G^1 Approximation of Conic Sections by Bernstein-Jacobi Hybrid Polynomial Curves

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***Abstract*—A G^1 approximation method of conic sections using Bernstein-Jacobi hybrid polynomial curves of arbitrary degree is proposed. Based on the method of weighted-sum-of-objective-function in the multi-objective optimization, the problem can be converted to a scale optimization. Applying weighted least-squares, we obtain the resulting curve. Meanwhile, by the orthogonality of Jacobi polynomials, the inverse of matrix is avoided. Finally, some examples and figures were offered to demonstrate the efficiency and the simplicity of our methods.**

***Index Terms*—Conic sections, Geometric continuity, Hybrid curves, Multi-objective optimization, Least-squares**

I. Introduction

ALTHOUGH rational Bézier curves are the standard in the Initial Graphics Exchange Specification (IGES), some Computer Aided Design (CAD) systems only use polynomial expressions to deal with parametric curves. This is because rational Bézier curves can't be differentiated and integrated easily [1]–[6]. Many research papers have been published about approximation of conic sections by Bézier curves since 1980s. Using geometric information such as point positions, tangents and curvatures, De Boor et al. [7] first applied Geometric Hermite Interpolation (GHI) method to accomplish a high accuracy approximation of circular arcs based on cubic Bézier curves. Floater [8] [9] studied approximation by quadratic splines and Bézier curves of odd degree n respectively. Both methods have the optimal approximation order $2n$. Fang [10] presented methods for approximating conic sections using quintic polynomial curves. The constructed quintic polynomial curve has G^3-continuity with the conic section at the end points and G^1-continuity at the parametric mid-point. Using the matrix form and the least squares method, Hu [11] researched G^1 approximation of conic sections by Bézier curves of arbitrary degree in L^2 norm, but the method requires to compute matrix inversion.

In this paper, we mainly interested in the G^1 approximation of conic sections by Bézier curves of arbitrary degree. In order to avoid calculating the inverse of matrix in L^2 norm, we construct a Jacobi-Berenstein hybrid polynomial curve. With the help of the weighted sum method in multi-objective optimization [12] [13] [14] and the weighted least squares method we obtain results.

The rest of the paper is organized as follow. In Section 2, some basic definitions and properties on conic sections and Jacobi-Bernstein polynomial curves were given. The problem of G^1-constrained approximation of the conic sections is described. In Section 3, using the weighted least-squares method we introduce an explicit algorithm to solve the problem. Approximation errors and numerical examples are presented in Section 4 to confirm the effectiveness of the method. Finally, in Section 5 we conclude this paper.

II. Preliminaries

A conic section can be represented in the standard rational Bézier form by

$$\begin{aligned}\mathbf{P}(t) &= \frac{\mathbf{c}(t)}{\omega(t)}\\ &= \frac{B_0^2(t)\mathbf{p}_0+B_1^2(t)\omega_1\mathbf{p}_1+B_2^2(t)\mathbf{p}_2}{B_0^2(t)+B_1^2(t)\omega_1+B_2^2(t)},\ t\in[0,1],\end{aligned} \tag{1}$$

where $\omega_1 \in \mathbb{R}^+$ is the weight, $\mathbf{p}_i = (x_i, y_i)$ are the control points and $B_i^n(t) = \binom{n}{i}t^i(1-t)^{n-i}$ are the Bernstein polynomials.

A Jacobi-Bernstein hybrid curve $\tilde{\mathbf{Q}}(t)$ of degree n can be expressed as

$$\begin{aligned}\tilde{\mathbf{Q}}(t) =& \sum_{i=0}^{r}\mathbf{q}_iB_i^n(t) + \varphi(t)\sum_{j=0}^{N}\tilde{\mathbf{q}}_jJ_j^{(r+1,s+1)}(u)\\ &+\sum_{i=n-s}^{n}\mathbf{q}_iB_i^n(t),\end{aligned} \tag{2}$$

where $N = n-(r+s+2)$, $u = 2t-1$, $\varphi(t) = t^{r+1}(1-t)^{s+1}$, $\mathbf{q}_i = (\bar{x}_i, \bar{y}_i)$ are the control points of the Bézier curves, $\tilde{\mathbf{q}}_i = (\tilde{x}_i, \tilde{y}_i)$ are the control points of the Jacobi curves and $J_j^{(r+1,s+1)}(u)$ are the Jacobi polynomials.

