Positive Solutions of a Four-point Fractional Boundary Value Problem

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***Abstract*—In this paper, based on the contraction map principle and the fixed point index theory, sufficient conditions are established for the uniqueness and existence results of positive solution for four-point boundary value problem of nonlinear differential equation with Caputo's fractional order derivative. It is interesting to note that the Krein-Rutmann theorem is also used in this paper. Such investigations will provide an important platform for gaining a deeper understanding of our environment.**

***Index Terms*—Caputo's fractional derivative, Positive solution, Four-point boundary value problem, Fixed point index.**

I. Introduction

FRACTIONAL derivatives arose in various fields of science and engineering such as control, porous media, electrochemistry, riscoelasticity, electromagnetic and other fields, see [1-5] and the reference therein. As we all known, many physical systems can be represented more accurately using fractional derivative formulations [6,7], thus many works on the basic theory of fractional calculus and fractional order differential equations have been established [6-10].

Recently, there have appeared a very large number of papers, which are devoted to the existence of positive solutions of nonlinear initial value problem or two point boundary value problem, and the solvability of nonlocal boundary value problem, see [11-23] and their references.

However, to the best of our knowledge, there are very few papers published on the positive solution with the nonlocal boundary value problem.

Motivated by the above, in the present paper, we consider the following four-point nonlocal boundary value problems of fractional order

$$^{C}D_{0+}^{\alpha}u(t)+f(t,u(t))=0, \quad 0<t<1, \tag{1}$$

$$u'(0)-\beta u'(\xi)=0, \quad u(1)+\gamma u'(\eta)=0, \tag{2}$$

where α is a real number with $1<\alpha\le 2,\ \ 0<\xi<\eta<1,\ 0<\beta<1,\ \gamma>0,\ {}^{C}D_{0+}^{\alpha}$ is the standard Caputo fractional derivative. The function $f\in C([0,1]\times[0,+\infty)\to[0,+\infty))$. By means of contraction map principle and the fixed point index theory, we establish the uniqueness and existence results of positive solution for the problem (1), (2). To the best knowledge of the authors, no work has been done to get positive solution of the problem (1), (2). It is interesting to note that the Krein-Rutmann theorem is also used in this paper.

The work presented in this paper has the following new features. First, the existence of positive solutions for the four-point fractional boundary value problems are considered. Second, the uniqueness results of positive solutions are obtained. Third, the Krein-Rutmann theorem is used in this paper.

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II. The preliminary lemmas

For the convenience of readers, we provide some background material in this section.

Definition 2.1 [20] The Riemann-Liouville fractional integral of order α for function y is defined as

$$I_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds, \quad \alpha>0.$$

Definition 2.2 [20] The Caputo's derivative for function y is defined as

$$^{C}D_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{y^{(n)}(s)ds}{(t-s)^{\alpha+1-n}}, \quad n=[\alpha]+1,$$

where $[\alpha]$ denotes the integer part of real number α.

Lemma 2.1*[20]* Let $\alpha>0$, then the fractional differential equation

$$^{C}D_{0+}^{\alpha}u(t)=0$$

has solutions

$$u(t)=c_1+c_2t+c_3t^2+\cdots+c_nt^{n-1},$$

$$c_i\in R, \quad i=1,2,\cdots,n, n=[\alpha]+1.$$

Lemma 2.2 *[9]* Let $\alpha>0$, then

$$I_{0+}^{\alpha}\,{}^{C}D_{0+}^{\alpha}u(t)=u(t)+c_1+c_2t+c_3t^2+\cdots+c_nt^{n-1}$$

for some $c_i\in R, \quad i=1,2,\cdots,n, n=[\alpha]+1.$

Lemma 2.3 *[9]* Let P be a cone in a Banach space X, and $\Omega(P)$ be a bounded open set in P. Suppose that $T:\overline{\Omega(P)}\to P$ is a completely continuous operator. If there exists $u_0\in P\backslash\{\theta\}$ such that

$$u-Tu\ne\mu u_0, \quad \forall u\in\partial\Omega(P), \quad \mu\ge 0,$$

then the fixed point index $i(T,\Omega(P),P)=0$.

