi(K_1\check{+}_{n-p} K_2))}\right)^{\frac{n-p}{(j-i)(n-1)}} \le \left(\frac{\widetilde{W}_i(\Psi D_1)}{\widetilde{W}_j(\Psi K_1)}\right)^{\frac{n-p}{(j-i)(n-1)}} + \left(\frac{\widetilde{W}_i(\Psi D_2)}{\widetilde{W}_j(\Psi K_2)}\right)^{\frac{n-p}{(j-i)(n-1)}}, \tag{1.6}$$

with equality if and only if D_1 and D_2 are dilates, and $\frac{\widetilde{W}_j(\Psi D_1)}{\widetilde{W}_j(\Psi D_2)} = \frac{\widetilde{W}_i(\Psi K_1)}{\widetilde{W}_i(\Psi K_2)}$.

Since the intersection body operator $\mathbf{I} : \mathcal{S}^n \to \mathcal{S}^n$ is a radial Blaschke Minkowski homomorphism, we change Ψ into the intersection body operator $\mathbf{I}$ in Theorem 1.1, Theorem 1.2, Theorem 1.3, we get the following Brunn-Minkowski inequalities for the dual quermassintegral quotient function of the intersection body operator $\mathbf{I}$.

Corollary 1.4 *Let $K_1, K_2, D_1, D_2 \in \mathcal{S}^n$, and let K_1 be a dilated copy of K_2, if i, j satisfy $0 \le i, j \le n-2$, $\frac{n-1}{p} > 1$ and $\frac{(n-j)(n-1)}{p} \ge 1 \ge \frac{(n-i)(n-1)}{p} \ge 0$, then*

$$\left(\frac{\widetilde{W}_j(\mathbf{I}(D_1\tilde{+}_p D_2))}{\widetilde{W}_i(\mathbf{I}(K_1\tilde{+}_p K_2))}\right)^{\frac{p}{(i-j)(n-1)}} \le \left(\frac{\widetilde{W}_j(\mathbf{I}D_1)}{\widetilde{W}_i(\mathbf{I}K_1)}\right)^{\frac{p}{(i-j)(n-1)}} + \left(\frac{\widetilde{W}_j(\mathbf{I}D_2)}{\widetilde{W}_i(\mathbf{I}K_2)}\right)^{\frac{p}{(i-j)(n-1)}},$$

with equality if and only if D_1 and D_2 are dilates, and $\frac{\widetilde{W}_j(\mathbf{I}D_1)}{\widetilde{W}_j(\mathbf{I}D_2)} = \frac{\widetilde{W}_i(\mathbf{I}K_1)}{\widetilde{W}_i(\mathbf{I}K_2)}$.

Corollary 1.5 *Let $K_1, K_2, D_1, D_2 \in \mathcal{S}^n$, and let K_1 be a dilated copy of K_2, if i, j satisfy $0 \le i, j \le n-2$, $p \ge 1$ and $\frac{(n-i)(n-1)}{p} \ge 1 \ge \frac{(n-j)(n-1)}{p} \ge 0$, then*

$$\left(\frac{\widetilde{W}_j(\mathbf{I}(D_1\hat{+}_{-p} D_2))}{\widetilde{W}_i(\mathbf{I}(K_1\hat{+}_{-p} K_2))}\right)^{\frac{p}{(j-i)(n-1)}} \le \left(\frac{\widetilde{W}_j(\mathbf{I}D_1)}{\widetilde{W}_i(\mathbf{I}K_1)}\right)^{\frac{p}{(j-i)(n-1)}} + \left(\frac{\widetilde{W}_j(\mathbf{I}D_2)}{\widetilde{W}_i(\mathbf{I}K_2)}\right)^{\frac{p}{(j-i)(n-1)}},$$

with equality if and only if D_1 and D_2 are dilates, and $\frac{\widetilde{W}_j(\mathbf{I}D_1)}{\widetilde{W}_j(\mathbf{I}D_2)} = \frac{\widetilde{W}_i(\mathbf{I}K_1)}{\widetilde{W}_i(\mathbf{I}K_2)}$.