Set $\rho(t) > 0$ is a weight function and F^x and F^y are the components of the vector equation

$$(F^x, F^y) = \int_0^1 \rho(t)\left(\mathbf{P}(t) - \tilde{\mathbf{Q}}(t)\right)^2 dt.$$

Based on the method of weighted-sum-of-objective-function, the problem of G^1 approximation of the conic section $\mathbf{P}(t)$ by Bézier curves is to find a Jacobi-Bernstein hybrid curve $\tilde{\mathbf{Q}}(t)$ of degree n so that

$$F\left(\lambda, \eta, \{\tilde{\mathbf{q}}_i\}_{i=0}^{n-(r+s+2)}\right) = \frac{1}{2}\left(F^x + F^y\right), \tag{3}$$

is minimized and the control points at the end points satisfy

$$\begin{aligned}\mathbf{q}_0 &= \mathbf{p}_0, & \mathbf{q}_n &= \mathbf{p}_2,\\ \mathbf{q}_1 &= \mathbf{p}_0 + \frac{2\omega}{n}\lambda\Delta\mathbf{p}_0, & \mathbf{q}_{n-1} &= \mathbf{p}_2 - \frac{2\omega}{n}\eta\Delta\mathbf{p}_1,\end{aligned} \tag{4}$$

where λ and η are free parameters.

Next, we review and derive several of mathematical preliminaries on Bernstein polynomials, classical Jacobi polynomials and conic sections which are used in the paper.

Lemma 1. Setting $B_{i_j}^{n_j}(t)$ be a Bernstein polynomial of degree n_j, multiplication of m Bernstein polynomials satisfy

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the following equation [6],

$$\prod_{j=1}^{m} B_{i_j}^{n_j}(t) = \frac{M}{\binom{N}{J}} B_J^N(t), \tag{5}$$

and the corresponding definite integral can be written as

$$\int_0^1 \prod_{j=1}^{m} B_{i_j}^{n_j}(t)dt = \frac{M}{(N+1)\binom{N}{J}}, \tag{6}$$

where $M = \prod_{j=1}^{m} \binom{n_j}{i_j}$, $N = \sum_{j=1}^{m} n_j$ and $J = \sum_{j=1}^{m} i_j$.

Lemma 2. Given two Bézier curves $\mathbf{X}(t) = \sum_{j=0}^{n} \mathbf{x}_j^n B_j^n(t)$ of degree n and $\mathbf{Y}(t) = \sum_{k=0}^{m} \mathbf{y}_k^m B_k^m(t)$ of degree m, we have [15]

$$\mathbf{X}(t)\mathbf{Y}(t) = \sum_{i=0}^{m+n} \mathbf{C}_i(\mathbf{y}_k^m, \mathbf{x}_j^n) B_i^{m+n}(t), \tag{7}$$

where

$$\mathbf{C}_i(\mathbf{y}_k^m, \mathbf{x}_j^n) = \sum_{l=\max(0,i-n)}^{\min(m,i)} \frac{\binom{m}{l}\binom{n}{i-l}}{\binom{m+n}{i}} \mathbf{y}_l^m \mathbf{x}_{i-l}^n. \tag{8}$$

Lemma 3. A Jacobi polynomial can be represented by Bernstein polynomials as follows [16]

$$J_j^{(r+1,s+1)}(u) = \sum_{i=0}^{j} a_i^j B_i^j(t), \tag{9}$$

where

$$a_i^j := (-1)^{(i+j)} \frac{\binom{j+r+1}{i}\binom{j+s+1}{j-i}}{\binom{j}{i}}. \tag{10}$$

Furthermore, using (7), (9) and (10), we obtain

$$\begin{aligned}
&\bar{G}_{i,j}^{\alpha,\beta}(\alpha,\beta,i,j)\\
&= \int_0^1 t^{\alpha}(1-t)^{\beta} J_i^{(r+1,s+1)}(u) J_j^{(r+1,s+1)}(u)dt\\
&= \int_0^1 B_{\alpha}^{\alpha}(t) B_0^{\beta}(t) \sum_{m=0}^{i} a_m^i B_m^i(t) \sum_{l=0}^{j} a_l^j B_l^j(t)dt\\
&= \sum_{l=0}^{i+j} C_l(a_m^i, a_l^j) \int_0^1 B_{\alpha}^{\alpha}(t) B_0^{\beta}(t) B_l^{i+j}(t)dt\\
&= \frac{1}{\alpha+\beta+i+j+1} \sum_{l=0}^{i+j} \frac{\binom{i+j}{l}}{\binom{\alpha+\beta+i+j}{\alpha+l}} C_l(a_m^i, a_l^j),
\end{aligned} \tag{11}$$

where $C_l(a_m^i, a_l^j)$ are scale forms of (8).

Moreover, when $\alpha = s+1$ and $\beta = r+1$, the equation $\bar{G}_{i,j}^{\alpha,\beta}(\alpha,\beta,i,j)$ has the orthogonality, that is

$$\begin{aligned}
\gamma_i^{(s+1,r+1)} &= \bar{G}_{i,j}^{\alpha,\beta}(s+1,r+1,i,j)\\
&= \begin{cases} \frac{1}{2i+r+s+3} \frac{\binom{i+s+1}{s+1}}{\binom{i+r+s+2}{s+1}} & i = j,\\ 0 & i \neq j. \end{cases}
\end{aligned} \tag{12}$$

