Dual Orlicz Harmonic Mixed Quermassintegrals

Tongyi Ma*, Weidong Wang

Abstract—**In the paper, the notion of dual harmonic mixed quermassintegrals in the classical Brunn-Minkowski theory is extended to that of Orlicz dual harmonic mixed quermassintegrals in the Orlicz-Brunn-Minkowski theory. The analogs of the classical dual Cauchy-Kubota formula, the dual Minkowski isoperimetric inequality and the dual Brunn-Minkowski inequality are established for this new dual Orlicz harmonic mixed quermassintegrals.**

Index Terms—**Dual Orlicz-Brunn-Minkowski theory, Orlicz dual harmonic quermassintegral, dual Orlicz Minkowski isoperimetric inequality, integral geometry, convex geometric analysis.**

I. Introduction

BEGINNING with the groundbreaking articles [13], [25], [26] and the very recent work [10], the classical Brunn-Minkowski theory of convex bodies (see, e.g., [6], [40], [41]) was extended to the Orlicz stage, which is known as the Orlicz Brunn-Minkowski theory. Analogous to the way that Orlicz spaces generalize L_p- spaces (see [36]), it represents a generalization of the L_p-Brunn-Minkowski theory, which emerged in the early 1960s (see [4]), began largely with the initial works [19], [20] in the mid 1990s, and expanded rapidly thereafter (see, e.g., [2], [3], [5], [11], [15], [18], [21], [22], [23], [24], [30], [31], [33], [34], [35], [37], [39], [42], [45], [46], [49]).

Lutwak's dual Brunn-Minkowski theory, introduced in the 1970s, helped achieving a major breakthrough in the solution of the Busemann-Petty problem in the 1990s. In contrast to the Brunn-Minkowski theory, in the dual theory, convex bodies are replaced by star-shaped sets, and projections onto subspaces are replaced by inter-sections with subspaces. The machinery of the dual theory includes dual mixed volumes and important auxiliary bodies known as intersection bodies (see, e.g., [7], [8], [9], [12], [17], [27], [28], [47]). Recently, the dual Orlicz-Brunn-Minkowski theory for star bodies was made by Zhu, Zhou and Xu [48], and later by Gardner, Hug, Weil and Ye [11]. Then followed by Jin, Yuan and Leng [16], [32], et al. In some respects, this is more delicate than the Orlicz-Brunn-Minkowski theory for convex bodies, partly due to the various flavors of star sets that have to be considered. Radial Orlicz addition for two or more star sets is introduced and its basic properties are established. At the same time, they established two fundamental inequalities, the dual Orlicz-Minkowski inequality and the dual Orlicz-Brunn-Minkowski inequality.

As usual, $x\cdot y$ denotes the standard inner product of x and y in $\mathbf{R}^n$; $B=\{x\in\mathbf{R}^n : x\cdot x\le 1\}$ and $S^{n-1}=\partial B$ denote the unit ball and unit sphere in $\mathbf{R}^n$, respectively. The volume of B is $\pi^{n/2}/\Gamma(1+n/2)$, where $\Gamma(\cdot)$ is the Gamma function. For a compact set K in $\mathbf{R}^n$ which is star shaped with respect to the origin, define the radial function ρ_K of K by $\rho_K(x)=\max\{\lambda\ge 0:\lambda x\in K\}$ for $x\in\mathbf{R}^n\backslash\{o\}$. If ρ_K is continuous we shall call K a star body (about the origin). A star body which is centrally symmetric with respect to the origin will be called a centered body. We shall use $\mathcal{S}_o^n$, and $\mathcal{S}_e^n$ to denote respectively the set of star bodies (about the origin) and the set of centered bodies. Two star bodies K and L are dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u\in S^{n-1}$. Obviously, for $K,L\in\mathcal{S}_o^n$,

$$K\subseteq L \quad \text{if and only if} \quad \rho_K\le\rho_L. \tag{1}$$

Hence, a star body is uniquely determined by its radial function. If $c>0$, we have

$$\rho(cK,x)=c\rho(K,x), \quad \text{for } x\in\mathbf{R}^n\backslash\{o\}. \tag{2}$$

More generally, from the definition of the radial function it follows immediately that for $T\in\mathrm{GL}(n)$ the radial function of the image $TK=\{Ty: y\in K\}$ of K is given by (see [40])

$$\rho(TK,x)=\rho(K,T^{-1}x), \quad \text{for } x\in\mathbf{R}^n\backslash\{o\}, \tag{3}$$

where T^{-1} is the inverse of T.

Lutwak [29] introduced the dual quermassintegrals of star bodies which is given by

$$\widetilde{W}_{n-i}(K)=\widetilde{V}_{n-i}(K,B)=\frac{1}{n}\int_{S^{n-1}}\rho_K^i(u)\mathrm{d}S(u) \tag{4}$$

for any $i\in\mathbf{R}$, where S is the spherical Lebesgue measure on S^{n-1} (i.e., the $(n-1)$-dimensional Hausdorff measure) and $\widetilde{W}_0(K)=V(K)$ is the volume of K, $\widetilde{W}_n(K)=V(B)=\omega_n=\pi^{n/2}/\Gamma(1+n/2)$ is the volume of the unit ball B in $\mathbf{R}^n$.

Denote by $\mathrm{vol}_i(\cdot)$ the i-dimensional volume. The importance of the dual quermassintegrals lies in the fact that the $(n-i)$th dual quermassintegral of a star body K is proportional to the mean of the i-dimensional volumes of the slices of K by the i-dimensional subspaces ξ of $\mathbf{R}^n$, that is (see [27]),

$$\widetilde{W}_{n-i}(K)=\frac{\omega_n}{\omega_i}\int_{G(n,i)}\mathrm{vol}_i(K\cap\xi)\mathrm{d}\mu_i(\xi), \tag{5}$$

$i=0,1,\cdots,n-1$, where $G(n,i)$, μ_i and $\mathrm{vol}_i(K\cap\xi)$ denote the Grassmannian manifold of i-dimensional linear subspaces of $\mathbf{R}^n$, the normalized Haar measure on $G(n,i)$ and the i-dimensional volume of slice of K by an $i-$dimensional subspace $\xi\subset\mathbf{R}^n$, respectively. Equation (5) is the well-known dual Cauchy-Kubota formula.

To study the first variation of dual quermassintegrals is an effective approach to find new geometric quantities or measures induced by star bodies.

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*T. Y. Ma is with the College of Mathematics and Statistics, Hexi University, Zhangye, 734000, P. R. China. e-mail: matongyi@126.com.

W. D. Wang is with the Department of Mathematics, China Three Gorges University, Yichang, 443002, P. R. China. e-mail: wdwxh722@163.com.

For star bodies $K, L \in \mathcal{S}_o^n$, $i \in \mathbf{R}, i \neq n$ and any $\varepsilon > 0$, then ith dual mixed quermassintegrals, $\widetilde{W}_i(K, L)$, is defined by:

$$\begin{aligned}\widetilde{W}_i(K,L) &= \frac{1}{n-i}\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K\widetilde{+}\varepsilon\circ L)-\widetilde{W}_i(K)}{\varepsilon}\\ &= \frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i-1}(u)\rho_L(u)\mathrm{d}S(u), \qquad (6)\end{aligned}$$

where $\lambda\circ K\widetilde{+}\mu\circ L = \{\lambda x\widetilde{+}\mu y : x \in K, y \in L, \lambda, \mu \in \mathbf{R}_+\}$ denotes the radial Minkowski linear combination, and

$$\rho(\lambda\circ K\widetilde{+}\mu\circ L,\cdot) = \lambda\rho(K,\cdot)+\mu\rho(L,\cdot).$$

Apparently, for $K, L \in \mathcal{S}_o^n$ and $i \in \mathbf{R}$ with $i \neq n$, the following integral representation for the ith dual mixed quermassintegrals, $\widetilde{W}_i(K, L)$, of K, L is obtained from (6):

$$\widetilde{W}_i(K,L) = \frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i-1}(u)\rho_L(u)\mathrm{d}S(u).$$

If $K, L \in \mathcal{S}_o^n$, and $\lambda, \mu \geq 0$ (not both zero), then for $p > 0$, the L_p-radial combination, $\lambda\circ K\widetilde{+}_p\mu\circ L \in \mathcal{S}_o^n$ is defined by:

$$\rho(\lambda\circ K\widetilde{+}_p\mu\circ L,\cdot)^p = \lambda\rho(K,\cdot)^p+\mu\rho(L,\cdot)^p.$$

For $K, L \in \mathcal{S}_o^n, p > 0$, real $i \neq n$ and any $\varepsilon > 0$, the ith dual L_p-mixed quermassintegrals, $\widetilde{W}_{p,i}(K, L)$, of K, L is defined by

$$\begin{aligned}\widetilde{W}_{p,i}(K,L) &= \frac{p}{n-i}\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K\widetilde{+}_p\varepsilon\circ L)-\widetilde{W}_i(K)}{\varepsilon}\\ &= \frac{1}{n}\int_{S^{n-1}}\rho_K(u)^{n-i-p}\rho_L(u)^p\mathrm{d}S(u). \qquad (7)\end{aligned}$$

Meanwhile, we define

$$\widetilde{W}_i(K) = \widetilde{W}_{0,i}(K,L) = \lim_{p\to 0^+}\widetilde{W}_{p,i}(K,L).$$

In particular, for $p > 0$,

$$\widetilde{V}_p(K,L) = \widetilde{W}_{p,0}(K,L) = \frac{1}{n}\int_{S^{n-1}}\rho_K(u)^{n-p}\rho_L(u)^p\mathrm{d}S(u)$$

is called the dual L_p-mixed volute of star bodies K and L

If $K, L \in \mathcal{S}_o^n$, and $\lambda, \mu \geq 0$ (not both zero), then for $p > 0$, the L_p-harmonic radial combination, $\lambda\diamond K\widetilde{+}_{-p}\mu\diamond L \in \mathcal{S}_o^n$ is defined by:

$$\rho(\lambda\diamond K\widetilde{+}_{-p}\mu\diamond L,\cdot)^{-p} = \lambda\rho(K,\cdot)^{-p}+\mu\rho(L,\cdot)^{-p}. \qquad (8)$$

When $p \geq 1$, it is easy to show that the L_p-harmonic radial combination reduces to Lutwak's L_p-harmonic radial combination (see [20]).

For $\varepsilon > 0, p > 0$ and real $i \neq n$, the ith dual L_p-harmonic mixed quermassintegrals of $K, L \in \mathcal{S}_o^n$ is defined by

$$\begin{aligned}\widetilde{W}_{-p,i}(K,L) &= \frac{-p}{n-i}\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K\widetilde{+}_{-p}\varepsilon\diamond L)-\widetilde{W}_i(K)}{\varepsilon}\\ &= \frac{1}{n}\int_{S^{n-1}}\rho_K(u)^{n+p-i}\rho_L(u)^{-p}\mathrm{d}S(u).\end{aligned} \qquad (9)$$

In particular, $\widetilde{W}_{-p,0}(K, L)$ is also denoted by $\widetilde{V}_{-p}(K, L)$, the dual L_p-harmonic mixed volume of star bodies K and L. When $p \geq 1$, the dual L_p-harmonic mixed quermassintegrals reduces to Wang and Leng's dual L_p-harmonic mixed quermassintegrals (see [43]).

The aim of this paper is to extend the notion of dual L_p-harmonic mixed quermassintegrals to the Orlicz setting. The above cited works [11], [48], and especially the work [10], make it apparent that the time is ripe.

Throughout this paper, we consider convex (or concave) function $\phi : [0,\infty) \to [0,\infty)$, that is strictly increasing and satisfies $\phi(0) = 0$. Let Φ_1 be the class of convex and strictly increasing functions $\phi : [0,\infty) \to [0,\infty)$ such that $\lim_{t\to 0^+}\phi(t) = 0, \lim_{t\to\infty}\phi(t) = +\infty$, and $\phi(0) = 0$. Let Φ_2 be the class of concave and strictly increasing functions $\phi : [0,\infty) \to [0,\infty)$ such that $\lim_{t\to 0^+}\phi(t) = 0, \lim_{t\to\infty}\phi(t) = +\infty$, and $\phi(0) = 0$.