Corollary 1.6 *Let $K_1, K_2, D_1, D_2 \in \mathcal{S}^n$, and let K_1 be a dilated copy of K_2, if i, j satisfy $0 \le i, j \le n-2$, $j+1 \le p < n$, $i+1 \le p < n$, and $\frac{(n-i)(n-1)}{n-p} \ge 1 \ge \frac{(n-j)(n-1)}{n-p} \ge 0$, then*

$$\left(\frac{\widetilde{W}_i(\mathbf{I}(D_1\check{+}_{n-p} D_2))}{\widetilde{W}_j(\mathbf{I}(K_1\check{+}_{n-p} K_2))}\right)^{\frac{n-p}{(j-i)(n-1)}} \le \left(\frac{\widetilde{W}_i(\mathbf{I}D_1)}{\widetilde{W}_j(\mathbf{I}K_1)}\right)^{\frac{n-p}{(j-i)(n-1)}} + \left(\frac{\widetilde{W}_i(\mathbf{I}D_2)}{\widetilde{W}_j(\mathbf{I}K_2)}\right)^{\frac{n-p}{(j-i)(n-1)}},$$

with equality if and only if D_1 and D_2 are dilates, and $\frac{\widetilde{W}_j(\mathbf{I}D_1)}{\widetilde{W}_j(\mathbf{I}D_2)} = \frac{\widetilde{W}_i(\mathbf{I}K_1)}{\widetilde{W}_i(\mathbf{I}K_2)}$.

In fact, the more general Brunn-Minkowski inequalities for the dual quermassintegral quotient function than Theorem 1.1, Theorem 1.2, Theorem 1.3 will be established in Section 3.

II. Notations and Background Materials

2.1 Dual mixed quermassintegral

Let $\mathcal{S}^n$ denote the set of star bodies (i.e. compact sets, star shaped with respect to the origin) in Euclidean space $\mathcal{R}^n$. We reserve the letter u for unit vector, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1}. We shall use $V(K)$ for the n-dimensional volume of body K.

For a compact set K in $\mathcal{R}^n$ which is star shaped with respect to the origin, we define the radial function $\rho(K, u)$ of K by(see [1])

$$\rho(K, u) = \max\{\lambda \ge 0 : \lambda u \in K\}, u \in S^{n-1}. \tag{2.1}$$

If $\rho(K, \cdot)$ is positive and continuous, K will be called a star body. Let δ denote the radial Hausdorff metric, i.e., if $K_1, K_2 \in \mathcal{S}^n$, then $\delta(K_1, K_2) = |\rho(K_1, \cdot) - \rho(K_2, \cdot)|_\infty$.

In 1975, Lutwak (see [3]) gave the notion of dual mixed volumes as follows: For $K_1, K_2, \cdots, K_n \in \mathcal{S}_o^n$, the dual mixed volume, $\widetilde{V}(K_1, K_2, \cdots, K_n)$, of $K_1, K_2, \cdots, K_n$ is defined by

$$\widetilde{V}(K_1, \cdots, K_n) = \frac{1}{n}\int_{S^{n-1}} \rho(K_1, u)\cdots\rho(K_n, u)dS(u). \tag{2.2}$$

Taking $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$ in (2.2), we write $\widetilde{V}_i(K, L) = \widetilde{V}(K, n-i; L, i)$, where K appears $n-i$ times and L appears i times, then

$$\widetilde{V}_i(K, L) = \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^{n-i}\rho(L, u)^i dS(u). \tag{2.3}$$

Let $L = B$ in (2.3) and notice $\rho(B, \cdot) = 1$, and allowing i is any real, then the dual quermassintegral can be defined as follows: For $K \in \mathcal{S}_o^n$ and if i is any real, the dual quermassintegral, $\widetilde{W}_i(K)$, of K is given by (see [3])

$$\widetilde{W}_i(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^{n-i}dS(u). \tag{2.4}$$

Let $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = B, K_n = L$ in (2.3), then we write $\widetilde{W}_i(K, L) = \widetilde{V}(K, n-i-1; B, i; L, 1)$, where K appears $n-i-1$ times, B appears i times and L appears 1 time. We call $\widetilde{W}_i(K, L)$ the dual mixed quermassintegral of K and L.

And we get the dual Minkowski inequality(see [7]): For $K, L \in \mathcal{S}^n$, if i satisfies $0 \le i \le n-2$, then

$$\widetilde{W}_i(K, L)^{n-i} \le \widetilde{W}_i(K)^{n-i-1}\widetilde{W}_i(L), \tag{2.5}$$

with equality if and only if K and L are dilates.