Similarly, given a Jacobi polynomial $J_j^{(r+1,s+1)}(u)$ and a Bernstein polynomial $B_k^n(t)$, we have

$$\begin{aligned}
G_{i,j}^{\alpha,\beta}(\alpha,\beta,k,j) &= \int_0^1 t^{\alpha}(1-t)^{\beta} B_k^n(t) J_j^{(r+1,s+1)}(u)dt\\
&= \frac{1}{\alpha+\beta+n+j+1} \sum_{i=0}^{j} (-1)^{(i+j)} \frac{\binom{n}{k}\binom{j+r+1}{i}\binom{j+s+1}{j-i}}{\binom{\alpha+\beta+n+j}{\alpha+k+i}}.
\end{aligned} \tag{13}$$

Lemma 4. Let $\mathbf{X}(t) = \sum_{i=0}^{n} \mathbf{x}_i^n B_i^n(t)$ be a Bézier curve of degree n and $\mathbf{P}(t)$ be a conic section given by equation (1), we have

$$\begin{aligned}
\boldsymbol{\xi}(n, \mathbf{x}_i^n) &= \int_0^1 \mathbf{X}(t)\mathbf{P}(t)dt\\
&= \sum_{i=2}^{n+2} \sum_{s=1}^{i-1} \binom{n+2}{i} \frac{a a_s \Delta^i \mathbf{c}_0^{n+2}}{i-s} + a\mathbf{A}\\
&\times \begin{cases} \frac{4}{\sqrt{4a-1}} \arctan \frac{1}{\sqrt{4a-1}}, & a > \frac{1}{4},\\ \frac{2}{\sqrt{1-4a}} \ln \left| \frac{1-\sqrt{1-4a}}{1+\sqrt{1-4a}} \right|, & a < \frac{1}{4},\\ -4, & a = \frac{1}{4}, \end{cases}
\end{aligned} \tag{14}$$

where

$a = \frac{1}{2(1-\omega_1)}$,

$a_i = 2^{-(i-1)} \sum_{s=0}^{[(i-1)/2]} \binom{i}{2s+1} (1-4a)^s, \ i \leqslant n+2,$

$a_{n+3} = 2^{-(n-2)} a \sum_{s=0}^{[n/2]} \binom{n-1}{2s+1} (1-4a)^s,$

and

$\mathbf{A} = \mathbf{c}_0^{n+2} + \frac{(n+2)\Delta \mathbf{c}_0^{n+2}}{2} + \left(\frac{a_{n+2}}{2} + a_{n+3}\right) \sum_{i=2}^{n+2} \binom{n+2}{i} \Delta^i \mathbf{c}_0^{n+2}$.

Finally, for a Jacobi polynomial $J_j^{(r+1,s+1)}(u)$ and a conic section $\mathbf{P}(t)$, by (14), it yields

$$\begin{aligned}
&\boldsymbol{\xi}(r+s+j+2, c_k)\\
&= \int_0^1 t^{r+1}(1-t)^{s+1} J_j^{(r+1,s+1)}(u)\mathbf{P}(t)dt\\
&= \int_0^1 \mathbf{P}(t) \sum_{i=0}^{j} (-1)^{(i+j)} \frac{\binom{j+r+1}{i}\binom{j+s+1}{j-i}}{\binom{r+s+j+2}{r+i+1}} B_{r+i+1}^{r+s+j+2}(t)dt\\
&= \int_0^1 \mathbf{P}(t) \sum_{k=r+1}^{r+j+1} c_k B_k^{r+s+j+2}(t)dt\\
&= \int_0^1 \mathbf{P}(t) \sum_{k=0}^{r+s+j+2} c_k B_k^{r+s+j+2}(t)dt
\end{aligned}$$

where

$$c_k = \begin{cases} (-1)^{(k-r-1+j)} \frac{\binom{j+r+1}{k-r-1}\binom{j+s+1}{j-k+r+1}}{\binom{r+s+j+2}{k}}, & k = r+1, ..., r+j+1,\\ 0, & \text{others}. \end{cases}$$

III. G^1 POLYNOMIAL APPROXIMATION OF CONIC SECTIONS

The G^1 approximation of conic sections means $r = s = 1$. According to the method of weighted least squares [16], derivatives of $F(\cdot)$ with respective to points $\tilde{\mathbf{q}}_k$ must be zero, so we have

$$\int_0^1 \rho(t)\varphi(t)\Big(\mathbf{P}(t) - \tilde{\mathbf{Q}}(t)\Big) J_k^{(2,2)}(2t-1)dt = 0.$$

Letting $\rho(t) = \frac{1}{\varphi(t)}$ and substituting (3), (12) and (13) into the above equation, we obtain

$$\begin{aligned}
\tilde{\mathbf{q}}_k =& \frac{1}{\gamma_k^{(2,2)}} \Bigg\{ \int_0^1 \Bigg[\mathbf{P}(t) - \Bigg(\sum_{i=0}^{1} B_i^n(t)\mathbf{p}_0 + \sum_{i=n-1}^{n} B_i^n(t)\mathbf{p}_2 \Bigg)\\
&- \frac{2\omega}{n} \left(\lambda B_1^n(t)\Delta \mathbf{p}_0 - \eta B_{n-1}^n(t)\Delta \mathbf{p}_1 \right) \Bigg] J_k^{(2,2)}(u)dt \Bigg\}\\
=& \frac{1}{\gamma_k^{(2,2)}} \Bigg[\boldsymbol{\xi}(k, a_i^k) - \sum_{i=0}^{1} \mathbf{p}_0 G_{i,k}^{0,0} - \sum_{i=n-1}^{n} \mathbf{p}_2 G_{i,k}^{0,0}\\
&- \frac{2\omega}{n} \left(\lambda \Delta \mathbf{p}_0 G_{1,k}^{0,0} - \eta \Delta \mathbf{p}_1 G_{n-1,k}^{0,0} \right) \Bigg].
\end{aligned}$$