Lemma 2.4 Let P be a cone in a Banach space X. Suppose that $T:\ P\to P$ is a completely continuous operator. If there exists a bounded open set $\Omega(P)$ such that each solution of

$$u=\sigma Tu, \quad u\in P, \quad \sigma\in[0,1]$$

satisfies $u\in\Omega(P)$, then the fixed point index $i(T,\Omega(P),P)=1$.

Lemma 2.5 *[9]* Suppose that $A:C[0,1]\to C[0,1]$ is a completely continuous linear operator and $A(P)\subset P$. If there exist $\psi\in C[0,1]\backslash(-P)$ and a constant $c>0$ such that $cA\psi\ge\psi$, then the spectral radius $r(A)\ne 0$ and A has a positive eigenfunction ϕ_1 corresponding to its first eigenvalue $\lambda_1=(r(A))^{-1}$.

Lemma 2.6 If $y \in C[0,1],\ 1 < \alpha \le 2$, then the unique solution of

$$^{C}D_{0+}^{\alpha}u(t) + y(t) = 0, \quad 0 < t < 1, \tag{3}$$

$$u'(0) - \beta u'(\xi) = 0, \quad u(1) + \gamma u'(\eta) = 0, \tag{4}$$

is

$$u(t) = \int_0^1 G(t,s)y(s)ds,$$

where

$$G(t,s) = \begin{cases} -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ +\frac{\beta(1+\gamma-t)(\xi-s)^{\alpha-2}}{(1-\beta)\Gamma(\alpha-1)} + \frac{\gamma(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & s \le \xi,\ s \le t, \\ \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\beta(1+\gamma-t)(\xi-s)^{\alpha-2}}{(1-\beta)\Gamma(\alpha-1)} \\ +\frac{\gamma(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & s \le \xi,\ t \le s, \\ -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\gamma(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & \xi \le s \le \eta,\ s \le t, \\ \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\gamma(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & \xi \le s \le \eta,\ t \le s, \\ -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & \eta \le s,\ s \le t, \\ \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & \eta \le s,\ t \le s. \end{cases} \tag{5}$$

Proof We can apply Lemma 2.2 and Definition 2.1 to reduce $^{C}D_{0+}^{\alpha}u(t) + y(t) = 0$ to an equivalent integral equation

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + c_1 + c_2 t,$$

for some $c_1,\ c_2 \in R$. By condition (4), one has

$$c_2 = -\frac{\beta}{\Gamma(\alpha-1)}\int_0^\xi (\xi-s)^{\alpha-2}y(s)ds + c_2\beta,$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds + c_1 + c_2$$

$$-\frac{\gamma}{\Gamma(\alpha-1)}\int_0^\eta (\eta-s)^{\alpha-2}y(s)ds + c_2\gamma = 0,$$

so, we have

$$c_1 = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds + \frac{\beta(1+\gamma)}{(1-\beta)\Gamma(\alpha-1)}\int_0^\xi (\xi-s)^{\alpha-2}y(s)ds + \frac{\gamma}{\Gamma(\alpha-1)}\int_0^\eta (\eta-s)^{\alpha-2}y(s)ds$$

$$c_2 = \frac{-\beta}{(1-\beta)\Gamma(\alpha-1)}\int_0^\xi (\xi-s)^{\alpha-2}y(s)ds.$$

Therefore, the unique solution of this problem is

$$\begin{aligned} u(t) &= -\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t (t-s)^{\alpha-1}y(s)ds \\ &\quad + \tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^1 (1-s)^{\alpha-1}y(s)ds \\ &\quad + \tfrac{\beta(1+\gamma-t)}{(1-\beta)\Gamma(\alpha-1)}\textstyle\int_0^\xi (\xi-s)^{\alpha-2}y(s)ds \\ &\quad + \tfrac{\gamma}{\Gamma(\alpha-1)}\textstyle\int_0^\eta (\eta-s)^{\alpha-2}y(s)ds \\ &= \textstyle\int_0^1 G(t,s)y(s)ds. \end{aligned}$$

The proof is complete.

Lemma 2.7 The function $G(t,s)$ defined by (5) satisfies $G(t,s) \ge 0$, for $t, s \in (0,1)$.