2.2 Some combinations

For $K_1, K_2 \in \mathcal{S}^n$ and $\lambda_1, \lambda_2 \ge 0$ (not both 0), $p \ge 0$, the L_p radial combination, $\lambda_1 \cdot K_1 \tilde{+}_p \lambda_2 \cdot K_2$, of K_1 and K_2 is defined by (see [6], [2])

$$\rho(\lambda_1 \cdot K_1 \tilde{+}_p \lambda_2 \cdot K_2, u)^p = \lambda_1\rho(K_1, u)^p + \lambda_2\rho(K_2, u)^p. \tag{2.6}$$

For $K_1, K_2 \in \mathcal{S}^n$ and $\lambda_1, \lambda_2 \ge 0$ (not both 0), $p \ge 1$, the L_p harmonic radial combination, $\lambda_1 \cdot K_1 \hat{+}_{-p} \lambda_2 \cdot K_2$, of K_1 and K_2 is defined by (see [5])

$$\rho(\lambda_1 \cdot K_1 \hat{+}_{-p} \lambda_2 \cdot K_2, u)^{-p} = \lambda_1\rho(K_1, u)^{-p} + \lambda_2\rho(K_2, u)^{-p}. \tag{2.7}$$

For $K_1, K_2 \in \mathcal{S}^n$ and $\lambda_1, \lambda_2 \ge 0$ (not both 0), $p \ge 1$, the L_p radial Blaschke combination, $\lambda_1 \circ K_1 \check{+}_{n-p} \lambda_2 \circ K_2$, of K_1 and K_2 is defined by (see [9])

$$\begin{aligned}&\rho(\lambda_1 \circ K_1 \check{+}_{n-p} \lambda_2 \circ K_2, u)^{n-p} \\ &= \lambda_1\rho(K_1, u)^{n-p} + \lambda_2\rho(K_2, u)^{n-p}.\end{aligned} \tag{2.8}$$

The special cases $p = 1$ in (2.6),(2.7) and (2.8), are just the classic linear combinations.

2.3 Intersection body

For $K \in \mathcal{S}^n$, we get the intersection body $\mathbf{I}K$ of K, whose radial function satisfies

$$\rho(\mathbf{I}K, u) = v(K \cap u^{\perp}), u \in S^{n-1} \tag{2.9}$$

where the v is the $n-1$-dimensional volume and $u^{\perp}$ is the $n-1$-dimensional subspace of $\mathcal{R}^n$ orthogonal to u. It is called the intersection body of K (see [4]).

By using the polar coordinate formula for volume, it is trivial to verify that(see [4])

$$\rho(\mathbf{I}K, u) = \frac{1}{n-1}\int_{S^{n-1}\cap u^{\perp}} \rho(K, u)^{n-1}dS(u). \tag{2.10}$$

An important generalization of this definition is the mixed intersection body (see [12]). The mixed intersection body of $K_1, \cdots, K_{n-1} \in \mathcal{S}^n$, $\mathbf{I}(K_1, \cdots, K_{n-1})$, is defined by

$$\rho(\mathbf{I}(K_1, \cdots, K_{n-1}), u) = \widetilde{v}(K_1 \cap u^{\perp}, \cdots, K_{n-1} \cap u^{\perp}), \tag{2.11}$$

where the $\widetilde{v}$ denotes the $n-1$-dimensional dual mixed volume. If $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = L$, $i = 0, 1, \cdots, n-1$, then $\mathbf{I}(\underbrace{K, \cdots, K}_{n-i-1}, \underbrace{L, \cdots, L}_{i})$ is usually written as $\mathbf{I}_i(K, L)$. And if $L = B$, then $\mathbf{I}_i(K, L)$ is written as $\mathbf{I}_i(K)$ and called the ith intersection body of K. For $\mathbf{I}_0(K)$, we simply write $\mathbf{I}K$.

The following map is the representation for radial Blaschke Minkowski homomorphism:

A map $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ is a radial Blaschke Minkowski homomorphism if and only if there is a nonnegative measure $\mu \in \mathcal{M}_+(S^{n-1}, \hat{e})$ such that(see [7])

$$\rho(\Psi K, \cdot) = \rho^{n-1}(K, \cdot) * \mu, \tag{2.12}$$

where $\mathcal{M}_+(S^{n-1}, \hat{e})$ denotes the set of nonnegative zonal measures on S^{n-1}.

From (2.10) (2.12), we can see that the intersection body operator is a radial Blaschke Minkowski homomorphism. The generating measure of the intersection body operator $\mathbf{I}$ is the invariant measure $\mu_{S_o^{n-2}}$ which is concentrated on $S_o^{n-2} := S^{n-1} \cap \hat{e}^{\perp}$, i.e. (see [7],[8]),

$$\rho(\mathbf{I}K, \cdot) = \rho^{n-1}(K, \cdot) * \mu_{S_o^{n-2}}. \tag{2.13}$$

Appendix A
Main results

In order to prove Theorem 1.1, the following lemmas are needed.