Let $K, L \in \mathcal{S}_o^n$ with radial functions ρ_K, ρ_L, respectively. For $\alpha, \beta \geq 0$(not both zero) and $\phi \in \Phi_1$ or $\phi \in \Phi_2$, we define the Orlicz harmonic radial combination $\alpha\diamond K\widetilde{+}_{-\phi}\beta\diamond L$ of K and L by

$$\begin{aligned}&\rho(\alpha\diamond K\widetilde{+}_{-\phi}\beta\diamond L,u)^{-1}\\ &= \inf\left\{\lambda > 0 : \alpha\phi\left(\frac{1}{\lambda\rho_K(u)}\right)+\beta\phi\left(\frac{1}{\lambda\rho_L(u)}\right) \leq \phi(1)\right\},\end{aligned}$$

$u \in S^{n-1}$. The rest of this paper is organized as follows. In Section 2, we list some basic and well-known facts from Convex Geometry. Some basic properties of the Orlicz harmonic radial combination will be given in Section 3. In Section 4, we compute the Orlicz first variations of ith dual quermassintegrals

$$\begin{aligned}&\frac{-\phi_r'(1)}{n-i}\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K\widetilde{+}_{-\phi}\varepsilon\diamond L)-\widetilde{W}_i(K)}{\varepsilon}\\ &= \frac{1}{n}\int_{S^{n-1}}\phi\left(\frac{\rho_K}{\rho_L}\right)\rho_K(u)^{n-i}\mathrm{d}S(u),\end{aligned}$$

for $i = 0, 1, \cdots, n-1$. From this, if $i \neq n$ is any real, we introduce the following concept of Orlicz dual harmonic mixed quermassintegrals.

Definition 1.1. Let $K, L \in \mathcal{S}_o^n$ and $\phi \in \Phi_1$ or $\phi \in \Phi_2$. Then for real $i \neq n$,

$$\widetilde{W}_{-\phi,i}(K,L) = \frac{1}{n}\int_{S^{n-1}}\phi\left(\frac{\rho_K}{\rho_L}\right)\rho_K(u)^{n-i}\mathrm{d}S(u).$$

The quantity $\widetilde{W}_{-\phi,i}(K, L)$ is called the dual Orlicz mixed quermassintegrals of K and L. If $i = 0$, then $\widetilde{W}_{-\phi,0}(K, L)$ is just the dual Orlicz mixed volume $\widetilde{V}_{-\phi}(K, L)$ (see [11]), and if $\phi(t) = t^p$ with $p > 0$, then $\widetilde{W}_{-\phi,i}(K, L) = \widetilde{W}_{-p,i}(K, L)$ is the ith dual L_p-mixed quermassintegrals.

In Section 5, we show the probabilistic essence of dual Orlicz mixed quermassintegrals from integral geometry (see, e.g., [36], [38]). The classical dual Cauchy-Kubota formula has a natural Orlicz extension:

Theorem 1.2. Suppose $K, L \in \mathcal{S}_o^n$ and $\phi \in \Phi_1$ or $\phi \in \Phi_2$. Then for each $i = 1, \cdots, n-1$,

$$\begin{aligned}&\widetilde{W}_{-\phi,i}(K,L)\\ &= \frac{\omega_n}{\omega_{n-i}}\int_{G(n,n-i)}\widetilde{V}_{-\phi}^{(n-i)}(K\cap\xi, L\cap\xi)\mathrm{d}\mu_{n-i}(\xi),\end{aligned}$$

where $\widetilde{V}_{-\phi}^{(n-i)}(K\cap\xi, L\cap\xi)$ denotes the dual harmonic Orlicz mixed volume of the $(n-i)$-dimensional star bodies $K\cap\xi$ and $L\cap\xi$ in the subspace ξ.

In Section 6, the dual L_p-Minkowski inequality and the dual L_p-Brunn-Minkowski inequality for the dual quermass-integrals are generalized to the Orlicz setting, respectively.

Theorem 1.3. Suppose that $K, L \in \mathcal{S}_o^n$ and real $i \in \mathbf{R}$. If $\phi \in \Phi_1$, then for $i < n$ or $n < i < n+1$,

$$\widetilde{W}_{-\phi,i}(K,L) \geq \widetilde{W}_i(K)\phi\left(\left(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\right)^{\frac{1}{n-i}}\right). \tag{10}$$

If $\phi \in \Phi_2$, then for $i > n+1$, the inequality (10) is reversed. If ϕ is strictly convex (or strictly concave), the equality holds in every inequality if and only if K and L are dilates of each other.

Theorem 1.4. Suppose that $K, L \in \mathcal{S}_o^n$ and real $i \in \mathbf{R}$. If $\phi \in \Phi_1$, then for $i < n$ or $n < i < n+1$,

$$\phi\left(\left(\frac{\widetilde{W}_i(K\widetilde{+}_{-\phi}L)}{\widetilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right) + \phi\left(\left(\frac{\widetilde{W}_i(K\widetilde{+}_{-\phi}L)}{\widetilde{W}_i(L)}\right)^{\frac{1}{n-i}}\right) \leq \phi(1). \tag{11}$$

If $\phi \in \Phi_2$, then for $i > n+1$, the inequality (11) is reversed. If ϕ is strictly convex (or strictly concave), the equality holds in every inequality if and only if K and L are dilates of each other.

II. Preliminaries

We say that the sequence $\{\phi_i\}$, of $\phi_i \in \Phi_1$ (or $\phi_i \in \Phi_2$), is such that $\phi_i \to \phi_0 \in \Phi_1$ (or Φ_2) provided

$$|\phi_i - \phi_0|_I := \max_{t\in I} |\phi_i(t) - \phi_0(t)| \to 0,$$

for every compact interval $I \subset \mathbf{R}$.

We define a metric $\tilde{\delta}$ on $\mathcal{S}_o^n$, the radial Hausdorff metric, as follows: If $K, L \in \mathcal{S}_o^n$, then

$$\widetilde{\delta}(K,L) = \sup_{u\in S^{n-1}} |\rho_K(u) - \rho_L(u)| := ||\rho(K,\cdot) - \rho(L,\cdot)||_\infty.$$

A sequence $\{K_i\}$ of star bodies is said to be convergent to K if $\widetilde{\delta}(K_i, K) \to 0,$ as $i \to \infty$. Therefore, a sequence of star bodies K_i converges to K if and only if the sequence of radial functions $\rho(K_i,\cdot)$ converges uniformly to $\rho(K,\cdot)$.

For $K, L \in \mathcal{S}_o^n, p \geq 1$, the dual L_p-harmonic mixed volume $\widetilde{V}_{-p}(K,L)$ of K and L is defined by

$$\widetilde{V}_{-p}(K,L) = \frac{-p}{n} \lim_{\varepsilon\to 0^+} \frac{V(K\widetilde{+}_{-p}\varepsilon \diamond L) - V(K)}{\varepsilon}. \tag{12}$$

In [20] Lutwak also proved the following integral representation for the dual L_p-harmonic mixed volume: For $K, L \in \mathcal{S}_o^n$, and $p \geq 1$,

$$\widetilde{V}_{-p}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho_K(u)^{n+p}\rho_L(u)^{-p}\mathrm{d}S(u). \tag{13}$$

Remark 2.1. In fact, the condition $p \geq 1$ of inequalities (12) and (13) can be extended to $p > 0$.

In [43], Wang and Leng proposed the concept of the dual L_p-harmonic mixed quermassintegrals, and proved the following analog of the Minkowski inequality for the dual L_p-harmonic mixed quermassintegrals: If $K, L \in S_o^n, p \geq 1$, then for $i < n$ or $n < i < n+p$,

$$\widetilde{W}_{-p,i}(K,L) \geq \widetilde{W}_i(K)^{\frac{n+p-i}{n-i}}\widetilde{W}_i(L)^{\frac{-p}{n-i}}, \tag{14}$$

and for $i > n+p$, inequality (14) is reversed. The equality holds in every inequality if and only if K and L are dilates each other.

Wang and Leng also proved the dual L_p-Brunn-Minkowski inequality for the dual L_p-harmonic mixed quermassintegrals: Suppose $K, L \in S_o^n, p \geq 1$ and $\alpha, \beta > 0$. If real $i < n$ or $n < i < n+p$, then (see [44])

$$\begin{aligned}&\widetilde{W}_i(\alpha \diamond K\widetilde{+}_{-p}\beta \diamond L)^{-p/(n-i)}\\ \geq\ & \alpha\widetilde{W}_i(K)^{-p/(n-i)} + \beta\widetilde{W}_i(L)^{-p/(n-i)},\end{aligned} \tag{15}$$

For $i > n+p$ inequality (15) is reversed. the equality holds in every inequality if and only if K and L are dilates each other.

Remark 2.2. In fact, the condition $p \geq 1$ in inequalities (14) and (15) can be extended to $p > 0$.

The dual cone-quermassintegral measure, $\widetilde{W}_K(\omega)$, of the star body K is a Borel measure on S^{n-1} defined for a Borel set $\omega \subseteq S^{n-1}$ by

$$\widetilde{W}_K(\omega) = \frac{1}{n}\int_\omega \rho_K^{n-i}\mathrm{d}S.$$

For $K \in \mathcal{S}_o^n$, it will be convenient to use the dual quermassintegral-normalized conical measure $V_{n-i}^*(K,\cdot)$ defined by

$$\widetilde{W}_i(K)\mathrm{d}V_{n-i}^*(K,\cdot) = \frac{1}{n}\rho_K^{n-i}\mathrm{d}S. \tag{16}$$

Note that the dual quermassintegral-normalized conical measure $V_{n-i}^*(K,\cdot)$ is a probability measure on S^{n-1}.

Suppose that μ is a probability measure on a space X and $f : X \to I \subset \mathbf{R}$ is a μ-integrable function, where I is a possibly infinite interval. Jensen's inequality states that if $\phi : I \to \mathbf{R}$ is a convex function, then

$$\int_X \phi\big(f(x)\big)\mathrm{d}\mu(x) \geq \phi\left(\int_X f(x)\mathrm{d}\mu(x)\right). \tag{17}$$

If $\phi : I \to \mathbf{R}$ is a concave function, then inequality (17) is reversed. If ϕ is strictly convex (or strictly concave), the equality holds in every inequality if and only if $f(x)$ is constant for μ-almost all $x \in X$ (see [14]).

III. Orlicz harmonic radial combination

We first define the Orlicz harmonic radial combination.

Definition 3.1. Let $K, L \in \mathcal{S}_o^n$ with radial functions ρ_K, ρ_L, respectively. For $\alpha, \beta \geq 0$(not both zero) and $\phi \in \Phi_1$ or $\phi \in \Phi_2$, define the Orlicz harmonic radial combination $\alpha \diamond K\widetilde{+}_{-\phi}\beta \diamond L$ of K and L as the star body whose radial function at $u \in S^{n-1}$ is given by

$$\begin{aligned}&\rho(\alpha \diamond K\widetilde{+}_{-\phi}\beta \diamond L, u)^{-1}\\ =\ & \inf\left\{\lambda > 0 : \alpha\phi\left(\frac{1}{\lambda\rho_K(u)}\right) + \beta\phi\left(\frac{1}{\lambda\rho_L(u)}\right) \leq \phi(1)\right\}.\end{aligned} \tag{18}$$

It is noted that $\rho(\alpha \diamond K\widetilde{+}_{-\phi}\beta \diamond L, u)$ can be defined for all $u \in S^{n-1}$ by the equation

$$\alpha\phi\left(\frac{\rho(\alpha \diamond K\widetilde{+}_{-\phi}\beta \diamond L, u)}{\rho_K(u)}\right) + \beta\phi\left(\frac{\rho(\alpha \diamond K\widetilde{+}_{-\phi}\beta \diamond L, u)}{\rho_L(u)}\right) = \phi(1).$$

If $\phi(t) = t^p$ with $0 < p < \infty$, then $\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L = \alpha \diamond K \widetilde{+}_{-p} \beta \diamond L$.