Setting

$$\mathbf{A}_k = \frac{1}{\gamma_k^{(2,2)}}\left[\boldsymbol{\xi}(k, a_i^k) - \sum_{i=0}^{1}\mathbf{p}_0 G_{i,k}^{0,0} - \sum_{i=n-1}^{n}\mathbf{p}_2 G_{i,k}^{0,0}\right],$$

$\mathbf{B}_k = \frac{2\omega\Delta\mathbf{p}_0 G_{1,k}^{0,0}}{n\gamma_k^{(2,2)}}$ and $\mathbf{C}_k = \frac{2\omega\Delta\mathbf{p}_1 G_{n-1,k}^{0,0}}{n\gamma_k^{(2,2)}}$,
$\tilde{\mathbf{q}}_k$ can be rewritten as

$$\tilde{\mathbf{q}}_k = \mathbf{A}_k - \lambda\mathbf{B}_k + \eta\mathbf{C}_k. \tag{15}$$

From now on, we can calculate all the control points of the approximation curve $\tilde{\mathbf{Q}}(t)$ by equation (15). In order to obtain the values of λ and η, we substituting (15) into the objective function (3) and letting $\rho(t) = 1$ yield

$$\begin{cases} \frac{\partial F^x}{\partial\lambda} + \frac{\partial F^y}{\partial\lambda} = 0 \\ \frac{\partial F^x}{\partial\eta} + \frac{\partial F^y}{\partial\eta} = 0 \end{cases} \tag{16}$$

and

$$\frac{\partial(F_x, F_y)}{\partial\lambda} = \lambda\Pi_1 - \eta\Pi_2 - \Pi_3,$$

$$\frac{\partial(F_x, F_y)}{\partial\eta} = \lambda\Pi_4 - \eta\Pi_5 - \Pi_6,$$

where

$$\begin{aligned}\mathbf{\Pi}_1 &= \int_0^1 \left(\frac{2\omega}{n}\Delta\mathbf{p}_0 B_1^n(t) - \varphi(t)\sum_{j=0}^{n-4}\mathbf{B}_j J_j^{(2,2)}(u)\right)^2 dt \\ &= \frac{(2\omega\Delta\mathbf{p}_0)^2}{n(4n^2-1)} - \frac{4\omega\Delta\mathbf{p}_0}{n}\sum_{j=0}^{n-4}\mathbf{B}_j G_{1,j}^{2,2} + \sum_{j=0}^{n-4}\sum_{k=0}^{n-4}\mathbf{B}_j\mathbf{B}_k\bar{G}_{j,k}^{4,4},\end{aligned}$$

$$\begin{aligned}\mathbf{\Pi}_2 =& \mathbf{\Pi}_4 \\ =& \int_0^1 \left(\frac{2\omega}{n}\Delta\mathbf{p}_1 B_{n-1}^n(t) - \phi(t)\sum_{j=0}^{n-4}\mathbf{C}_j J_j^{(2,2)}(u)\right) \\ &\times \left(\frac{2\omega}{n}\Delta\mathbf{p}_0 B_1^n(t) - \phi(t)\sum_{j=0}^{n-4}\mathbf{B}_j J_j^{(2,2)}(u)\right) dt \\ =& \frac{(2\omega)^2\Delta\mathbf{p}_0\Delta\mathbf{p}_1}{(2n+1)\binom{2n}{n}} + \sum_{j=0}^{n-4}\sum_{k=0}^{n-4}\mathbf{C}_j\mathbf{B}_k\bar{G}_{j,k}^{4,4} \\ &- \frac{2\omega}{n}\sum_{j=0}^{n-4}\left(\Delta\mathbf{p}_0\mathbf{C}_j G_{1,j}^{2,2} + \Delta\mathbf{p}_1\mathbf{B}_j G_{n-1,j}^{2,2}\right),\end{aligned}$$