Lemma 3.1 *Let* $\Psi : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ *be a mixed radial Blaschke Minkowski homomorphism. If* $K_1, K_2 \in \mathcal{S}^n$, *integers* i, j *satisfy* $0 \le i, j \le n-2$ *and* $\frac{n-j-1}{p} > 1$, *then*

$$\begin{aligned}&\widetilde{W}_i(\Psi_j(K_1 \tilde{+}_p K_2))^{\frac{p}{(n-i)(n-j-1)}} \\ &\le \widetilde{W}_i(\Psi_j K_1)^{\frac{p}{(n-i)(n-j-1)}} + \widetilde{W}_i(\Psi_j K_2)^{\frac{p}{(n-i)(n-j-1)}},\end{aligned} \tag{3.1}$$

with equality if and only if K_1 and K_2 are dilates.

Proof For $K_1, K_2 \in \mathcal{S}^n$, if i, j satisfy $0 \le i, j \le n-2$ and $\frac{n-j-1}{p} > 1$, according to (1.2)(2.5)(2.6) and Minkowski inequality , we have

$$\widetilde{W}_i(Q, \Psi_j(K_1 \tilde{+}_p K_2))^{\frac{p}{(n-j-1)}} = \widetilde{W}_j(K_1 \tilde{+}_p K_2, \Psi_i Q)^{\frac{p}{(n-j-1)}}$$

$$= \left(\frac{1}{n}\int_{S^{n-1}} \rho(K_1 \tilde{+}_p K_2, u)^{n-j-1}\rho(\Psi_i Q, u)dS(u)\right)^{\frac{p}{(n-j-1)}}$$

$$= \left(\frac{1}{n}\int_{S^{n-1}} ((\rho(K_1, u)^p + \rho(K_2, u)^p)^{\frac{n-j-1}{p}}\rho(\Psi_i Q, u)dS(u)\right)^{\frac{p}{(n-j-1)}}$$

$$\leq \left(\frac{1}{n}\int_{S^{n-1}}\rho(K_1,u)^{n-j-1}\rho(\Psi_i Q,u)dS(u)\right)^{\frac{p}{(n-j-1)}}$$
$$+\left(\frac{1}{n}\int_{S^{n-1}}\rho(K_2,u)^{n-j-1}\rho(\Psi_i Q,u)dS(u)\right)^{\frac{p}{(n-j-1)}}$$
$$=\widetilde{W}_j(K_1,\Psi_i Q)^{\frac{p}{(n-j-1)}}+\widetilde{W}_j(K_2,\Psi_i Q)^{\frac{p}{(n-j-1)}}$$
$$=\widetilde{W}_i(Q,\Psi_j K_1)^{\frac{p}{(n-j-1)}}+\widetilde{W}_i(Q,\Psi_j K_2)^{\frac{p}{(n-j-1)}}$$
$$\leq\left(\widetilde{W}_i(Q)^{\frac{n-i-1}{n-i}}\widetilde{W}_i(\Psi_j K_1)^{\frac{1}{n-i}}\right)^{\frac{p}{(n-j-1)}}$$
$$+\left(\widetilde{W}_i(Q)^{\frac{n-i-1}{n-i}}\widetilde{W}_i(\Psi_j K_1)^{\frac{1}{n-i}}\right)^{\frac{p}{(n-j-1)}}, \quad (3.2)$$

let $Q=\Psi_j(K_1\tilde{+}_p K_2)$ in (3.2), we have

$$\widetilde{W}_i(\Psi_j(K_1\tilde{+}_p K_2))^{\frac{p}{(n-i)(n-j-1)}}$$
$$\leq\widetilde{W}_i(\Psi_j K_1)^{\frac{p}{(n-i)(n-j-1)}}+\widetilde{W}_i(\Psi_j K_2)^{\frac{p}{(n-i)(n-j-1)}}.$$

According to the equality conditions, equality holds in (3.2) if and only if K_1 and K_2 are dilates.

Lemma 3.2(Dresher's inequality[10]) *Let $f_1,f_2,g_1,g_2\geq 0$, E is a bounded measurable subset in $\mathcal{R}^n$. If $p\geq 1\geq r\geq 0$, then*

$$\left(\frac{\int_E(f_1+f_2)^p dx}{\int_E(g_1+g_2)^r dx}\right)^{\frac{1}{p-r}}\leq\left(\frac{\int_E f_1^p dx}{\int_E g_1^r dx}\right)^{\frac{1}{p-r}}+\left(\frac{\int_E f_2^p dx}{\int_E g_2^r dx}\right)^{\frac{1}{p-r}}, \quad (3.3)$$

equality holds if and only if $\frac{f_1}{f_2}=\frac{g_1}{g_2}$.