From (3) and the definition of Orlicz harmonic radial combination, we have:

Proposition 3.2. Suppose that $K, L \in \mathcal{S}_o^n$, and $\alpha, \beta \geq 0$. If $\phi \in \Phi_1$ or $\phi \in \Phi_2$, then for $T \in \mathrm{GL}(n)$,

$$T(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L) = \alpha \diamond TK \widetilde{+}_{-\phi} \beta \diamond TL.$$

Proof. For $u \in S^{n-1}$, by (3) and the Definition 3.1 of Orlicz harmonic radial combination, we have

$$\begin{aligned}
& \rho(\alpha \diamond TK \widetilde{+}_{-\phi} \beta \diamond TL, u)^{-1} \\
= \;& \inf\left\{\lambda > 0 : \alpha\phi\left(\frac{1}{\lambda\rho_{TK}(u)}\right) + \beta\phi\left(\frac{1}{\lambda\rho_{TL}(u)}\right) \leq \phi(1)\right\} \\
= \;& \inf\left\{\lambda > 0 : \alpha\phi\left(\frac{1}{\lambda\rho_{K}(T^{-1}u)}\right) + \beta\phi\left(\frac{1}{\lambda\rho_{L}(T^{-1}u)}\right) \leq \phi(1)\right\} \\
= \;& \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, T^{-1}u)^{-1} \\
= \;& \rho(T(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L), u)^{-1}.
\end{aligned}$$

Since $K, L \in \mathcal{S}_o^n, 0 < \rho_K < \infty$ and $0 < \rho_L < \infty$, then $\frac{1}{\lambda\rho_K} \to 0$ and $\frac{1}{\lambda\rho_L} \to 0$ as $\lambda \to \infty$. By this and the assumption that ϕ is strictly increasing in $(0, \infty)$, the function

$$\lambda \mapsto \alpha\phi\left(\frac{1}{\lambda\rho_K}\right) + \beta\phi\left(\frac{1}{\lambda\rho_L}\right)$$

is strictly decreasing and continuous in $(0, \infty)$. Thus, we have:

Lemma 3.3. Suppose that $K, L \in \mathcal{S}_o^n$ and $u \in S^{n-1}$. If $\phi \in \Phi_1$ or $\phi \in \Phi_2$, then

$$\alpha\phi\left(\frac{1}{\lambda\rho_K(u)}\right) + \beta\phi\left(\frac{1}{\lambda\rho_L(u)}\right) = \phi(1)$$

if and only if

$$\lambda = \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u)^{-1}.$$

For $K \in \mathcal{S}_o^n$, define the real numbers R_K and r_K by

$$R_K = \max_{u \in S^{n-1}} \rho_K(u) \quad \text{and} \quad r_K = \min_{u \in S^{n-1}} \rho_K(u).$$

Note that $0 < r_K < R_K < \infty$, for all $K \in \mathcal{S}_o^n$. If $K, L \in \mathcal{S}_o^n$, let $R = \max\{R_K, R_L\}$ and $r = \min\{r_K, r_L\}$. For $\alpha, \beta \geq 0$, let $M = \max\{\alpha, \beta\}, m = \min\{\alpha, \beta\}$ and $c = \alpha + \beta$. Since ϕ is continuous and strictly increasing in $(0, \infty)$, hence the inverse ϕ^{-1} is also continuous and increasing in $(0, \infty)$.

Lemma 3.4. Suppose that $K, L \in \mathcal{S}_o^n$. Then

(1) If $\phi \in \Phi_1$, then for all $u \in S^{n-1}$,

$$r\phi^{-1}\left(\frac{\phi(1)}{2M}\right) \leq \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u) \leq R\phi^{-1}\left(\frac{\phi(1)}{c}\right).$$

(2) If $\phi \in \Phi_2$, then for all $u \in S^{n-1}$,

$$r\phi^{-1}\left(\frac{\phi(1)}{c}\right) \leq \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u) \leq R\phi^{-1}\left(\frac{\phi(1)}{2m}\right).$$

Proof. Since two assertions can be proved similarly, we only give that the proof of (1). Suppose $u \in S^{n-1}$ and $\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u)^{-1} = \lambda$. By Lemma 3.3 and the fact that ϕ is strictly increasing on $(0, \infty)$, we have

$$\begin{aligned}
\phi(1) &= \alpha\phi\left(\tfrac{1}{\lambda\rho_K(u)}\right) + \beta\phi\left(\tfrac{1}{\lambda\rho_L(u)}\right) \\
&\leq M\phi\left(\tfrac{1}{\lambda r_K}\right) + M\phi\left(\tfrac{1}{\lambda r_L}\right) \\
&\leq 2M\phi\left(\tfrac{1}{\lambda r}\right).
\end{aligned}$$

Since the inverse ϕ^{-1} of ϕ is strictly increasing on $(0, \infty)$, we have the lower bound for $\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u)$:

$$\frac{1}{\lambda} \geq r\phi^{-1}\left(\frac{\phi(1)}{2M}\right).$$

On the other hand, from Lemma 3.3 and Jensen's inequality, together with the convexity and the strictly increasing on $(0, \infty)$ of ϕ, we have

$$\begin{aligned}
\frac{\phi(1)}{\alpha+\beta} &= \tfrac{\alpha}{\alpha+\beta}\phi\left(\tfrac{1}{\lambda\rho_K(u)}\right) + \tfrac{\beta}{\alpha+\beta}\phi\left(\tfrac{1}{\lambda\rho_L(u)}\right) \\
&\geq \tfrac{\alpha}{\alpha+\beta}\phi\left(\tfrac{1}{\lambda R_K}\right) + \tfrac{\beta}{\alpha+\beta}\phi\left(\tfrac{1}{\lambda R_L}\right) \\
&\geq \phi\left(\tfrac{\alpha}{\alpha+\beta} \cdot \tfrac{1}{\lambda R_K} + \tfrac{\beta}{\alpha+\beta} \cdot \tfrac{1}{\lambda R_L}\right) \\
&\geq \phi\left(\tfrac{1}{\lambda R}\right).
\end{aligned}$$

Then we obtain the upper estimate:

$$\frac{1}{\lambda} \leq R\phi^{-1}\left(\frac{\phi(1)}{c}\right).$$

We now show that the Orlicz harmonic radial combination of two star bodies is also a star body.

Lemma 3.5. Suppose that $\phi \in \Phi_1$ or $\phi \in \Phi_2$ and $\alpha, \beta \geq 0$ (not both zero). If $K, L \in \mathcal{S}_o^n$, then $\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L \in \mathcal{S}_o^n$.

Proof. Let $u_0 \in S^{n-1}$. For any subsequence $\{u_i\}_{i\in\mathbf{N}} \subset S^{n-1}$ such that $u_i \to u_0$ as $i \to \infty$, we need to show

$$\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u_i) \to \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u_0), \quad \text{as } i \to \infty.$$

Let

$$\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u_i) = \frac{1}{\lambda_i} := \mu_i, \quad \lambda_i > 0.$$

Then Lemma 3.4 gives

$$r\phi^{-1}\left(\frac{\phi(1)}{2M}\right) \leq \mu_i \leq R\phi^{-1}\left(\frac{\phi(1)}{c}\right), \quad \text{for } \phi \in \Phi_1,$$

and

$$r\phi^{-1}\left(\frac{\phi(1)}{c}\right) \leq \mu_i \leq R\phi^{-1}\left(\frac{\phi(1)}{2m}\right), \quad \text{for } \phi \in \Phi_2.$$

Since $K, L \in \mathcal{S}_o^n$, we have $0 < r_K \leq R_K < \infty, 0 < r_L \leq R_L < \infty$. Thus, there exist a, b such that $0 < a \leq \mu_i \leq b < \infty$, for all i. To show that the bounded sequence $\{\mu_i\}_{i\in\mathbf{N}}$ converges to $\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u_0)$, we show that every convergent subsequence of $\{\mu_i\}_{i\in\mathbf{N}}$ converges to $\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u_0)$. Denote an arbitrary convergent subsequence of $\{\mu_i\}_{i\in\mathbf{N}}$ by $\{\mu_i\}_{i\in\mathbf{N}}$ as well, and suppose that for this subsequence

$$\mu_i \to \mu_0 \quad \text{as} \quad i \to \infty.$$

It is clear that $a \le \mu_0 \le b$. Lemma 3.3 and the fact $\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u_i) = \mu_i$ show that

$$\alpha\phi\left(\frac{\mu_i}{\rho_K(u_i)}\right) + \beta\phi\left(\frac{\mu_i}{\rho_L(u_i)}\right) = \phi(1).$$

Since ρ_K and ρ_L are continuous on S^{n-1}, together with the continuity of ϕ and $\mu_i \to \mu_0$, it follows that

$$\alpha\phi\left(\frac{\mu_0}{\rho_K(u_0)}\right) + \beta\phi\left(\frac{\mu_0}{\rho_L(u_0)}\right) = \phi(1).$$

By Lemma 3.3, we have

$$\mu_0 = \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u_0).$$

This shows

$$\rho(\alpha\diamond K\widetilde{+}_{-\phi}\beta\diamond L, u_i) \to \rho(\alpha\diamond K\widetilde{+}_{-\phi}\beta\diamond L, u_0), \quad \text{as } i \to \infty.$$

Therefore, the continuity of $\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, \cdot)$ is proved and $\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L \in \mathcal{S}_o^n$.

From the Definition 3.1 of the Orlicz harmonic radial combination, for $c > 0$, we have

$$\begin{aligned}
&\rho(\alpha \diamond (cK)\widetilde{+}_{-\phi}\beta \diamond (cL), u)^{-1} \\
=\ & \inf\left\{\lambda > 0 : \alpha\phi\left(\frac{1}{\lambda\rho_{cK}(u)}\right)\right. \\
&\left. +\beta\phi\left(\frac{1}{\lambda\rho_{cL}(u)}\right) \le \phi(1)\right\} \\
=\ & c^{-1}\inf\left\{c\lambda > 0 : \alpha\phi\left(\frac{1}{c\lambda\rho_K(u)}\right)\right. \\
&\left. +\beta\phi\left(\frac{1}{c\lambda\rho_L(u)}\right) \le \phi(1)\right\} \\
=\ & c^{-1}\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u)^{-1}.
\end{aligned}$$

This gives that

$$\rho(\alpha \diamond (cK)\widetilde{+}_{-\phi}\beta \diamond (cL), u) = c\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u). \quad (19)$$

Next, we show that the Orlicz harmonic radial combination $\widetilde{+}_{-\phi} : \mathcal{S}_0^n \to \mathcal{S}_o^n$ is continuous.

Lemma 3.6. Suppose that $\phi \in \Phi_1$ or $\phi \in \Phi_2$. If $K_i, L_i \in \mathcal{S}_o^n$ and $K_i \to K \in \mathcal{S}_o^n, L_i \to L \in \mathcal{S}_o^n$, as $i \to \infty$, then

$$\alpha \diamond K_i \widetilde{+}_{-\phi} \beta \diamond L_i \to \alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, \quad \text{as } i \to \infty$$

for all α and β.

Proof. Suppose $u \in S^{n-1}$. We will show that

$$\begin{aligned}
&\rho(\alpha \diamond K_i \widetilde{+}_{-\phi} \beta \diamond L_i, u) \\
\to\ & \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u), \quad \text{as } i \to \infty. \quad (20)
\end{aligned}$$

Let

$$\rho(\alpha \diamond K_i \widetilde{+}_{-\phi} \beta \diamond L_i, u) = \frac{1}{\lambda_i} := \mu_i, \quad \lambda_i > 0,$$

and write $R_i = \max\{R_{K_i}, R_{L_i}\}$ and $r_i = \min\{r_{K_i}, r_{L_i}\}$. Then Lemma 3.4 gives

$$r_i\phi^{-1}\left(\frac{\phi(1)}{2M}\right) \le \mu_i \le R_i\phi^{-1}\left(\frac{\phi(1)}{c}\right), \quad \text{for } \phi \in \Phi_1,$$

and

$$r_i\phi^{-1}\left(\frac{\phi(1)}{c}\right) \le \mu_i \le R_i\phi^{-1}\left(\frac{\phi(1)}{2m}\right), \quad \text{for } \phi \in \Phi_2.$$

Since $K_i \to K \in \mathcal{S}_o^n$ and $L_i \to L \in \mathcal{S}_o^n$, as $i \to \infty$, we have $R_{K_i} \to R_K < \infty, R_{L_i} \to R_L < \infty$, and $r_{K_i} \to r_K > 0, r_{L_i} \to r_L > 0$. By the fact that the functions $R_i = \max\{R_{K_i}, R_{L_i}\}$ and $r_i = \min\{r_{K_i}, r_{L_i}\}$ are continuous, we have $R_i \to R < \infty, r_i \to r > 0$. Thus, there exist a, b such that

$$0 < a \le \mu_i \le b < \infty, \quad \text{for all } i \in \mathbf{N}. \quad (21)$$

To show that the bounded sequence $\{\mu_i\}_{i\in\mathbf{N}}$ converges to $\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u)$, we show that every convergent subsequence of $\{\mu_i\}_{i\in\mathbf{N}}$ converges to $\rho(\alpha\diamond K\widetilde{+}_{-\phi}\beta\diamond L, u)$. Denote an arbitrary convergent subsequence of $\{\mu_i\}_{i\in\mathbf{N}}$ by $\{\mu_i\}_{i\in\mathbf{N}}$ as well, and suppose that for this subsequence we have

$$\mu_i \to \mu_0 \quad \text{as } i \to \infty.$$

It is clear that $a \le \mu_0 \le b$. Let $\widetilde{K}_i = \mu_i^{-1}K_i$ and $\widetilde{L}_i = \mu_i^{-1}L_i$. Since $K_i \to K, L_i \to L$, and $\mu_i^{-1} \to \mu_0^{-1}$, we have $\widetilde{K}_i = \mu_i^{-1}K_i \to \mu_0^{-1}K$ and $\widetilde{L}_i = \mu_i^{-1}L_i \to \mu_0^{-1}L$.