$$\begin{aligned}\mathbf{\Pi}_3 =& \int_0^1 \left(\frac{2\omega}{n}\Delta\mathbf{p}_0 B_1^n(t) - \phi(t)\sum_{j=0}^{n-4}\mathbf{B}_j J_j^{(2,2)}(u)\right) \\ &\times \left(\mathbf{P}(t) - \sum_{i=0}^{1} B_i^n(t)\mathbf{p}_0 - \sum_{i=n-1}^{n} B_i^n(t)\mathbf{p}_2 \right. \\ &\left. -\varphi(t)\sum_{j=0}^{n-4}\mathbf{A}_j J_j^{(2,2)}(u)\right) dt \\ =& \frac{2\omega\Delta\mathbf{p}_0}{n}\left[\boldsymbol{\xi}(n, u_i^\lambda) - \frac{n}{2n+1}\left(\sum_{i=0}^{1}\frac{\binom{n}{i}\mathbf{p}_0}{\binom{2n}{i+1}} + \sum_{i=n-1}^{n}\frac{\binom{n}{i}\mathbf{p}_2}{\binom{2n}{i+1}}\right)\right. \\ &\left. - \sum_{j=0}^{n-4}\mathbf{A}_j G_{1,j}^{2,2}\right] - \sum_{j=0}^{n-4}\boldsymbol{B}_j\left(\boldsymbol{\xi}(j+4, c_k) - \sum_{i=0}^{1}\mathbf{p}_0 G_{i,j}^{2,2}\right. \\ &\left. - \sum_{i=n-1}^{n}\mathbf{p}_2 G_{i,j}^{2,2} - \sum_{i=0}^{n-4}\mathbf{A}_i\bar{G}_{i,j}^{4,4}\right),\end{aligned}$$

$$\begin{aligned}\mathbf{\Pi}_5 &= \int_0^1 \left(\frac{2\omega}{n}\Delta\mathbf{p}_1 B_{n-1}^n(t) - \varphi(t)\sum_{j=0}^{n-4}\mathbf{C}_j J_j^{(2,2)}(u)\right)^2 dt \\ &= \frac{(2\omega\Delta\mathbf{p}_1)^2}{n(4n^2-1)} - \frac{4\omega\Delta\mathbf{p}_1}{n}\sum_{j=0}^{n-4}\mathbf{C}_j G_{n-1,j}^{2,2} + \sum_{j=0}^{n-4}\sum_{k=0}^{n-4}\mathbf{C}_j\mathbf{C}_k\bar{G}_{j,k}^{4,4}\end{aligned}$$

and

$$\begin{aligned}\mathbf{\Pi}_6 =& \int_0^1 \left(\mathbf{P}(t) - \sum_{i=0}^{1}\mathbf{p}_0 B_i^n(t) - \sum_{i=n-1}^{n}\mathbf{p}_2 B_i^n(t)\right. \\ &\left. -\phi(t)\sum_{j=0}^{n-4}\mathbf{C}_j J_j^{(2,2)}(u)\right) \times \left(\frac{2\omega}{n}\Delta\mathbf{p}_1 B_{n-1}^n(t)\right. \\ &\left. -\phi(t)\sum_{j=0}^{n-4}\mathbf{C}_j J_j^{(2,2)}(u)\right) dt \\ =& \frac{2\omega\Delta\mathbf{p}_1}{n} \times \left[\boldsymbol{\xi}(n, u_i^\eta) - \frac{n}{(2n+1)}\left(\sum_{i=0}^{1}\frac{\binom{n}{i}\mathbf{p}_0}{\binom{2n}{n+i-1}}\right.\right. \\ &\left.\left. + \sum_{i=n-1}^{n}\frac{\binom{n}{i}\mathbf{p}_2}{\binom{2n}{n+i-1}}\right) - \sum_{j=0}^{n-4}\mathbf{A}_j G_{n-1,j}^{2,2}\right] \\ &- \sum_{j=0}^{n-4}\mathbf{C}_j\left(\boldsymbol{\xi}(j+4, c_k) - \sum_{i=0}^{1}\mathbf{p}_0 G_{i,j}^{2,2}\right. \\ &\left. - \sum_{i=0}^{1}\mathbf{p}_2 G_{i,j}^{2,2} - \sum_{k=0}^{n-4}\mathbf{A}_k\bar{G}_{j,k}^{4,4}\right),\end{aligned}$$

where

$$u_i^\lambda = \begin{cases} 1 & i=1 \\ 0 & \text{Others} \end{cases}$$

and

$$u_i^\eta = \begin{cases} 1 & i=n\text{-}1 \\ 0 & \text{Others} \end{cases}.$$

Finally, we can obtain control points $\tilde{\mathbf{q}}_i$ of equation (2) from the system of linear equations (16).

IV. Numerical examples

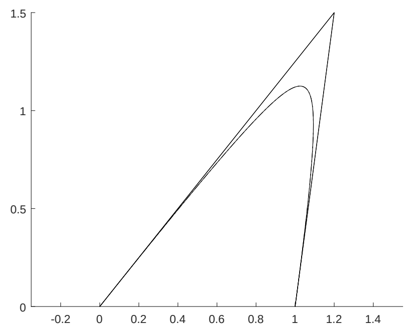
In this section, we provide two examples to show the effective of our method. For each example, we use discrete Hausdorff distances to express approximation error.