Theorem 3.3 *Let $\Psi:\underbrace{\mathcal{S}^n\times\cdots\times\mathcal{S}^n}_{n-1}\to\mathcal{S}^n$ be a mixed radial Blaschke Minkowski homomorphism. For $K_1,K_2,D_1,D_2\in\mathcal{S}^n$, and let K_1 be a dilated copy of K_2, if i,j,q satisfy $0\leq i,j,q\leq n-2$ and $\frac{n-q-1}{p}>1$, $\frac{(n-j)(n-q-1)}{p}\geq 1\geq\frac{(n-i)(n-q-1)}{p}\geq 0$, then we get*

$$\left(\frac{\widetilde{W}_j(\Psi_q(D_1\tilde{+}_p D_2))}{\widetilde{W}_i(\Psi_q(K_1\tilde{+}_p K_2))}\right)^{\frac{p}{(i-j)(n-q-1)}}$$
$$\leq\left(\frac{\widetilde{W}_j(\Psi_q D_1)}{\widetilde{W}_i(\Psi_q K_1)}\right)^{\frac{p}{(i-j)(n-q-1)}}+\left(\frac{\widetilde{W}_j(\Psi_q D_2)}{\widetilde{W}_i(\Psi_q K_2)}\right)^{\frac{p}{(i-j)(n-q-1)}}, \quad (3.4)$$

with equality if and only if D_1 and D_2 are dilates, and $\frac{\widetilde{W}_j(\Psi_q D_1)}{\widetilde{W}_j(\Psi_q D_2)}=\frac{\widetilde{W}_i(\Psi_q K_1)}{\widetilde{W}_i(\Psi_q K_2)}$.

Proof For $K_1,K_2,D_1,D_2\in\mathcal{S}^n$, K_1 and K_2 are dilates, if i,j,q satisfy $0\leq i,j,q\leq n-2$ and $\frac{n-q-1}{p}>1$, according to (3.1), we have

$$\widetilde{W}_i(\Psi_q(K_1\tilde{+}_p K_2))=\left(\widetilde{W}_i(\Psi_q K_1)^{\frac{p}{(n-i)(n-q-1)}}\right.$$
$$\left.+\widetilde{W}_i(\Psi_q K_2)^{\frac{p}{(n-i)(n-q-1)}}\right)^{\frac{(n-i)(n-q-1)}{p}}, \quad (3.5)$$

and

$$\widetilde{W}_j(\Psi_q(D_1\tilde{+}_p D_2))\leq\left(\widetilde{W}_j(\Psi_q D_1)^{\frac{p}{(n-j)(n-q-1)}}\right.$$
$$\left.+\widetilde{W}_j(\Psi_q D_2)^{\frac{p}{(n-j)(n-q-1)}}\right)^{\frac{(n-j)(n-q-1)}{p}}. \quad (3.6)$$

If $\frac{(n-j)(n-q-1)}{p}\geq 1\geq\frac{(n-i)(n-q-1)}{p}\geq 0$, according to (3.5)(3.6), using (3.3), we get

$$\left(\frac{\widetilde{W}_j(\Psi_q(D_1\tilde{+}_p D_2))}{\widetilde{W}_i(\Psi_q(K_1\tilde{+}_p K_2))}\right)^{\frac{p}{(i-j)(n-q-1)}}$$
$$\leq\left(\frac{\widetilde{W}_j(\Psi_q D_1)}{\widetilde{W}_i(\Psi_q K_1)}\right)^{\frac{p}{(i-j)(n-q-1)}}+\left(\frac{\widetilde{W}_j(\Psi_q D_2)}{\widetilde{W}_i(\Psi_q K_2)}\right)^{\frac{p}{(i-j)(n-q-1)}}, \quad (3.7)$$

by the equality conditions, equality holds in (3.7) if and only if D_1 and D_2 are dilates, and $\frac{\widetilde{W}_j(\Psi_q D_1)}{\widetilde{W}_j(\Psi_q D_2)}=\frac{\widetilde{W}_i(\Psi_q K_1)}{\widetilde{W}_i(\Psi_q K_2)}$.

Since the intersection body operator $\mathbf{I_q}:\mathcal{S}^n\to\mathcal{S}^n$ is a radial Blaschke Minkowski homomorphism, we change Ψ_q into the intersection body operator $\mathbf{I_q}$ in Theorem 3.3, we get the following Brunn-Minkowski inequality for the dual quermassintegral quotient function of the intersection body operator $\mathbf{I_q}$.