Now (19), and the fact $\rho(\alpha\diamond K_i\widetilde{+}_{-\phi}\beta\diamond L_i, u) = \mu_i$, show that $\rho(\alpha \diamond \widetilde{K}_i \widetilde{+}_{-\phi} \beta \diamond \widetilde{L}_i, u) = 1$. That is,

$$\alpha\phi\left(\frac{1}{\rho_{\widetilde{K}_i}(u)}\right) + \beta\phi\left(\frac{1}{\rho_{\widetilde{L}_i}(u)}\right) = \phi(1), \quad \text{for all } i \in \mathbf{N}.$$

Since $\widetilde{K}_i = \mu_i^{-1}K_i \to \mu_0^{-1}K$ and $\widetilde{L}_i = \mu_i^{-1}L_i \to \mu_0^{-1}L$, together with the continuity of ϕ, and (2), it follows that

$$\alpha\phi\left(\frac{\mu_0}{\rho_K(u)}\right) + \beta\phi\left(\frac{\mu_0}{\rho_L(u)}\right) = \phi(1).$$

By Lemma 3.3, we have

$$\mu_0 = \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u).$$

This shows

$$\rho(\alpha\diamond K_i\widetilde{+}_{-\phi}\beta\diamond L_i, u) \to \rho(\alpha\diamond K\widetilde{+}_{-\phi}\beta\diamond L, u), \quad \text{as } i \to \infty.$$

Now the pointwise convergence (20) has been proved.

We will show that the convergence (20) is uniform for any $u_0 \in S^{n-1}$. Assume that $\rho(\alpha \diamond K_i \widetilde{+}_{-\phi} \beta \diamond L_i, \cdot)$ does not converge uniformly to $\rho(\alpha\diamond K\widetilde{+}_{-\phi}\beta\diamond L, \cdot)$. Then, there exists a $\varepsilon_0 > 0$ and an $N_0 \in \mathbf{N}$ such that, for $i \ge N_0$,

$$|\rho(\alpha \diamond K_i \widetilde{+}_{-\phi} \beta \diamond L_i, u_i) - \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u_i)| \ge \varepsilon_0. \quad (22)$$

Since S^{n-1} is compact, for some $u_0 \in S^{n-1}$, there exists a subsequence $\{u_i\}_{i\in\mathbf{N}} \subset S^{n-1}$ such that $u_i \to u_0$ as $i \to \infty$.

From Lemma 3.4, there exist an $N_1 \in \mathbf{N}$ and positive a, b such that (21) holds for $i \ge N_1$. Then for every number $\varepsilon > 0$, there exists a positive μ_0' such that

$$|\rho(\alpha \diamond K_i \widetilde{+}_{-\phi} \beta \diamond L_i, u_i) - \mu_0'| < \varepsilon,$$

whenever $i \ge N = \max\{N_0, N_1\}$, this means that

$$\rho(\alpha \diamond K_i \widetilde{+}_{-\phi} \beta \diamond L_i, u_i) \to \mu_0' \quad \text{as} \quad i \to \infty.$$

From (22), we have

$$|\mu_0' - \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u_0)| \ge \varepsilon_0.$$

This implies

$$\mu_0' \ne \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u_0). \quad (23)$$

Let $\mu_i' = \rho(\alpha \diamond K_i \widetilde{+}_{-\phi} \beta \diamond L_i, u_i)$ and $\mu_i' \to \mu_0'$ as $i \to \infty$. By Lemma 3.3, we have

$$\alpha\phi\left(\frac{\mu_i'}{\rho_{K_i}(u_i)}\right) + \beta\phi\left(\frac{\mu_i'}{\rho_{L_i}(u_i)}\right) = \phi(1).$$

Together with the facts that $K_i \to K, L_i \to L$ and $\mu_i' \to \mu_0'$ as $i \to \infty$, we get that

$$\alpha\phi\left(\frac{\mu_0'}{\rho_K(u_0)}\right) + \beta\phi\left(\frac{\mu_0'}{\rho_L(u_0)}\right) = \phi(1).$$

By Lemma 3.3 again, we have

$$\mu_0' = \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u_0).$$

Contradicts to (23), we have

$$\rho(\alpha \diamond K_i \widetilde{+}_{-\phi} \beta \diamond L_i, u) \to \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u)$$

uniformly on S^{n-1} and hence

$$\alpha \diamond K_i \widetilde{+}_{-\phi} \beta \diamond L_i \to \alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, \quad \text{as } i \to \infty.$$

We will see that the Orlicz harmonic radial combination $\widetilde{+}_{-\phi}$ is continuous in α and β.

Lemma 3.7. Suppose that $\phi \in \Phi_1$ or $\phi \in \Phi_2$. If $\alpha_i, \beta_i \geq 0$ and $\alpha_i \to \alpha, \beta_i \to \beta$, as $i \to \infty$, then

$$\alpha_i \diamond K \widetilde{+}_{-\phi} \beta_i \diamond L \to \alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, \quad \text{as } i \to \infty,$$

for all $K, L \in \mathcal{S}_o^n$.

Proof. Suppose that $u \in \mathcal{S}^{n-1}$ and $K, L \in \mathcal{S}_o^n$. Next we will show that

$$\begin{aligned} &\rho(\alpha_i \diamond K \widetilde{+}_{-\phi} \beta_i \diamond L, u) \\ \to \ &\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u), \quad \text{as } i \to \infty. \end{aligned} \tag{24}$$

Let

$$\rho(\alpha_i \diamond K \widetilde{+}_{-\phi} \beta_i \diamond L, u) = \mu_i.$$

By Lemma 3.4 we have

$$r\phi^{-1}\left(\frac{\phi(1)}{2M_i}\right) \leq \mu_i \leq R\phi^{-1}\left(\frac{\phi(1)}{c_i}\right), \quad \text{for } \phi \in \Phi_1,$$

and

$$r\phi^{-1}\left(\frac{\phi(1)}{c_i}\right) \leq \mu_i \leq R\phi^{-1}\left(\frac{\phi(1)}{2m_i}\right), \quad \text{for } \phi \in \Phi_2.$$

Since $\alpha_i \to \alpha, \beta_i \to \beta$ as $i \to \infty$ and the facts that the functions $M_i = \max\{\alpha_i, \beta_i\}, m_i = \min\{\alpha_i, \beta_i\}$ and $c_i = \alpha_i + \beta_i$ are continuous, we have $M_i \to M, m_i \to m$ and $c_i \to c$, as $i \to \infty$. Since the inverse ϕ^{-1} of ϕ is also continuous and increasing in $(0, \infty)$, there exist a, b such that $0 < a \leq \mu_i \leq b < \infty$, for all i. To show that the bound sequence $\{\mu_i\}_{i \in \mathbf{N}}$ converges to $\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u)$, we show that every convergent subsequence of $\{\mu_i\}_{i \in \mathbf{N}}$ converges to $\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u)$. Denote an arbitrary convergent subsequence of $\{\mu_i\}_{i \in \mathbf{N}}$ by $\{\mu_i\}_{i \in \mathbf{N}}$ as well, and suppose that for this subsequence we have

$$\mu_i \to \mu_0 \quad \text{as } i \to \infty.$$

It is clear that $0 < a \leq \mu_0 \leq b < \infty$. Since $\rho(\alpha_i \diamond K \widetilde{+}_{-\phi} \beta_i \diamond L, u) = \mu_i$, that is,

$$\alpha\phi\left(\frac{\mu_i}{\rho_K(u)}\right) + \beta\phi\left(\frac{\mu_i}{\rho_L(u)}\right) = \phi(1), \quad \text{for all } i \in \mathbf{N}.$$

Since $\alpha_i \to \alpha$ and $\beta_i \to \beta$, together with the continuity of ϕ, and $\mu_i \to \mu_0$ as $i \to \infty$, it follows that

$$\alpha\phi\left(\frac{\mu_0}{\rho_K(u)}\right) + \beta\phi\left(\frac{\mu_0}{\rho_L(u)}\right) = \phi(1).$$

By Lemma 3.3, we have

$$\mu_0 = \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u).$$

This shows

$$\rho(\alpha_i \diamond K \widetilde{+}_{-\phi} \beta_i \diamond L, u) \to \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u), \quad \text{as } i \to \infty.$$

Now the pointwise convergence (24) has been proved.

To show the convergence (24) is uniform on S^{n-1}, we assume that $\rho(\alpha_i \diamond K \widetilde{+}_{-\phi} \beta_i \diamond L, \cdot)$ does not converge uniformly to $\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, \cdot)$. Then, there exist a positive ε_0 and an $N_0 \in \mathbf{N}$ such that, for $i \geq N_0$,

$$|\rho(\alpha_i \diamond K \widetilde{+}_{-\phi} \beta_i \diamond L, u_i) - \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u_i)| \geq \varepsilon_0. \tag{25}$$

Since S^{n-1} is compact, for $u_0 \in S^{n-1}$, there exists a subsequence $\{u_i\}_{i \in \mathbf{N}} \subset S^{n-1}$ such that $u_i \to u_0$ as $i \to \infty$.

From Lemma 3.4, there exist an $N_1 \in \mathbf{N}$ and positive a, b such that, for $i \geq N_1$,

$$0 < a \leq \rho(\alpha_i \diamond K \widetilde{+}_{-\phi} \beta_i \diamond L, u_i) \leq b < \infty.$$

Then, for any $\varepsilon > 0$, there exists a positive μ_0' such that for all $i \geq N = \max\{N_0, N_1\}$,

$$|\rho(\alpha_i \diamond K \widetilde{+}_{-\phi} \beta_i \diamond L, u_i) - \mu_0'| < \varepsilon,$$

this means that

$$\rho(\alpha_i \diamond K \widetilde{+}_{-\phi} \beta_i \diamond L, u_i) \to \mu_0', \quad \text{as} \ \ i \to \infty.$$

From (25), we have

$$|\mu_0' - \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u_0)| \geq \varepsilon_0.$$

This implies

$$\mu_0' \neq \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u_0). \tag{26}$$

Let $\mu_i' = \rho(\alpha_i * K \widetilde{+}_{-\phi} \beta_i * L, u_i)$. By Lemma 3.3, we have

$$\alpha_i\phi\left(\frac{\mu_i'}{\rho_K(u_i)}\right) + \beta_i\phi\left(\frac{\mu_i'}{\rho_L(u_i)}\right) = \phi(1).$$

This, together with the facts that $\alpha_i \to \alpha, \beta_i \to \beta, \mu_i' \to \mu_0'$ and the continuity of ϕ, gives

$$\alpha\phi\left(\frac{\mu_0'}{\rho_K(u_0)}\right) + \beta\phi\left(\frac{\mu_0'}{\rho_L(u_0)}\right) = \phi(1).$$

By Lemma 3.3 again, we have

$$\mu_0' = \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u_0).$$

This contradicts to (26). Therefore,

$$\rho(\alpha_i \diamond K \widetilde{+}_{-\phi} \beta_i \diamond L, u) \to \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u), \quad \text{as } i \to \infty.$$

uniformly on S^{n-1}, and

$$\alpha_i \diamond K \widetilde{+}_{-\phi} \beta_i \diamond L \to \alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, \quad \text{as } i \to \infty.$$

The following lemma shows that the Orlicz harmonic radial combination and the L_1-harmonic radial combination are closely related.

Lemma 3.8. Let $K, L \in \mathcal{S}_o^n$ and $0 < \alpha < 1$. If $\phi \in \Phi_1$, then

$$(1-\alpha) \diamond K \widetilde{+}_{-\phi} \alpha \diamond L \subseteq (1-\alpha) \diamond K \widetilde{+}_{-1} \alpha \diamond L. \tag{27}$$

If $\phi \in \Phi_2$, then

$$(1-\alpha) \diamond K \widetilde{+}_{-\phi} \alpha \diamond L \supseteq (1-\alpha) \diamond K \widetilde{+}_{-1} \alpha \diamond L.$$

If ϕ is strictly convex (strictly concave), the equality holds in every inequality if and only if K and L are dilates of each other.