Example 1 (Also Example 1. in [11]) Given a conic section $\mathbf{P}(t)$ with control points $(0, 0)$, $(1.2, 1.5)$, $(1, 0)$ and the weight $\omega_1 = 3$. Table I lists λ, η and error obtained by Hu's method and ours method, respectively. The resulting curves of degree $n = 4$ are shown in the left-hand side of Fig. 1 and the corresponding error distance curves are illustrated in the right-hand side of Fig. 1. Fig. 2 shows the resulting curves with degree $n = 6$ and corresponding error distance curves.

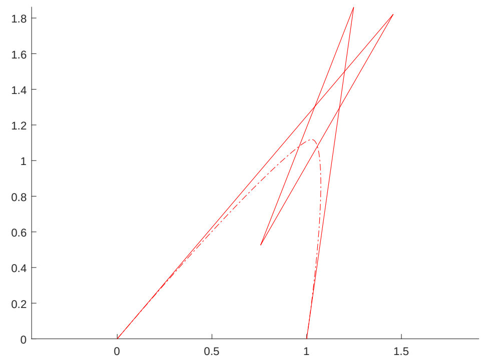
TABLE I
Approximation of conic sections with polynomials of degree 4 and 6

n	Hu's method			Our method		
	λ	η	ε	λ	η	ε
4	0.8091	0.8273	2.14×10^{-2}	0.8168	0.8368	1.94×10^{-2}
6	0.9438	0.9507	5.1×10^{-3}	0.9503	0.9524	5.1×10^{-3}

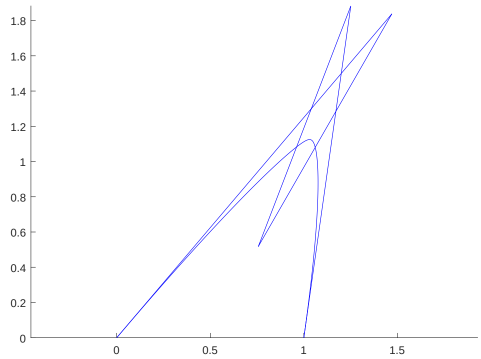
Example 2 (Also Example 2. in [11]) Given a conic section $\mathbf{P}(t)$ with control points $(0, 0)$, $(0.2, 0.8)$, $(1, 0)$ and the weight $\omega_1 = 0.2$ and $\omega_1 = -0.2$, respectively. Table II lists λ, η and error obtained by Hu's method and ours method, respectively. The resulting curves are shown in (a) of Fig. 3 , the corresponding error distance curves for $\omega_1 = 0.2$ are illustrated in (b) of Fig. 3 and the corresponding error distance curves for $\omega_1 = 0.2$ are illustrated in (c) of Fig. 3.



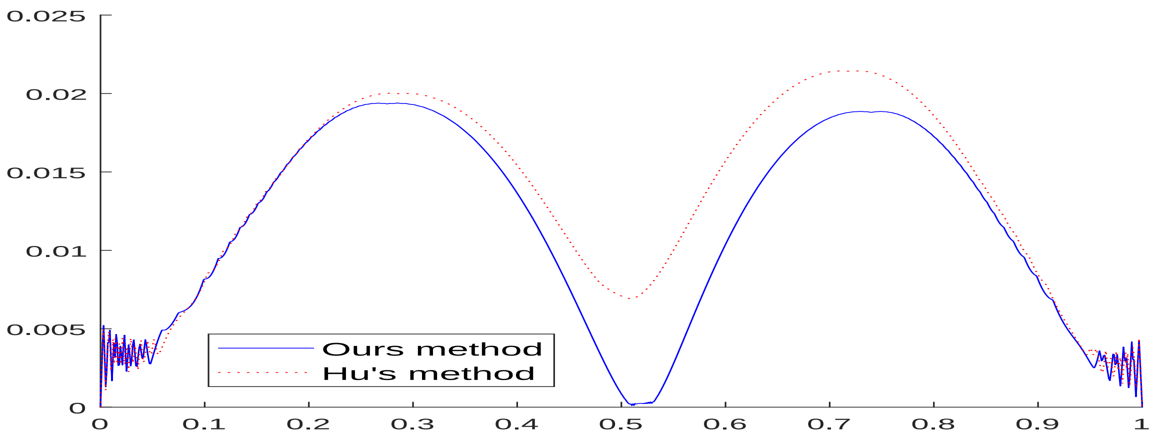
(a)



(b)



(c)



(d)

Fig. 1. (a) The conic section. (b) The resulting curve of degree 4 using Hu's method. (c) The resulting curve of degree 4 using ours method (d) The corresponding error distance curves. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

TABLE II
APPROXIMATION OF CONIC SECTIONS WITH POLYNOMIALS OF DEGREE 5

ω_1	Hu's method			Our method		
	λ	η	ε	λ	η	ε
0.2	0.8458	0.9221	2.7×10^{-3}	0.8634	0.9415	2.4×10^{-3}
-0.2	0.3185	0.1163	9.2×10^{-3}	0.3223	0.1261	7.2×10^{-3}

V. Conclusion

In this paper, we have studied G^1-constrained approximation of conic section with arbitrary degree Bernstein-Jacobi hybrid polynomial curves. As [11] explained our method is to minimize the L_2-error distance rather than to minimize the bound on the Hausdorff error distance, So Hausdorff error distance is larger than that by the method [18], [19] for the quartic Bézier curves. One of our work is to generalize our method to conic section approximated by the quartic Bernstein-Jacobi hybrid polynomial curves based on the bound on the Hausdorff error distance.