Proof For $0 \le s \le \xi \le 1$,

$$\begin{aligned} g(t,s) &\ge -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &\quad + \frac{\beta(1+\gamma-t)(\xi-s)^{\alpha-2}}{(1-\beta)\Gamma(\alpha-1)} + \frac{\gamma(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\ &\ge -\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\beta\gamma(\xi-s)^{\alpha-2}}{(1-\beta)\Gamma(\alpha-1)} \\ &\quad + \frac{\gamma(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)} \ge 0. \end{aligned}$$

For $\xi \le s \le \eta$,

$$\begin{aligned} g(t,s) &\ge -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\gamma(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\ &\ge -\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\gamma(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)} \ge 0. \end{aligned}$$

For $\eta \le s \le 1$,

$$\begin{aligned} g(t,s) &\ge -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &\ge -\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} = 0. \end{aligned}$$

The proof is complete.

III. The uniqueness result

Consider the Banach space $E = C[0,1]$ with the norm

$$\|u\| = \max_{t\in[0,1]} |u(t)|.$$

Let E be endowed with the ordering $u \le v$ if $u(t) \le v(t)$ for all $t \in [0,1]$. Denote the cone $P \subset E$ by

$$P = \{u \in E:\ u(t) \ge 0, \quad t \in [0,1]\}.$$

For $u \in P$, define the operator $T:\ P \to P$ by

$$(Tu)(t) = \int_0^1 G(t,s)f(s,u(s))ds, \tag{6}$$

clearly the problem (1), (2) has a solution u if and only if u solves the operator equation $u = Tu$.

Lemma 3.1 The operator $T:\ P \to P$ defined by (6) is completely continuous.

Proof The operator $T:\ P \to P$ is continuous in view of the continuity of $G(t,s)$ and $f(t,u)$. Let $B_l = \{u \in P: \|u\| \le l\},\ L = \max_{0\le t\le 1,\ u\in B_l} |f(t,u)| + 1$, for each $u \in B_l$, we have

$$\begin{aligned} |(Tu)(t)| &= \left|\textstyle\int_0^1 G(t,s)(f(s,u(s))ds\right| \le L\textstyle\int_0^1 G(t,s)ds \\ &\le L\Bigg(\int_0^\xi \Bigg[\frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1} \\ &\quad + \frac{\beta(1+\gamma)}{(1-\beta)\Gamma(\alpha-1)}(\xi-s)^{\alpha-2} \\ &\quad + \frac{\gamma}{\Gamma(\alpha-1)}(\eta-s)^{\alpha-2}\Bigg]ds \\ &\quad + \int_\xi^\eta \Bigg[\frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1} + \frac{\gamma}{\Gamma(\alpha-1)}(\eta-s)^{\alpha-2}\Bigg]ds \\ &\quad + \int_\eta^1 \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}ds \\ &= L\Bigg(\frac{1}{\Gamma(\alpha+1)} + \frac{\beta(1+\gamma)\xi^{\alpha-1}}{(1-\beta)\Gamma(\alpha)} + \frac{\gamma\eta^{\alpha-1}}{\Gamma(\alpha)}\Bigg). \end{aligned}$$

This shows that T maps bounded sets into bounded sets in P.

Let $B_l \subset P$ be a bounded set, $t_1, t_2 \in [0,1]$, $t_1 < t_2$, for any $u \in B_l$, we have

$$\begin{aligned}|(Tu)(t_2) - (Tu)(t_1)| \\ &= |\textstyle\int_0^1 (G(t_2,s) - G(t_1,s))(f(s,u(s))ds| \\ &\le L\Big(\textstyle\int_0^1 |G(t_2,s) - G(t_1,s)|ds\Big).\end{aligned}$$

$G(t,s)$ is uniformly continuous in $[0,1]\times[0,1]$ because of the continuity of $G(t,s)$. So, for any $\varepsilon > 0$, there exists $\delta > 0$, whenever $|t_2 - t_1| < \delta$, we have $|G(t_2,s) - G(t_1,s)| < \frac{\varepsilon}{L}$. So, $|(Tu)(t_2) - (Tu)(t_1)| < \varepsilon$, which implies that $\{Tu : u \in B_\tau\}$ is equicontinuous.