Proof. Since two assertions can be proved similarly, we only give the proof of (27). Let $K_\alpha = (1-\alpha) \diamond K \widetilde{+}_{-\phi} \alpha \diamond L$. By Lemma 3.3 and convexity of ϕ, we have

$$\begin{aligned} \phi(1) &= (1-\alpha)\phi\left(\frac{\rho_{K_\alpha}}{\rho_K(u)}\right) + \alpha\phi\left(\frac{\rho_{K_\alpha}}{\rho_L(u)}\right) \\ &\geq \phi\left(\frac{(1-\alpha)\rho_K^{-1}(u) + \alpha\rho_L^{-1}(u)}{\rho_{K_\alpha}^{-1}(u)}\right) \\ &= \phi\left(\frac{\rho((1-\alpha) \diamond K \widetilde{+}_{-1} \alpha \diamond L, u)^{-1}}{\rho_{K_\alpha}^{-1}(u)}\right). \end{aligned}$$

Since ϕ is strictly increasing on $(0, \infty)$, then we have

$$\rho(K_\alpha, u)^{-1} \geq \rho((1-\alpha) \diamond K \widetilde{+}_{-1} \alpha \diamond L, u)^{-1}.$$

This is,

$$\rho((1-\alpha) \diamond K \widetilde{+}_{-1} \alpha \diamond L, u) \geq \rho(K_\alpha, u).$$

By (1), we obtain the desired inclusion. From the equality condition in Jensen's inequality (17), if ϕ is strictly convex, then equation holds in (27) if and only if K and L are dilates of each other.

IV. Dual Orlicz harmonic mixed quermassintegrals

We denote the right derivative of a real-valued function f by f'_r. For $\phi \in \Phi_1$ (or $\phi \in \Phi_2$), there is $\phi'_r(1) > 0$ because ϕ is convex (or concave) and strictly increasing.

Lemma 4.1. Let $K, L \in \mathcal{S}_o^n$ and $\phi \in \Phi_1$ or $\phi \in \Phi_2$. Then the convergence in

$$\begin{aligned} &\lim_{\varepsilon \to 0^+} \frac{\rho_{K \widetilde{+}_{-\phi} \varepsilon \diamond L}(u) - \rho_K(u)}{\varepsilon} \\ &= -\frac{\rho_K(u)}{\phi'_r(1)} \phi\left(\frac{\rho_K(u)}{\rho_L(u)}\right) \end{aligned} \tag{28}$$

is uniform on S^{n-1}.

Proof. Suppose that $\varepsilon > 0, K, L \in \mathcal{S}_o^n$, and $u \in S^{n-1}$. Let

$$\rho_{K_\varepsilon} = \rho_{K \widetilde{+}_{-\phi} \varepsilon \diamond L}(u) = \rho(\varepsilon, u).$$

Then, by Lemma 3.7, we have

$$\rho_{K_\varepsilon} \to \rho_K(u) \quad \text{as } \varepsilon \to 0.$$

By Lemma 3.3, we have

$$\phi\left(\frac{\rho_{K_\varepsilon}}{\rho_K(u)}\right) + \varepsilon\phi\left(\frac{\rho_{K_\varepsilon}}{\rho_L(u)}\right) = \phi(1).$$

Then

$$\frac{\rho_{K_\varepsilon}}{\rho_K(u)} = \phi^{-1}\left(\phi(1) - \varepsilon\phi\left(\frac{\rho_{K_\varepsilon}}{\rho_L(u)}\right)\right).$$

Let

$$t = \phi^{-1}\left(\phi(1) - \varepsilon\phi\left(\frac{\rho_{K_\varepsilon}}{\rho_L(u)}\right)\right) \tag{29}$$

and note that $t \to 1^+$ as $\varepsilon \to 0^+$. Thus

$$\frac{\rho_{K_\varepsilon}^{-1}(u) - \rho_K^{-1}(u)}{\rho_{K_\varepsilon}^{-1}(u)} = 1 - \frac{\rho_K^{-1}(u)}{\rho_{K_\varepsilon}^{-1}(u)} = 1 - t. \tag{30}$$

Together with (30) and Lemma 3.7, we obtain

$$\begin{aligned} &\lim_{\varepsilon \to 0^+} \frac{\rho^{-1}(\varepsilon, u) - \rho_K^{-1}(u)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0^+} \frac{\rho_{K_\varepsilon}^{-1}(u)}{\varepsilon} \cdot \frac{\rho^{-1}(\varepsilon, u) - \rho_K^{-1}(u)}{\rho_{K_\varepsilon}^{-1}(u)} \\ &= \lim_{\varepsilon \to 0^+} \rho_{K_\varepsilon}^{-1}(u) \cdot \phi\left(\frac{\rho_{K_\varepsilon}(u)}{\rho_L(u)}\right) \\ &\quad \times \frac{\frac{\rho^{-1}(\varepsilon,u) - \rho_K^{-1}(u)}{\rho_{K_\varepsilon}^{-1}(u)}}{\phi(1) - \left(\phi(1) - \varepsilon\phi\left(\frac{\rho_{K_\varepsilon}(u)}{\rho_L(u)}\right)\right)} \\ &= \rho_K^{-1}(u) \cdot \phi\left(\frac{\rho_K(u)}{\rho_L(u)}\right) \cdot \lim_{t \to 1^+} \frac{1-t}{\phi(1) - \phi(t)} \\ &= \frac{\rho_K^{-1}(u)}{\phi'_r(1)} \phi\left(\frac{\rho_K(u)}{\rho_L(u)}\right). \end{aligned}$$

From this and Lemma 3.7, it follows that

$$\lim_{\varepsilon \to 0^+} \frac{\rho(\varepsilon, u) - \rho_K(u)}{\varepsilon} = -\frac{\rho_K(u)}{\phi'_r(1)} \phi\left(\frac{\rho_K(u)}{\rho_L(u)}\right). \tag{31}$$

Then the pointwise limit (28) has been proved.

Moreover, the convergence is uniform for any $u \in S^{n-1}$. Indeed, by (29) and (31), it suffices to recall that by Lemma 3.7,

$$\lim_{\varepsilon \to 0^+} \rho_{K \widetilde{+}_{-\phi} \varepsilon \diamond L}(u) = \rho_K(u),$$

uniformly for $u \in S^{n-1}$.

We are ready to derive the variational formula of dual quermassintegral for the Orlicz harmonic radial combination.

Theorem 4.2. Let $K, L \in \mathcal{S}_o^n, \phi \in \Phi_1$ or $\phi \in \Phi_2$, and $i = 0, 1, \cdots, n-1$. Then

$$\begin{aligned} &\frac{-\phi'_r(1)}{n-i} \lim_{\varepsilon \to 0^+} \frac{\widetilde{W}_i(K \widetilde{+}_{-\phi} \varepsilon \diamond L) - \widetilde{W}_i(K)}{\varepsilon} \\ &= \frac{1}{n} \int_{S^{n-1}} \phi\left(\frac{\rho_K(u)}{\rho_L(u)}\right) \rho_K(u)^{n-i} \mathrm{d}S(u). \end{aligned}$$

Proof. Suppose that $K, L \in \mathcal{S}_o^n, \varepsilon > 0$, and $u \in S^{n-1}$. By Lemma 3.7 and Lemma 4.1, it follows that

$$\begin{aligned} &\lim_{\varepsilon \to 0^+} \int_{S^{n-1}} \frac{\rho_{K_\varepsilon}(u)^{n-i} - \rho_K(u)^{n-i}}{\varepsilon} \mathrm{d}S(u) \\ &= \lim_{\varepsilon \to 0^+} \int_{S^{n-1}} \frac{\rho_{K_\varepsilon}(u) - \rho_K(u)}{\varepsilon} \\ &\quad \times \left(\sum_{j=0}^{n-i-1} \rho_{K_\varepsilon}(u)^j \rho_K(u)^{n-i-j-1}\right) \mathrm{d}S(u) \\ &= \int_{S^{n-1}} \lim_{\varepsilon \to 0^+} \frac{\rho_{K_\varepsilon}(u) - \rho_K(u)}{\varepsilon} \\ &\quad \times \lim_{\varepsilon \to 0^+} \left(\sum_{j=0}^{n-i-1} \rho_{K_\varepsilon}(u)^j \rho_K(u)^{n-i-j-1}\right) \mathrm{d}S(u) \end{aligned}$$

$$\begin{aligned} &= (n-i)\int_{S^{n-1}} \lim_{\varepsilon\to 0^+} \frac{\rho_{K_\varepsilon}(u)-\rho_K(u)}{\varepsilon} \\ &\quad \times \rho_K(u)^{n-i-1}\mathrm{d}S(u) \\ &= \frac{-(n-i)}{\phi_r'(1)} \int_{S^{n-1}} \phi\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\rho_K(u)^{n-i}\mathrm{d}S(u). \end{aligned}$$

Hence

$$\begin{aligned} &\frac{-\phi_r'(1)}{n-i} \lim_{\varepsilon\to 0^+} \frac{\widetilde{W}_i(K\widetilde{+}_{-\phi}\varepsilon\diamond L)-\widetilde{W}_i(K)}{\varepsilon} \\ &= \frac{-\phi_r'(1)}{(n-i)n} \lim_{\varepsilon\to 0^+}\int_{S^{n-1}} \frac{\rho_{K_\varepsilon}(u)^{n-i}-\rho_K(u)^{n-i}}{\varepsilon}\mathrm{d}S(u) \\ &= \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\rho_K(u)^{n-i}\mathrm{d}S(u). \end{aligned}$$

We complete the proof of Theorem 4.2.

From this, if $i \neq n$ is any real, we can define the dual Orlicz harmonic mixed quermassintegrals.

Definition 4.3. Let $K, L \in \mathcal{S}_o^n, \phi \in \Phi_1$ or $\phi \in \Phi_2$ and any real $i \neq n$. The geometric quantity

$$\widetilde{W}_{-\phi,i}(K,L) = \int_{S^{n-1}} \phi\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\widetilde{W}_K(u)$$

is called the *dual Orlicz harmonic mixed quermassintegral* of K and L regarding ϕ. The *normalized dual Orlicz harmonic mixed quermassintegral* $\overline{W}_{-\phi}(K,L)$, of K and L regarding ϕ, is defined by

$$\begin{aligned} \overline{W}_{-\phi,i}(K,L) &= \phi^{-1}\left(\frac{\widetilde{W}_{-\phi,i}(K,L)}{\widetilde{W}_i(K)}\right) \\ &= \phi^{-1}\left(\int_{S^{n-1}} \phi\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}V_{n-i}^*(K,u)\right). \end{aligned}$$

When $\phi(t) = t^p$, with $p \geq 1$. The dual Orlicz harmonic mixed quermassintegral reduces to Wang and Leng's dual L_p-harmonic mixed quermassintegral (see [43]):

$$\widetilde{W}_{-p,i}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n+p-i}(u)\rho_L^{-p}(u)\mathrm{d}S(u),$$

for all $K, L \in \mathcal{S}_o^n$. Where real $i \neq n, n+p$.

From Definition 4.3 and the variational formula of Theorem 4.2, we can define that for $K, L \in \mathcal{S}_o^n, \phi \in \Phi_1$ or $\phi \in \Phi_2$, and $i \in \mathbf{R}$ with $i \neq n$, the dual Orlicz harmonic mixed quermassintegral, $\widetilde{W}_{-\phi,i}(K,L)$, of K, L is

$$\begin{aligned} &\widetilde{W}_{-\phi,i}(K,L) \\ &= \frac{-\phi_r'(1)}{n-i} \lim_{\varepsilon\to 0^+} \frac{\widetilde{W}_i(K\widetilde{+}_{-\phi}\varepsilon\diamond L)-\widetilde{W}_i(K)}{\varepsilon}. \quad (32) \end{aligned}$$

In particular, take $i = 0$ in (32), then the above formula of the dual Orlicz harmonic mixed quermassintegral reduces to the following formula of the dual Orlicz harmonic mixed volume:

$$\widetilde{V}_{-\phi}(K,L) = \frac{-\phi_r'(1)}{n} \lim_{\varepsilon\to 0^+} \frac{V(K\widetilde{+}_{-\phi}\varepsilon\diamond L)-V(K)}{\varepsilon}. \quad (33)$$

Some basic facts are observed for $\widetilde{W}_{-\phi,i}(K,L)$:

Proposition 4.4. Let $K, L, L_1, L_2 \in \mathcal{S}_o^n$ and $\phi \in \Phi_1$ or $\phi \in \Phi_2$.

(1) $\widetilde{W}_{-\phi,i}(K,K) = \phi(1)\widetilde{W}_i(K)$.

(2) $\widetilde{W}_{-\phi,0}(K,L) = \widetilde{V}_{-\phi}(K,L)$.

(3) If $\phi(t) = t^p$ and $p > 0$, then $\widetilde{W}_{-\phi,i}(K,L) = \widetilde{W}_{-p,i}(K,L)$.

(4) $\widetilde{W}_{-\phi,i}(TK,TL) = \widetilde{W}_{-\phi,i}(K,L)$, for all $T \in \mathrm{O}(n)$. where $\mathrm{O}(n)$ denote orthogonal transformation group in $\mathbf{R}^n$.