References

[1] T. W. Sederberg, M. Kakimoto, "Approximating rational curves using polynomial curves," in *NURBS for curve and surface Design, SIAM, Philadelphia, 1991*, pp. 149-158.

[2] B. G. Lee, Y. B. Park, "Approximate conversion of rational Bézier curves," *The journal of the Korean Society for Industrial and Applied Mathematics*, vol. 2, no. 1, pp. 88-93, 1998.

[3] S. Lewanowicz, P. Woźny, P. Keller, "Polynomial approximation of rational Bézier curves with constraints," *Numerical Algorithms*, vol. 59, no. 4, pp. 607-622, 2012.

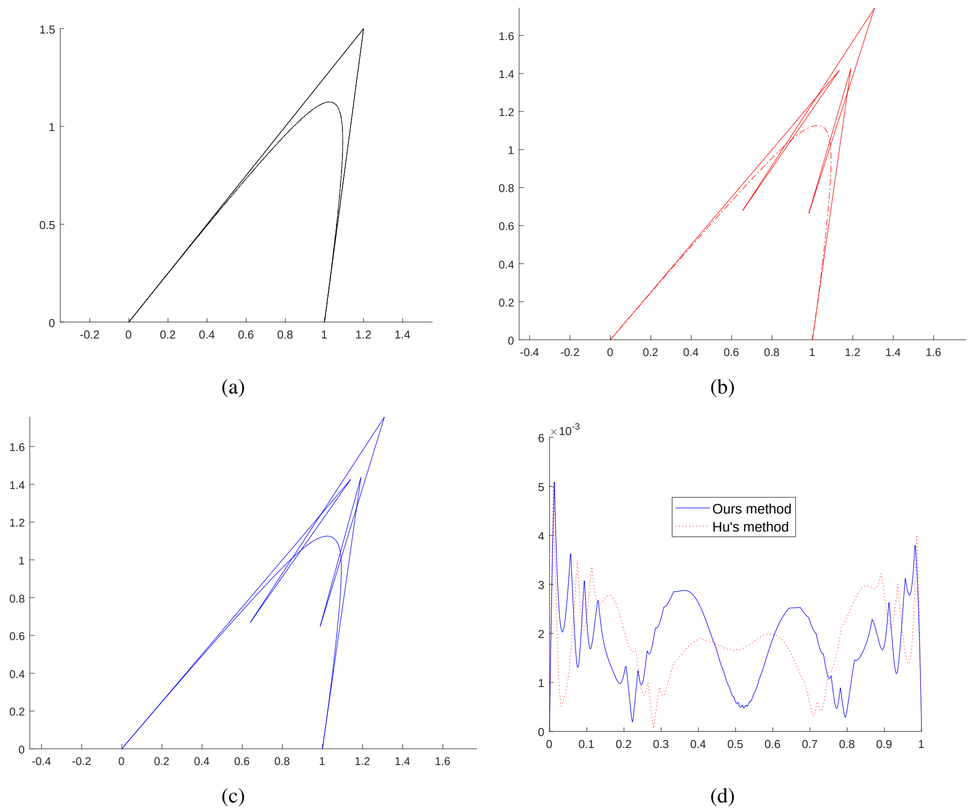


Fig. 2. (a) The conic section. (b) The resulting curve of degree 6 using Hu's method. (c) The resulting curve of degree 6 using ours method (d) The corresponding error distance curves. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