Corollary 3.4 *Let $\mathbf{I}_q$ be intersection body operator, $K_1,K_2,D_1,D_2\in\mathcal{S}^n$, and let K_1 be a dilated copy of K_2, if i,j,q satisfy $0\leq i,j,q\leq n-2$ and $\frac{n-q-1}{p}>1$, $\frac{(n-j)(n-q-1)}{p}\geq 1\geq\frac{(n-i)(n-q-1)}{p}\geq 0$, then*

$$\left(\frac{\widetilde{W}_j(\mathbf{I}_q(D_1\tilde{+}_p D_2))}{\widetilde{W}_i(\mathbf{I}_q(K_1\tilde{+}_p K_2))}\right)^{\frac{p}{(i-j)(n-q-1)}}$$
$$\leq\left(\frac{\widetilde{W}_j(\mathbf{I}_q D_1)}{\widetilde{W}_i(\mathbf{I}_q K_1)}\right)^{\frac{p}{(i-j)(n-q-1)}}+\left(\frac{\widetilde{W}_j(\mathbf{I}_q D_2)}{\widetilde{W}_i(\mathbf{I}_q K_2)}\right)^{\frac{p}{(i-j)(n-q-1)}},$$

with equality if and only if D_1 and D_2 are dilates, and $\frac{\widetilde{W}_j(\mathbf{I}_q D_1)}{\widetilde{W}_j(\mathbf{I}_q D_2)}=\frac{\widetilde{W}_i(\mathbf{I}_q K_1)}{\widetilde{W}_i(\mathbf{I}_q K_2)}$.

Taking $q=0$ in Theorem 3.3 and Corollary 3.4, we get Theorem 1.1 and Corollary 1.4 respectively.

Next, we will establish the Brunn-Minkowski inequality for the dual quermassintegral quotient function of Ψ_q about L_p harmonic radial combination, and the following Lemma will be needed.

Lemma 3.5[9] *Let $\Psi:\underbrace{\mathcal{S}^n\times\cdots\times\mathcal{S}^n}_{n-1}\to\mathcal{S}^n$ be a mixed radial Blaschke Minkowski homomorphism. If $K_1,K_2\in\mathcal{S}^n$, integers i,j satisfy $0\leq i,j\leq n-2$ and $p\geq 1$, then*

$$\widetilde{W}_i(\Psi_j(K_1\hat{+}_{-p}K_2))^{-\frac{p}{(n-i)(n-j-1)}}$$
$$\geq\widetilde{W}_i(\Psi_j K_1)^{-\frac{p}{(n-i)(n-j-1)}}+\widetilde{W}_i(\Psi_j K_2)^{-\frac{p}{(n-i)(n-j-1)}}, \quad (3.8)$$

with equality if and only if K_1 and K_2 are dilates.

Theorem 3.6 *Let $\Psi:\underbrace{\mathcal{S}^n\times\cdots\times\mathcal{S}^n}_{n-1}\to\mathcal{S}^n$ be a mixed radial Blaschke Minkowski homomorphism. For $K_1,K_2,D_1,D_2\in\mathcal{S}^n$, and let K_1 be a dilated copy of K_2, if integers i,j,q satisfy $0\leq i,j,q\leq n-2$, $p\geq 1$ and $\frac{(n-i)(n-q-1)}{p}\geq 1\geq\frac{(n-j)(n-q-1)}{p}\geq 0$, then*

$$\left(\frac{\widetilde{W}_j(\Psi_q(D_1\hat{+}_{-p}D_2))}{\widetilde{W}_i(\Psi_q(K_1\hat{+}_{-p}K_2))}\right)^{\frac{p}{(j-i)(n-q-1)}}$$

$$\leq \left(\frac{\widetilde{W}_j(\Psi_q D_1)}{\widetilde{W}_i(\Psi_q K_1)}\right)^{\frac{p}{(j-i)(n-q-1)}} + \left(\frac{\widetilde{W}_j(\Psi_q D_2)}{\widetilde{W}_i(\Psi_q K_2)}\right)^{\frac{p}{(j-i)(n-q-1)}}, \tag{3.9}$$

with equality if and only if D_1 and D_2 are dilates, and $\frac{\widetilde{W}_j(\Psi_q D_1)}{\widetilde{W}_j(\Psi_q D_2)} = \frac{\widetilde{W}_i(\Psi_q K_1)}{\widetilde{W}_i(\Psi_q K_2)}$.