(5) If $L_1 \subseteq L_2$, then $\widetilde{W}_{-\phi,i}(K,L_1) \geq \widetilde{W}_{-\phi,i}(K,L_2)$.

Proof. We only give the proof of (4), and proof will be not given for (1), (2), (3) and (5).

From Proposition 3.2 and (32), we have, for $T \in O(n)$,

$$\begin{aligned} &\widetilde{W}_{-\phi,i}(TK,TL) \\ &= \frac{-\phi_r'(1)}{n-i} \lim_{\varepsilon\to 0^+} \frac{\widetilde{W}_i(TK\widetilde{+}_{-\phi}\varepsilon\diamond TL)-\widetilde{W}_i(TK)}{\varepsilon} \\ &= \frac{-\phi_r'(1)}{n-i} \lim_{\varepsilon\to 0^+} \frac{\widetilde{W}_i(T(K\widetilde{+}_{-\phi}\varepsilon\diamond L))-\widetilde{W}_i(K)}{\varepsilon} \\ &= \frac{-\phi_r'(1)}{n-i} \lim_{\varepsilon\to 0^+} \frac{\widetilde{W}_i(K\widetilde{+}_{-\phi}\varepsilon\diamond L)-\widetilde{W}_i(K)}{\varepsilon} \\ &= \widetilde{W}_{-\phi,i}(K,L). \end{aligned}$$

The next lemma shows the continuity of dual Orlicz harmonic mixed quermassintegrals.

Lemma 4.5. Let $K_j, L_k, K, L \in \mathcal{S}_o^n$ and $j, k \in \mathbf{N}$.

(1) If $K_j \to K, L_k \to L$ as $j, k \to \infty$, then $\widetilde{W}_{-\phi,i}(K_j,L_k) \to \widetilde{W}_{-\phi,i}(K,L)$.

(2) Let $\phi_j, \phi \in \Phi_1$ or $\phi_j, \phi \in \Phi_2$. If $\phi_j \to \phi$ as $j \to \infty$, then $\widetilde{W}_{-\phi_j,i}(K,L) \to \widetilde{W}_{-\phi,i}(K,L)$.

Proof. Note that $K_j \to K, L_k \to L$ implies

$$\rho_{K_j} \to \rho_K \quad \text{and} \quad \rho_{L_k} \to \rho_L$$

is uniform on S^{n-1}. In addition, from the continuity of ϕ, the convergence in

$$\phi\left(\frac{\rho_{K_j}}{\rho_{L_k}}\right) \to \phi\left(\frac{\rho_K}{\rho_L}\right)$$

is uniform on S^{n-1}. Thus, we derive (1) immediately.

Take

$$r_M = \max_{u\in S^{n-1}} \frac{\rho_K(u)}{\rho_L(u)} \quad \text{and} \quad r_m = \min_{u\in S^{n-1}} \frac{\rho_K(u)}{\rho_L(u)}.$$

Note that $\phi_j \to \phi$ implies $\phi_j|_{[r_m,r_M]} \to \phi|_{[r_m,r_M]}$ uniformly. Therefore, $\phi_j\big(\frac{\rho_K}{\rho_L}\big) \to \phi\big(\frac{\rho_K}{\rho_L}\big)$ uniformly on S^{n-1}, which implies (2).

From the definition of $\overline{W}_{-\phi,i}(K,L)$, (1) of Proposition 4.4, and together with the face that ϕ and ϕ^{-1} are strictly increasing on $(0,\infty)$, we have as follows:

Proposition 4.6. Let $K, L \in \mathcal{S}_o^n$ and $\phi, \phi_1, \phi_2 \in \Phi_1$ or Φ_2.

(1) $\overline{W}_{-\phi,i}(K,K) = 1$.

(2) If $\phi_1 \leq \phi_2$, then $\overline{W}_{-\phi_1,i}(K,L) \leq \overline{W}_{-\phi_2,i}(K,L)$.

V. The Generalized Dual Cauchy-Kubota Formula

In this section, we show the probabilistic essence of dual Orlicz harmonic mixed quermassintegrals. The starting point is the dual Cauchy-Kubota formula.

Recall that for a star body $K \in \mathcal{S}_o^n$,

$$\widetilde{W}_i(K) = \frac{\omega_n}{\omega_{n-i}}\int_{G(n,n-i)} \mathrm{vol}_{n-i}(K\cap\xi)\mathrm{d}\mu_{n-i}(\xi), \quad (34)$$

where $i = 1, \cdots, n-1$.

We generalize this formula to the Orlicz setting. For this aim, the next lemma is needed.

Lemma 5.1. Let $K, L \in \mathcal{S}_o^n, \phi \in \Phi_1$ (or $\phi \in \Phi_2$), and $i = 1, \cdots, n-1$. Then for each $\xi \in G(n,i)$ and $\varepsilon > 0$,

$$(K \widetilde{+}_{-\phi} \varepsilon \diamond L) \cap \xi = (K \cap \xi) \widetilde{+}_{-\phi} \varepsilon \diamond (L \cap \xi).$$

Proof. Let $\xi \in G(n,i)$ be arbitrary but fixed, and let

$$S^{i-1} = S^{n-1} \cap \xi.$$

For any $u \in S^{i-1}$ and $Q \in \mathcal{S}_o^n$, it follows that

$$\rho_Q(u) = \rho_{Q\cap\xi}(u).$$

Thus, applying the definition of $K \widetilde{+}_{-\phi} \varepsilon \diamond L$ to $u \in S^{i-1}$, it gives

$$\begin{aligned} & \phi\left(\frac{\rho((K \widetilde{+}_{-\phi} \varepsilon \diamond L) \cap \xi, u)}{\rho_{K\cap\xi}(u)}\right) \\ + \quad & \varepsilon\phi\left(\frac{\rho((K \widetilde{+}_{-\phi} \varepsilon \diamond L) \cap \xi, u)}{\rho_{L\cap\xi}(u)}\right) = \phi(1). \end{aligned}$$

On the other hand, from the definition of $(K \cap \xi) \widetilde{+}_{-\phi} \varepsilon \diamond (L \cap \xi)$ defined in ξ, it gives

$$\begin{aligned} & \phi\left(\frac{\rho((K \cap \xi) \widetilde{+}_{-\phi} \varepsilon \diamond (L \cap \xi), u)}{\rho_{K\cap\xi}(u)}\right) \\ & + \varepsilon\phi\left(\frac{\rho((K \cap \xi) \widetilde{+}_{-\phi} \varepsilon \diamond (L \cap \xi), u)}{\rho_{L\cap\xi}(u)}\right) = \phi(1). \end{aligned}$$

Hence, $(K \widetilde{+}_{-\phi} \varepsilon \diamond L) \cap \xi$ and $(K \cap \xi) \widetilde{+}_{-\phi} \varepsilon \diamond (L \cap \xi)$ is a same star body in ξ.

Theorem 5.2 provides a probabilistic approach to define dual Orlicz harmonic mixed quermassintegrals.

Theorem 5.2. Suppose that $K, L \in \mathcal{S}_o^n$ and $\phi \in \Phi_1$ or $\phi \in \Phi_2$. Then for each $i = 1, \cdots, n-1$,

$$\widetilde{W}_{-\phi,i}(K, L) = \frac{\omega_n}{\omega_{n-i}} \int_{G(n,n-i)} \widetilde{V}_{-\phi}^{(n-i)}(K\cap\xi, L\cap\xi) \mathrm{d}\mu_{n-i}(\xi),$$

where $\widetilde{V}_{-\phi}^{(n-i)}(K\cap\xi, L\cap\xi)$ denotes the dual Orlicz harmonic mixed volume of the $(n-i)$-dimensional star bodies $K \cap \xi$ and $L \cap \xi$ in the subspace ξ.

Proof. From (32), the dual Cauchy-Kubota formula (34) and Lemma 5.1, it follows that

$$\begin{aligned} & \widetilde{W}_{-\phi,i}(K, L) \\ = \quad & \frac{-\phi_r'(1)}{n-i} \lim_{\varepsilon \to 0^+} \frac{\widetilde{W}_i(K \widetilde{+}_{-\phi} \varepsilon \diamond L) - \widetilde{W}_i(K)}{\varepsilon} \\ = \quad & \frac{-\phi_r'(1)}{n-i} \cdot \frac{\omega_n}{\omega_{n-i}} \int_{G(n,n-i)} \\ \times \quad & \lim_{\varepsilon \to 0^+} \frac{\mathrm{vol}_{n-i}((K \widetilde{+}_{-\phi} \varepsilon \diamond L) \cap \xi) - \mathrm{vol}_{n-i}(K \cap \xi)}{\varepsilon} \\ \times \quad & \mathrm{d}\mu_{n-i}(u) \\ = \quad & \frac{-\phi_r'(1)}{n-i} \cdot \frac{\omega_n}{\omega_{n-i}} \int_{G(n,n-i)} \\ \times \quad & \lim_{\varepsilon \to 0^+} \frac{\mathrm{vol}_{n-i}((K \cap \xi) \widetilde{+}_{-\phi} \varepsilon \diamond (L \cap \xi)) - \mathrm{vol}_{n-i}(K \cap \xi)}{\varepsilon} \\ \times \quad & \mathrm{d}\mu_{n-i}(u). \end{aligned}$$

By (33) and the above integrand depends smoothly on ε (for small ε). Hence, it gives

$$\begin{aligned} & \widetilde{W}_{-\phi,i}(K, L) \\ = \quad & \frac{\omega_n}{\omega_{n-i}} \int_{G(n,n-i)} \widetilde{V}_{-\phi}^{(n-i)}(K \cap \xi, L \cap \xi) \mathrm{d}\mu_{n-i}(\xi), \end{aligned}$$

as desired.

Up to a constant, the quantity $\widetilde{W}_{-\phi,i}(K, L)$ is the expectation of the random variable

$$\begin{aligned} & \widetilde{V}_{-\phi}^{(n-i)}(K \cap \cdot, L \cap \cdot) : G(n, n-i) \to (0, \infty), \\ & \quad \xi \mapsto \widetilde{V}_{-\phi}^{(n-i)}(K \cap \xi, L \cap \xi), \end{aligned}$$

which is defined on the probability space $(G(n, n-i), \mathcal{B}, \mu_{n-i})$ (where $\mathcal{B}$ is the Borel sigma-algebra on $G(n, n-i)$).

Letting $\phi(t) = t^p$ with $p > 0$ in Theorem V, it yields the formula

$$\begin{aligned} & \widetilde{W}_{-p,i}(K, L) \\ = \quad & \frac{\omega_n}{\omega_{n-i}} \int_{G(n,n-i)} \widetilde{V}_{-p}^{(n-i)}(K \cap \xi, L \cap \xi) \mathrm{d}\mu_{n-i}(\xi). \end{aligned}$$

Remark 5.3. We generalize the dual Cauchy-Kubota formula to $1 \le q \le i < n$ states that for a star body $K \in \mathcal{S}_o^n$ and $1 \le q \le i < n$,

$$\widetilde{W}_i(K) = \frac{\omega_n}{\omega_{n-q}} \int_{G(n,n-q)} \widetilde{W}_{i-q}^{(n-q)}(K \cap \xi) \mathrm{d}\mu_{n-q}(\xi), \quad (35)$$

where $\widetilde{W}_{i-q}^{(n-q)}$ denotes the (i-q)th dual harmonic quermassintegral in the subspace ξ.

From (35), (32), and using an argument similar to that in Theorem 5.2, we can obtain the following theorem.

Theorem 5.4. Suppose that $K, L \in \mathcal{S}_o^n$ and $\phi \in \Phi_1$ or $\phi \in \Phi_2$. Then for $1 \le q \le i < n$,

$$\begin{aligned} & \widetilde{W}_{-\phi,i}(K, L) \\ = \quad & \frac{\omega_n}{\omega_{n-q}} \int_{G(n,n-q)} \widetilde{W}_{-\phi,i-q}^{(n-q)}(K \cap \xi, L \cap \xi) \mathrm{d}\mu_{n-q}(\xi), \end{aligned} \quad (36)$$

where $\widetilde{W}_{-\phi,i-q}^{(n-q)}(K \cap \xi, L \cap \xi)$ denotes the dual Orlicz harmonic mixed quermassintegral of the (n-q)-dimensional star bodies $K \cap \xi$ and $L \cap \xi$ in the subspace ξ.