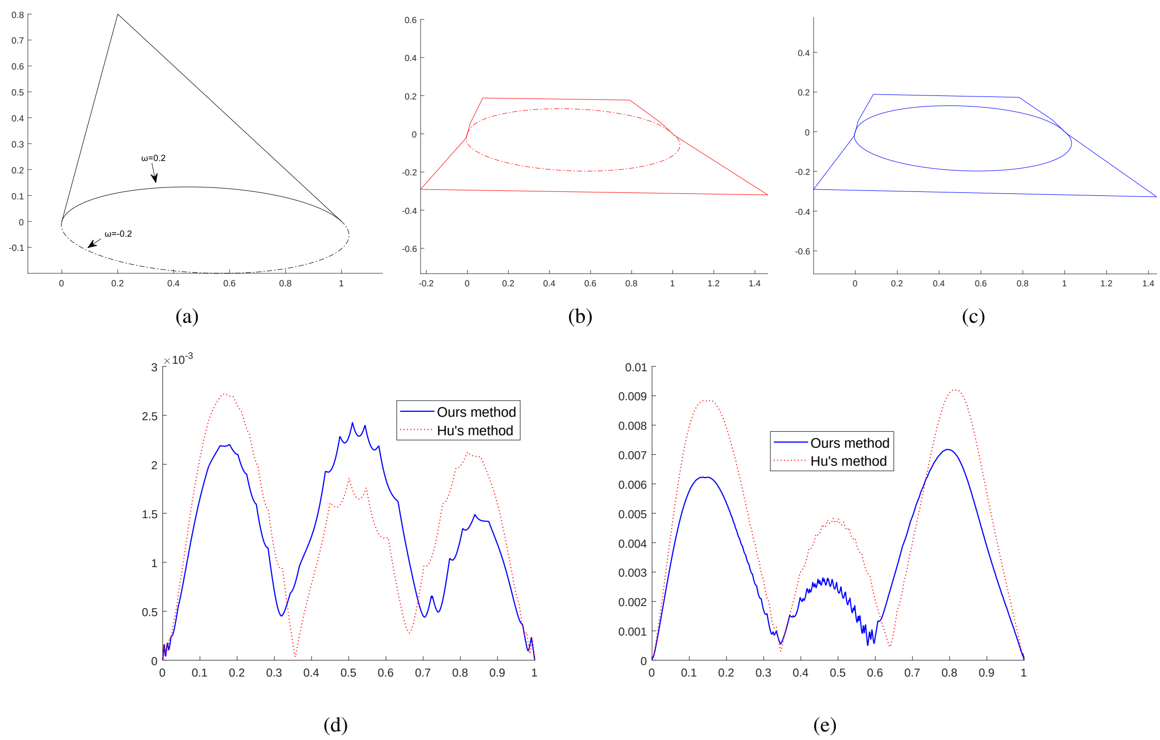


Fig. 3. (a) A whole ellipse. (b) Bzier curves of degree 5 using Hu's method. (c) Bzier curves of degree 5 using Ours method. (d) The corresponding error distance curves for $\omega = 0.2$. (e)The corresponding error distance curves for $\omega = -0.2$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

[4] L.A. Piegl, W. Tiller, K. Rajab, ”It is time to drop the ”R” from NURBS,” *Engineering with Computers*, vo. 30, no. 4, pp. 703-714, 2014.
[5] Q. Q. Hu, H. X. Xu, ”Constrained polynomial approximation of rational Bézier curves using reparameterization,” *Journal of Computational and Applied Mathematics*, vol. 249, pp. 133-143, 2013.
[6] M. Shi, J. S. Deng, ”Polynomial approximation of rational Bézier curves,” *Journal of Computational Information Systems*, vol. 11, no. 20, pp. 7339-7346, 2015.
[7] C. de Boor, K. Hllig, M. A. Sabin, ”High accuracy geometric Hermite interpolation,” *Computer Aided Geometric Design*, vol. 4, no. 4, pp. 269-278, 1987.
[8] M. Floater, ”High order approximation of conic sections by quadratic splines,” *Computer Aided Geometric Design*, vol. 12, no. 6, pp. 617-637, 1995.
[9] M. Floater, ”An O(h^{2n}) Hermite approximation for conic sections,” *Computer Aided Geometric Design*, vol. 14, no. 2, pp. 135-151, 1997.
[10] L. Fang, ”G^3 approximation of conic sections by quintic polynomial curves,” *Computer Aided Geometric Design*, vol. 16, no. 8, pp.755-766, 1999.
[11] Q. Q. Hu, ”Explicit G^1 approximation of conic section using Bézier curves of arbitrary degree,” *Journal of Computational and Applied Mathematics*, vol. 292, pp. 505-512, 2016.
[12] K. Deb, ”Multi-Objective Optimization using Evolutionary Algorithms,” *John Wiley & Sons LTD*, 2001.
[13] M. Shi, B. S. Kang, ”Degree Reduction for Bézier Curve Based on Genetic Algorithms,” *Computer Applications and Software*, vol. 15, no. 9, pp.15-17, 2003 (In Chinese).
[14] M. Shi, ”Degree Reduction of Disk Rational Bézier Curves Using Multi-Objective Optimization Techniques,” *IAENG International Journal of Applied Mathematics*, vol. 45, no. 4, pp. 392-397, 2015.
[15] R. T. Farouki, V. T. Rajan, ”Algorithms for polynomials in Bernstein form,” *Computer Aided Geometric Design*, vol. 5, no. 1, pp. 1-26, 1998.
[16] G. Szegő, ”Orthgonal polynomials, 4th.”, *American Mathematical Society*, 1975.
[17] Q. Q. Hu, ”Approximating conic sections by constrained Bézier curves of arbitrary degree,” *Journal of Computational and Applied Mathematics*, vol. 236, pp. 2813-2821, 2012.
[18] Q.Q. Hu, ”G^1 approximation of conic sections by quartic Bézier curves,” *Computers and Mathematics with Applications*, vol. 68, no. 12, pp. 1882-1891, 2014.
[19] X. L. Han, X. Guo, ”Optimal parameter values for approximating conic sections by the quartic Bézier curves,” *Journal of Computational and Applied Mathematics*, vol. 322, pp. 86-95, 2017.