Proof For $K_1, K_2, D_1, D_2 \in \mathcal{S}^n$, K_1 and K_2 are dilates, for i, j, q satisfy $0 \leq i, j, q \leq n-2$, $p \geq 1$, according to (3.8), then

$$\widetilde{W}_i(\Psi_q(K_1 \hat{+}_{-p} K_2))^{-1} = \left(\widetilde{W}_i(\Psi_q K_1)^{-1\frac{p}{(n-i)(n-q-1)}} + \widetilde{W}_i(\Psi_q K_2)^{-1\frac{p}{(n-i)(n-q-1)}}\right)^{\frac{(n-i)(n-q-1)}{p}}, \tag{3.10}$$

$$\widetilde{W}_j(\Psi_q(D_1 \hat{+}_{-p} D_2))^{-1} \geq \left(\widetilde{W}_j(\Psi_q D_1)^{-1\frac{p}{(n-j)(n-q-1)}} + \widetilde{W}_j(\Psi_q D_2)^{-1\frac{p}{(n-j)(n-q-1)}}\right)^{\frac{(n-j)(n-q-1)}{p}}. \tag{3.11}$$

And if $\frac{(n-i)(n-q-1)}{p} \geq 1 \geq \frac{(n-j)(n-q-1)}{p} \geq 0$, according to (3.10)(3.11), using (3.3), we have

$$\begin{aligned}
&\left(\frac{\widetilde{W}_j(\Psi_q(D_1 \hat{+}_{-p} D_2))}{\widetilde{W}_i(\Psi_q(K_1 \hat{+}_{-p} K_2))}\right)^{\frac{p}{(j-i)(n-q-1)}} \\
&= \left(\frac{\widetilde{W}_i(\Psi_q(K_1 \hat{+}_{-p} K_2))^{-1}}{\widetilde{W}_j(\Psi_q(D_1 \hat{+}_{-p} D_2))^{-1}}\right)^{\frac{p}{(j-i)(n-q-1)}} \\
&\leq \left(\frac{\widetilde{W}_i(\Psi_q K_1)^{-1}}{\widetilde{W}_j(\Psi_q D_1)^{-1}}\right)^{\frac{p}{(j-i)(n-q-1)}} + \left(\frac{\widetilde{W}_i(\Psi_q K_2)^{-1}}{\widetilde{W}_j(\Psi_q D_2)^{-1}}\right)^{\frac{p}{(j-i)(n-q-1)}} \\
&= \left(\frac{\widetilde{W}_j(\Psi_q D_1)}{\widetilde{W}_i(\Psi_q K_1)}\right)^{\frac{p}{(j-i)(n-q-1)}} + \left(\frac{\widetilde{W}_j(\Psi_q D_2)}{\widetilde{W}_i(\Psi_q K_2)}\right)^{\frac{p}{(j-i)(n-q-1)}},
\end{aligned} \tag{3.12}$$

by the equality conditions of the Dresher's inequality, equality holds in (3.12) if and only if D_1 and D_2 are dilates, and $\frac{\widetilde{W}_j(\Psi_q D_1)}{\widetilde{W}_j(\Psi_q D_2)} = \frac{\widetilde{W}_i(\Psi_q K_1)}{\widetilde{W}_i(\Psi_q K_2)}$.

Changing Ψ_q into the intersection body operator $\mathbf{I}_q$ in Theorem 3.6, we get the following Brunn-Minkowski inequality for the dual quermassintegral quotient function of the intersection body operator.

Corollary 3.7 *For $K_1, K_2, D_1, D_2 \in \mathcal{S}^n$, and let K_1 be a dilated copy of K_2, if integers i, j satisfy $0 \leq i, j, q \leq n-2$, $p \geq 1$ and $\frac{(n-i)(n-q-1)}{p} \geq 1 \geq \frac{(n-j)(n-q-1)}{p} \geq 0$, then*

$$\left(\frac{\widetilde{W}_j(\mathbf{I}_q(D_1 \hat{+}_{-p} D_2))}{\widetilde{W}_i(\mathbf{I}_q(K_1 \hat{+}_{-p} K_2))}\right)^{\frac{p}{(j-i)(n-1)}} \leq \left(\frac{\widetilde{W}_j(\mathbf{I}_q D_1)}{\widetilde{W}_i(\mathbf{I}_q K_1)}\right)^{\frac{p}{(j-i)(n-1)}} + \left(\frac{\widetilde{W}_j(\mathbf{I}_q D_2)}{\widetilde{W}_i(\mathbf{I}_q K_2)}\right)^{\frac{p}{(j-i)(n-1)}}, \tag{3.13}$$

with equality if and only if D_1 and D_2 are dilates, and $\frac{\widetilde{W}_j(\mathbf{I}_q D_1)}{\widetilde{W}_j(\mathbf{I}_q D_2)} = \frac{\widetilde{W}_i(\mathbf{I}_q K_1)}{\widetilde{W}_i(\mathbf{I}_q K_2)}$.