VI. Dual Orlicz-Brunn-Minkowski inequalities

We now establish the following dual Orlicz-Minkowski inequality:

Theorem 6.1. Suppose that $K, L \in \mathcal{S}_o^n$. If $\phi \in \Phi_1$, then for real $i < n$ or $n < i < n+1$, we have

$$\widetilde{W}_{-\phi,i}(K, L) \ge \widetilde{W}_i(K) \phi\left(\left(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\right)^{\frac{1}{n-i}}\right). \quad (37)$$

If $\phi \in \Phi_2$, then for real $i > n+1$, we have

$$\widetilde{W}_{-\phi,i}(K, L) \le \widetilde{W}_i(K) \phi\left(\left(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\right)^{\frac{1}{n-i}}\right). \quad (38)$$

If ϕ is strictly convex (or strictly concave), the equality holds in every inequality if and only if K and L are dilates of each other.

Proof. Since two assertions can be proved similarly, we only give the proof of (37).

Since the dual quermassintegral-normalized conical measure $V_{n-i}^*(K, \cdot)$ defined by (16) is a probability measure on S^{n-1}, then by Jensen's inequality (17), the integral formulas of dual harmonic mixed quermassintegral (9), the L_p-dual

Minkowski inequality (14), and the fact that $\phi \in \Phi_1$ is increasing on $(0,\infty)$, we obtain

$$\begin{aligned}
&\frac{\widetilde{W}_{-\phi,i}(K,L)}{\widetilde{W}_i(K)}\\
=\ & \frac{1}{n\widetilde{W}_i(K)}\int_{S^{n-1}}\phi\bigg(\frac{\rho_K(u)}{\rho_L(u)}\bigg)\rho_K(u)^{n-i}\mathrm{d}S(u)\\
\geq\ & \phi\bigg(\frac{1}{n\widetilde{W}_i(K)}\int_{S^{n-1}}\frac{\rho_K(u)}{\rho_L(u)}\cdot\rho_K(u)^{n-i}\mathrm{d}S(u)\bigg)\\
=\ & \phi\bigg(\frac{\widetilde{W}_{-1,i}(K,L)}{\widetilde{W}_i(K)}\bigg)\\
\geq\ & \phi\bigg(\frac{\widetilde{W}_i(K)^{\frac{n+1-i}{n-i}}\widetilde{W}_i((L)^{\frac{-1}{n-i}}}{\widetilde{W}_i(K)}\bigg)\\
=\ & \phi\bigg(\bigg(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\bigg)^{\frac{1}{n-i}}\bigg).
\end{aligned}$$

This gives the desired inequality.

Suppose that equality holds in (37). Since ϕ is strictly increasing, we have equality in the L_p-dual Minkowski inequality. So there is $c>0$ such that $L=cK$ and hence

$$\rho_L(u)=c\rho_K(u),$$

for all $u\in S^{n-1}$.

Conversely, when $L=cK$, by Definition 4.3, we have

$$\begin{aligned}
\widetilde{W}_{-\phi,i}(K,L) &= \widetilde{W}_i(K)\phi(1/c)\\
&= \widetilde{W}_i(K)\phi\bigg(\bigg(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\bigg)^{\frac{1}{n-i}}\bigg).
\end{aligned}$$

When $\phi(t)=t^p$ with $p>0$. The dual Orlicz-Minkowski inequality (37) and (38) reduces to dual L_p-Minkowski inequality for the dual harmonic L_p-mixed quermassintegral: If $p\geq 1$, then for $i<n$ or $n<i<n+1$,

$$\widetilde{W}_{-p,i}(K,L)\geq\widetilde{W}_i(K)^{\frac{n+p-i}{n-i}}\widetilde{W}_i(L)^{\frac{-p}{n-i}}. \tag{39}$$

If $0<p<1$, then for $i>n+1$, inequality (39) is reversed. The equality holds in every inequality if and only if K and L are dilates each other.

Remark 6.2. By comparing (14) and (39), we see that inequality (39) is different from Wang and Leng's inequality (14) (see [43]).

The following uniqueness is a direct consequence of the dual Orlicz-Minkowski inequality (37) (or (38)).

Corollary 6.3. Suppose that $\phi\in\Phi_1$ or $\phi\in\Phi_2$, and $\mathcal{M}\subset\mathcal{S}_o^n$ such that $K,L\in\mathcal{M}$. If

$$\widetilde{W}_{-\phi,i}(M,K)=\widetilde{W}_{-\phi,i}(M,L),\quad \text{for all } M\in\mathcal{M}, \tag{40}$$

or

$$\frac{\widetilde{W}_{-\phi,i}(K,M)}{\widetilde{W}_i(K)}=\frac{\widetilde{W}_{-\phi,i}(L,M)}{\widetilde{W}_i(L)},\quad \text{for all } M\in\mathcal{M}, \tag{41}$$

then $K=L$

Proof. We only prove the case of $\phi\in\Phi_1$.

Suppose that (40) holds. If we take K for M, then from Definition 4.3 and (1) of Proposition 4.4, we obtain

$$\phi(1)\widetilde{W}_i(K)=\widetilde{W}_{-\phi,i}(K,K)=\widetilde{W}_{-\phi,i}(K,L).$$

Hence, from the dual Orlicz-Minkowski inequality (37) we have

$$\phi(1)\geq\phi\bigg(\bigg(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\bigg)^{\frac{1}{n-i}}\bigg).$$

with equality if and only if K and L are dilates of each other. Since ϕ is strictly increasing on $(0,\infty)$, we have

$$\widetilde{W}_i(L)\geq\widetilde{W}_i(K),$$

with equality if and only if K and L are dilates of each other. If taking L for M we similarly have $\widetilde{W}_i(L)\leq\widetilde{W}_i(K)$. Hence, $\widetilde{W}_i(K)=\widetilde{W}_i(L)$ and from the equality conditions we can conclude that K and L are dilates of each other. However, since they have the same dual quermassintegral they must be equal.

Next, suppose (41) holds. If we take K for M, then from Definition 4.3 and (1) of Proposition 4.4, we obtain

$$\phi(1)=\frac{\widetilde{W}_{-\phi,i}(K,K)}{\widetilde{W}_i(K)}=\frac{\widetilde{W}_{-\phi,i}(L,K)}{\widetilde{W}_i(L)}.$$

Then, from the dual Orlicz-Minkowski inequality (37) we have

$$\phi(1)\geq\bigg(\bigg(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\bigg)^{\frac{1}{n-i}}\bigg),$$

with equality if and only if K and L are dilates of each other. Since ϕ is strictly increasing on $(0,\infty)$, we have

$$\widetilde{W}_i(K)\geq\widetilde{W}_i(L),$$

with equality if and only if K and L are dilates of each other. If we take L for M we similarly have $\widetilde{W}_i(K)\leq\widetilde{W}_i(L)$. Hence, $\widetilde{W}_i(K)=\widetilde{W}_i(L)$ and from the equality conditions we can conclude that K and L are dilates of each other. However, since they have the same dual quermassintegral they must be equal.

Lemma 6.4. Suppose that $\phi\in\Phi_1$ or $\phi\in\Phi_2$, and $K,L\in\mathcal{S}_o^n$.

(1) If K and L are dilates, then for each $\alpha,\beta>0$, K and $\alpha\diamond K\widetilde{+}_{-\phi}\beta\diamond L$ are dilates.

(2) Suppose $\alpha,\beta>0$. If K and $\alpha\diamond K\widetilde{+}_{-\phi}\beta\diamond L$ are dilates, then K and L are dilates.

Proof. To prove (1), assume that $L=\varepsilon K$ for some constant $\varepsilon>0$. Let $\widetilde{C}_S$ denote the class

$$\{\rho_K|_{S^{n-1}}:K\in\mathcal{S}_o^n\}.$$

The definition of Orlicz harmonic radial combination implies that the function $\rho(\alpha\diamond K\widetilde{+}_{-\phi}\beta\diamond L,\cdot)$ is the unique solution to the equation

$$\alpha\phi\bigg(\frac{f}{\rho_K}\bigg)+\beta\phi\bigg(\frac{f}{\varepsilon\rho_K}\bigg)=\phi(1),\quad f\in\widetilde{C}_S.$$

On the other hand, it is obvious to prove that there exists a unique $\delta>0$ such that

$$\alpha\phi(\delta)+\beta\phi\bigg(\frac{\delta}{\varepsilon}\bigg)=\phi(1),$$

which immediately implies

$$\alpha\phi\bigg(\frac{\rho_{\delta K}}{\rho_K}\bigg)+\beta\phi\bigg(\frac{\rho_{\delta K}}{\varepsilon\rho_K}\bigg)=\phi(1).$$

Hence, $\alpha\diamond K\widetilde{+}_{-\phi}\beta\diamond L=\delta K$, which concludes (1).

To prove (2), assume $\alpha\diamond K\widetilde{+}_{-\phi}\beta\diamond L=\lambda K$ for some constant $\lambda>0$. Then for arbitrary $u\in S^{n-1}$,

$$\alpha\phi(\lambda)+\beta\phi\Big(\frac{\rho(\alpha\diamond K\widetilde{+}_{-\phi}\beta\diamond L,u)}{\rho_L(u)}\Big)=\phi(1),$$

which implies that

$$\phi\Big(\frac{\rho(\alpha\diamond K\widetilde{+}_{-\phi}\beta\diamond L,u)}{\rho_L(u)}\Big)$$

is constant for all $u\in S^{n-1}$. This and the injectivity of ϕ show that $\alpha\diamond K\widetilde{+}_{-\phi}\beta\diamond L$ and L are dilates.

We derive the dual Orlicz-Brunn-Minkowski inequality for dual Orlicz harmonic mixed quermassintegrals as follows:

Theorem 6.5. Suppose $K,L\in\mathcal{S}_o^n$, and $\alpha,\beta>0$. If $\phi\in\Phi_1$, then for $i<n$ or $n<i<n+1$, we have

$$\begin{aligned}&\alpha\phi\Big(\Big(\frac{\widetilde{W}_i(\alpha\diamond K\widetilde{+}_{-\phi}\beta\diamond L)}{\widetilde{W}_i(K)}\Big)^{\frac{1}{n-i}}\Big)\\ +\quad&\beta\phi\Big(\Big(\frac{\widetilde{W}_i(\alpha\diamond K\widetilde{+}_{-\phi}\beta\diamond L)}{\widetilde{W}_i(L)}\Big)^{\frac{1}{n-i}}\Big)\le\phi(1).\end{aligned}\tag{42}$$

If $\phi\in\Phi_2$, then for $i>n+1$, we have

$$\begin{aligned}&\alpha\phi\Big(\Big(\frac{\widetilde{W}_i(\alpha\diamond K\widetilde{+}_{-\phi}\beta\diamond L)}{\widetilde{W}_i(K)}\Big)^{\frac{1}{n-i}}\Big)\\ +\quad&\beta\phi\Big(\Big(\frac{\widetilde{W}_i(\alpha\diamond K\widetilde{+}_{-\phi}\beta\diamond L)}{\widetilde{W}_i(L)}\Big)^{\frac{1}{n-i}}\Big)\ge\phi(1).\end{aligned}\tag{43}$$

If ϕ is strictly convex (or strictly concave), the equality holds in every inequality if and only if K and L are dilates of each other.

Proof. Since two assertions can be proved similarly, we only give the proof of (42).

Let $K_\phi=\alpha\diamond K\widetilde{+}_{-\phi}\beta\diamond L$. From the formulas (4), Lemma 3.3 and the dual Orlicz-Minkowski inequality (37), it follows that

$$\begin{aligned}&\phi(1)\\ =\;&\frac{1}{n\widetilde{W}_i(K_\phi)}\int_{S^{n-1}}\phi(1)\rho_{K_\phi}^{n-i}(u)\mathrm{d}S(u)\\ =\;&\frac{1}{n\widetilde{W}_i(K_\phi)}\int_{S^{n-1}}\Big[\alpha\phi\Big(\frac{\rho_{K_\phi}(u)}{\rho_K(u)}+\beta\phi\Big(\frac{\rho_{K_\phi}(u)}{\rho_L(u)}\Big)\Big]\\ &\times\rho_{K_\phi}^{n-i}(u)\mathrm{d}S(u)\\ =\;&\frac{\alpha}{\widetilde{W}_i(K_\phi)}\widetilde{W}_{-\phi,i}(K_\phi,K)+\frac{\beta}{\widetilde{W}_i(K_\phi)}\widetilde{W}_{-\phi,i}(K_\phi,L)\\ \ge\;&\alpha\phi\Big(\Big(\frac{\widetilde{W}_i(K_\phi)}{\widetilde{W}_i(K)}\Big)^{\frac{1}{n-i}}\Big)+\beta\phi\Big(\Big(\frac{\widetilde{W}_i(K_\phi)}{\widetilde{W}_i(L)}\Big)^{\frac{1}{n-i}}\Big).\end{aligned}$$

We get the desired dual Orlicz-Brunn-Minkowski inequality (42). From Theorem 6.1 and Lemma 6.4, the equality conditions can be obtained immediately.

When $\phi(t)=t^p$ with $p>0$. the dual Orlicz-Brunn-Minkowski inequality reduces to dual L_p-Brunn-Minkowski inequality: If $p\ge1$, then for $i<n$ or $n<i<n+1$,

$$\begin{aligned}&\widetilde{W}_i(\alpha\diamond K\widetilde{+}_{-p}\beta\diamond L)^{-\frac{p}{n-i}}\\ \ge\;&\alpha\widetilde{W}_i(K)^{-\frac{p}{n-i}}+\beta\widetilde{W}_i(L)^{-\frac{p}{n-i}}.\end{aligned}\tag{44}$$

If $0<p<1$, then for $i>n+1$, inequality (44) is reversed. The equality holds in every inequality if and only if K and L are dilates of each other.

Remark 6.6. By comparing (15) and (44), we see that inequality (44) is different from Wang and Leng's inequality (15) (see [44]).

The next corollary is a weaker version of Theorem 6.5.

Corollary 6.7. Suppose that $K,L\in\mathcal{S}_o^n$ and $0<\alpha<1$. If $\phi\in\Phi_1$, then for real $i<n$ or $n<i<n+1$,

$$\widetilde{W}_i(\alpha\diamond K\widetilde{+}_{-\phi}(1-\alpha)\diamond L)\le\widetilde{W}_i(K)^\alpha\widetilde{W}_i(L)^{1-\alpha},\tag{45}$$

with equality if and only if $K=L$.

Proof. For brevity, let

$$K_\alpha=\alpha\diamond K\widetilde{+}_{-\phi}(1-\alpha)\diamond L.$$

Since $\phi\in\Phi_1$ is strictly increasing and convex on $(0,\infty)$, by (42) of Theorem 6.5 and the weighted arithmetic mean-geometric mean inequality, we have

$$\begin{aligned}&\phi(1)\\ \ge\;&\alpha\phi\Big(\Big(\frac{\widetilde{W}_i(K_\alpha)}{\widetilde{W}_i(K)}\Big)^{\frac{1}{n-i}}\Big)\\ &+(1-\alpha)\phi\Big(\Big(\frac{\widetilde{W}_i(K_\alpha)}{\widetilde{W}_i(L)}\Big)^{\frac{1}{n-i}}\Big)\\ \ge\;&\phi\Big(\alpha\Big(\frac{\widetilde{W}_i(K_\alpha)}{\widetilde{W}_i(K)}\Big)^{\frac{1}{n-i}}+(1-\alpha)\Big(\frac{\widetilde{W}_i(K_\alpha)}{\widetilde{W}_i(L)}\Big)^{\frac{1}{n-i}}\Big)\\ \ge\;&\phi\Big(\frac{\widetilde{W}_i(K_\alpha)^{\frac{1}{n-i}}}{\widetilde{W}_i(K)^{\frac{\alpha}{n-i}}\widetilde{W}_i(L)^{\frac{1-\alpha}{n-i}}}\Big).\end{aligned}$$

Therefore,

$$\widetilde{W}_i(K_\alpha)\le\widetilde{W}_i(K)^\alpha\widetilde{W}_i(L)^{1-\alpha}.$$

According to the condition of equality holds in inequality (42) and the weighted arithmetic mean-geometric mean inequality, we know that the equality holds in inequality (45) if and only if $K=L$.

If $\phi(t)=t^p$ with $p\ge1$, then the above corollary gives that for each $0<\alpha<1$ and $i<n$ or $n<i<n+1$,

$$\widetilde{W}_i(\alpha\diamond K\widetilde{+}_{-p}(1-\alpha)\diamond L)\le\widetilde{W}_i(K)^\alpha\widetilde{W}_i(L)^{1-\alpha},$$

with equality if and only if $K=L$.

Theorem 6.8. With the same assumptions of Theorem 6.1. The dual Orlicz-Minkowski inequality (37) (or (38)) $\Longleftrightarrow$the dual Orlicz-Brunn-Minkowski inequality (42) (or (43)).

Proof. We have proved the dual Orlicz-Brunn-Minkowski inequality (42) (or (43)) by the dual Orlicz-Minkowski inequality (37) (or (38)). Thus, we only need to prove the dual Orlicz-Minkowski inequality (37) by the dual Orlicz-Brunn-Minkowski inequality (42), the remainder of the argument is analogous to that in the first part and is left to the reader.

For $\varepsilon\ge0$, let $K_\varepsilon=K\widetilde{+}_{-\phi}\varepsilon\diamond L$. By the dual Orlicz-Brunn-Minkowski inequality, the following function

$$\begin{aligned}F(\varepsilon)\;=\;&\phi\Big(\Big(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\varepsilon)}\Big)^{-\frac{1}{n-i}}\Big)\\ &+\varepsilon\phi\Big(\Big(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K_\varepsilon)}\Big)^{-\frac{1}{n-i}}\Big)-\phi(1)\end{aligned}\tag{46}$$

is non-positive. Then by Lemma 3.7 and (46),

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\frac{F(\varepsilon)-F(0)}{\varepsilon}\\
=&\lim_{\varepsilon\to 0^+}\frac{\phi\Big(\Big(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\varepsilon)}\Big)^{-\frac{1}{n-i}}\Big)+\varepsilon\phi\Big(\Big(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K_\varepsilon)}\Big)^{-\frac{1}{n-i}}\Big)-\phi(1)}{\varepsilon}\\
=&\lim_{\varepsilon\to 0^+}\frac{\phi\Big(\Big(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\varepsilon)}\Big)^{-\frac{1}{n-i}}\Big)-\phi(1)}{\varepsilon}+\phi\Big(\Big(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\Big)^{-\frac{1}{n-i}}\Big)\\
=&\lim_{\varepsilon\to 0^+}\frac{\phi\Big(\Big(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\varepsilon)}\Big)^{-\frac{1}{n-i}}\Big)-\phi(1)}{\Big(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\varepsilon)}\Big)^{-\frac{1}{n-i}}-1}\\
&\times\lim_{\varepsilon\to 0^+}\frac{\Big(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\varepsilon)}\Big)^{-\frac{1}{n-i}}-1}{\varepsilon}+\phi\Big(\Big(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\Big)^{-\frac{1}{n-i}}\Big). \quad (47)
\end{aligned}$$

Let $\eta=\Big(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\varepsilon)}\Big)^{-\frac{1}{n-i}}$ and note that $\eta\to 1^+$ as $\varepsilon\to 0^+$. Consequently,

$$\begin{aligned}
\lim_{\varepsilon\to 0^+}\frac{\phi\Big(\Big(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\varepsilon)}\Big)^{-\frac{1}{n-i}}\Big)-\phi(1)}{\Big(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\varepsilon)}\Big)^{-\frac{1}{n-i}}-1} &= \lim_{\eta\to 1^+}\frac{\phi(\eta)-\phi(1)}{\eta-1}\\
&= \phi_r'(1). \quad (48)
\end{aligned}$$

By Lemma 3.7 and (32), we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\frac{\Big(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\varepsilon)}\Big)^{-\frac{1}{n-i}}-1}{\varepsilon} \quad (49)\\
=&\ \widetilde{W}_i(K)^{-\frac{1}{n-i}}\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K_\varepsilon)^{\frac{1}{n-i}}-\widetilde{W}_i(K)^{\frac{1}{n-i}}}{\varepsilon}\\
=&\ \frac{1}{(n-i)\widetilde{W}_i(K)}\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K_\varepsilon)-\widetilde{W}_i(K)}{\varepsilon}\\
=&\ -\frac{1}{\phi_r'(1)}\cdot\frac{\widetilde{W}_{-\phi,i}(K,L)}{\widetilde{W}_i(K)}. \quad (50)
\end{aligned}$$

From (48), (49), (50) and $F(\varepsilon)$ is non-positive, it follows that

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\frac{F(\varepsilon)-F(0)}{\varepsilon}\\
=&\ -\frac{\widetilde{W}_{-\phi,i}(K,L)}{\widetilde{W}_i(K)}+\phi\Big(\Big(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\Big)^{\frac{1}{n-i}}\Big)\\
\le&\ 0. \quad (51)
\end{aligned}$$

From the definition of function $F(\varepsilon)$, we have $F(0)=0$. Therefore, the equality holds in (51) if and only if $F(\varepsilon)=F(0)=0$, this implies that equality can be obtained from the equality condition of dual Orlicz-Brunn-Minkowski inequality.

Along with dual Orlicz harmonic mixed quermassintegrals, we introduce the following quantity.

Definition 6.9. For star bodies $K,L\in\mathcal{S}_o^n$ and $\phi\in\Phi_1$ or $\phi\in\Phi_2$, define

$$\begin{aligned}
&\widehat{W}_{-\phi,i}(K,L)\\
=&\ \inf\Big\{\lambda>0:\int_{S^{n-1}}\phi\Big(\frac{\rho_K(u)}{\lambda\rho_L(u)}\Big)\mathrm{d}V_{n-i}^*(K,u)\\
&\le\phi(1)\Big\}. \quad (52)
\end{aligned}$$

It can be checked that if $\phi(t)=t^p$ with $p>0$, then

$$\widehat{W}_{-p,i}(K,L)=\big(\widetilde{W}_{-p,i}(K,L)/\widetilde{W}_i(K)\big)^{\frac{1}{p}}.$$

From Definition VI and Definition IV, we have

$$\overline{W}_{-\phi,i}\big(K,\widehat{W}_{-\phi,i}(K,L)L\big)=1 \quad (53)$$

and

$$\widetilde{W}_{-\phi,i}\big(K,\widehat{W}_{-\phi,i}(K,L)L\big)=\phi(1)\widetilde{W}_i(K). \quad (54)$$

The quantity $\widehat{W}_{-\phi,i}(K,L)$ provides an approach to extend dual Minkowski's isoperimetric inequality to the Orlicz setting.

Theorem 6.10. Suppose that $K,L\in\mathcal{S}_o^n$. If $\phi\in\Phi_1$, then for each $i<n$ or $n<i<n+1$,

$$\widehat{W}_{-\phi,i}(K,L)\ge\Big(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\Big)^{\frac{1}{n-i}}, \quad (55)$$

while if $\phi\in\Phi_2$, the inequality is reversed. If ϕ is strictly convex (or concave), the equality holds in every inequalities if and only if K and L are dilates.

Proof. Since two assertions can be proved similarly, we only give the proof of (55).

From (54), (37) of Theorem 6.1 and the fact that

$$\widetilde{W}_i(\alpha K)=\alpha^{n-i}\widetilde{W}_i(K),\quad \alpha>0,$$

it follows that

$$\begin{aligned}
\phi(1) &= \frac{\widetilde{W}_{-\phi,i}(K,\widehat{W}_{-\phi,i}(K,L)L)}{\widetilde{W}_i(K)}\\
&\ge \phi\Big(\Big(\widetilde{W}_i(K)/\widetilde{W}_i(\widehat{W}_{-\phi,i}(K,L)L)\Big)^{\frac{1}{n-i}}\Big)\\
&= \phi\Bigg(\frac{\Big(\widetilde{W}_i(K)/\widetilde{W}_i(L)\Big)^{\frac{1}{n-i}}}{\widehat{W}_{-\phi,i}(K,L)}\Bigg).
\end{aligned}$$

Note that ϕ is strictly increasing on $(0,\infty)$. Hence, the desired inequality is obtained. If ϕ is strictly convex, by Theorem 6.1 again, the equality holds if and only if K and $\widehat{W}_{-\phi,i}(K,L)L$ are dilates.

The inequality in Theorem 6.1 and Definition 4.3 can be rewritten as

Corollary 6.11. Suppose that $K\in\mathcal{S}_o^n, i\in\mathbf{R}$. If $\phi\in\Phi_1$, then for $i<n$ or $n<i<n+1$,

$$\overline{W}_{-\phi,i}(K,L)\ge\Big(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\Big)^{\frac{1}{n-i}},$$

while if $\phi\in\Phi_2$, the inequality is reversed. If ϕ is strictly convex (or concave), each equality holds if and only if K and L are dilates.

In particular, $\overline{W}_{-\phi,0}(K,L)$ is $\overline{V}_{-\phi}(K,L)$, which is the normalized dual Orlicz harmonic mixed volume of K and L. That is,

$$\overline{V}_{-\phi}(K,L) = \phi^{-1}\left(\int_{S^{n-1}} \phi\left(\frac{\rho_K(u)}{\rho_L(u)}\right) \mathrm{d}V_n^*(K,u)\right).$$

Correspondingly, there is the inequality

$$\overline{V}_{-\phi}(K,L) \geq \left(\frac{V(K)}{V(L)}\right)^{\frac{1}{n}}, \quad \text{for } \phi \in \Phi_1.$$

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