Taking $q = 0$ in Theorem 3.6 and Corollary 3.7, we get Theorem 1.2 and Corollary 1.5 respectively.

In order to proof Theorem 3.9, we need the following Lemma.

Lemma 3.8[9] *Let $\Psi : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ be a mixed radial Blaschke Minkowski homomorphism. If $K_1, K_2 \in \mathcal{S}^n$, integers i, j satisfy $0 \leq i, j \leq n-2$ and $j+1 \leq p < n$, then*

$$\widetilde{W}_i(\Psi_j(K_1 \check{+}_{n-p} K_2))^{\frac{n-p}{(n-i)(n-j-1)}} \leq \widetilde{W}_i(\Psi_j K_1)^{\frac{n-p}{(n-i)(n-j-1)}} + \widetilde{W}_i(\Psi_j K_2)^{\frac{n-p}{(n-i)(n-j-1)}}, \tag{3.14}$$

with equality if and only if K_1 and K_2 are dilates.

Finally, we will establish the Brunn-Minkowski inequality for the dual quermassintegral quotient function of Ψ_q about L_p radial Blaschke combination as follows:

Theorem 3.9 *Let $\Psi : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ be a mixed radial Blaschke Minkowski homomorphism. For $K_1, K_2, D_1, D_2 \in \mathcal{S}^n$, let K_1 be a dilated copy of K_2, and if integers i, j, q satisfy $0 \leq i, j, q \leq n-2$, $q+1 \leq p < n$ and $\frac{(n-i)(n-q-1)}{n-p} \geq 1 \geq \frac{(n-j)(n-q-1)}{n-p} \geq 0$, then*

$$\left(\frac{\widetilde{W}_i(\Psi_q(D_1 \check{+}_{n-p} D_2))}{\widetilde{W}_j(\Psi_q(K_1 \check{+}_{n-p} K_2))}\right)^{\frac{n-p}{(j-i)(n-q-1)}} \leq \left(\frac{\widetilde{W}_i(\Psi_q D_1)}{\widetilde{W}_j(\Psi_q K_1)}\right)^{\frac{n-p}{(j-i)(n-q-1)}} + \left(\frac{\widetilde{W}_i(\Psi_q D_2)}{\widetilde{W}_j(\Psi_q K_2)}\right)^{\frac{n-p}{(j-i)(n-q-1)}}, \tag{3.15}$$

with equality if and only if D_1 and D_2 are dilates, and $\frac{\widetilde{W}_i(\Psi_q D_1)}{\widetilde{W}_i(\Psi_q D_2)} = \frac{\widetilde{W}_j(\Psi_q K_1)}{\widetilde{W}_j(\Psi_q K_2)}$.

It's similar to the proof of Theorem 3.3, using the Lemma 3.2, we can easily obtain Theorem 3.9.

Changing Ψ_q into the intersection body operator $\mathbf{I}_q$ in Theorem 3.9, we get

Corollary 3.10 *For $K_1, K_2, D_1, D_2 \in \mathcal{S}^n$, let K_1 be a dilated copy of K_2, and if integers i, j, q satisfy $0 \leq i, j, q \leq n-2$, $q+1 \leq p < n$ and $\frac{(n-i)(n-q-1)}{n-p} \geq 1 \geq \frac{(n-j)(n-q-1)}{n-p} \geq 0$, then*

$$\left(\frac{\widetilde{W}_i(\mathbf{I}_q(D_1 \check{+}_{n-p} D_2))}{\widetilde{W}_j(\mathbf{I}_q(K_1 \check{+}_{n-p} K_2))}\right)^{\frac{n-p}{(j-i)(n-q-1)}} \leq \left(\frac{\widetilde{W}_i(\mathbf{I}_q D_1)}{\widetilde{W}_j(\mathbf{I}_q K_1)}\right)^{\frac{n-p}{(j-i)(n-q-1)}} + \left(\frac{\widetilde{W}_i(\mathbf{I}_q D_2)}{\widetilde{W}_j(\mathbf{I}_q K_2)}\right)^{\frac{n-p}{(j-i)(n-q-1)}},$$

with equality if and only if D_1 and D_2 are dilates and $\frac{\widetilde{W}_i(\mathbf{I}_q D_1)}{\widetilde{W}_i(\mathbf{I}_q D_2)} = \frac{\widetilde{W}_j(\mathbf{I}_q K_1)}{\widetilde{W}_j(\mathbf{I}_q K_2)}$.

Taking $q = 0$ in Theorem 3.9 and Corollary 3.10, we get Theorem 1.3 and Corollary 1.6 respectively.

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