Therefore, the operator $T: P \to P$ is completely continuous by the Arzela-Ascoli theorem.

Theorem 3.2 Assume that $f(t,u)$ satisfies

$$|f(t,u) - f(t,v)| < \omega(t)|u-v|, \quad t \in [0,1], \quad u,v \in [0,+\infty). \tag{7}$$

If

$$\begin{aligned}&\textstyle\int_0^\xi [(1-\beta)\Gamma(\alpha-1)(1-s)^{\alpha-1} \\ &\quad + \beta(1+\gamma)\Gamma(\alpha)(\xi-s)^{\alpha-2} \\ &\quad + (1-\beta)\gamma\Gamma(\alpha)(\eta-s)^{\alpha-2}]\omega(s)ds \\ &\quad + \textstyle\int_\xi^\eta [(1-\beta)\Gamma(\alpha-1)(1-s)^{\alpha-1} \\ &\quad + (1-\beta)\gamma\Gamma(\alpha)(\eta-s)^{\alpha-2}]\omega(s)ds \\ &\quad + \textstyle\int_\eta^1 [(1-\beta)\Gamma(\alpha-1)(1-s)^{\alpha-1}]\omega(s)ds \\ &\quad < (1-\beta)\Gamma(\alpha)\Gamma(\alpha-1).\end{aligned} \tag{8}$$

Then problem (1), (2) has a unique positive solution.

Proof In the following, we will prove T^n is a contraction operator for n sufficiently large under the condition (7) and (8). Indeed, for $u, v \in P$, we have

$$\begin{aligned}|(Tu - Tv)(t)| &= \textstyle\int_0^1 G(t,s)\,|f(s,u(s)) - f(s,v(s))|\,ds \\ &< \|u-v\|\Big(\textstyle\int_0^\xi \Big[\frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1} \\ &\quad + \frac{\beta(1+\gamma)}{(1-\beta)\Gamma(\alpha-1)}(\xi-s)^{\alpha-2} \\ &\quad + \frac{\gamma}{\Gamma(\alpha-1)}(\eta-s)^{\alpha-2}\Big]\omega(s)ds \\ &\quad + \textstyle\int_\xi^\eta \Big[\frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1} \\ &\quad + \frac{\gamma}{\Gamma(\alpha-1)}(\eta-s)^{\alpha-2}\Big]\omega(s)ds \\ &\quad + \textstyle\int_\eta^1 \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}\omega(s)ds \\ &= \frac{\|u-v\|}{(1-\beta)\Gamma(\alpha)\Gamma(\alpha-1)} \\ &\Big(\textstyle\int_0^\xi [(1-\beta)\Gamma(\alpha-1)(1-s)^{\alpha-1} \\ &\quad + \beta(1+\gamma)\Gamma(\alpha)(\xi-s)^{\alpha-2} \\ &\quad + (1-\beta)\gamma\Gamma(\alpha)(\eta-s)^{\alpha-2}]\omega(s)ds \\ &\quad + \textstyle\int_\xi^\eta [(1-\beta)\Gamma(\alpha-1)(1-s)^{\alpha-1} \\ &\quad + (1-\beta)\gamma\Gamma(\alpha)(\eta-s)^{\alpha-2}]\omega(s)ds \\ &\quad + \textstyle\int_\eta^1 [(1-\beta)\Gamma(\alpha-1)(1-s)^{\alpha-1}]\omega(s)ds\Big).\end{aligned}$$

Denote

$$\begin{aligned}L = &\textstyle\int_0^\xi [(1-\beta)\Gamma(\alpha-1)(1-s)^{\alpha-1} \\ &+ \beta(1+\gamma)\Gamma(\alpha)(\xi-s)^{\alpha-2} \\ &+ (1-\beta)\gamma\Gamma(\alpha)(\eta-s)^{\alpha-2}]\omega(s)ds \\ &+ \textstyle\int_\xi^\eta [(1-\beta)\Gamma(\alpha-1)(1-s)^{\alpha-1} \\ &+ (1-\beta)\gamma\Gamma(\alpha)(\eta-s)^{\alpha-2}]\omega(s)ds \\ &+ \textstyle\int_\eta^1 [(1-\beta)\Gamma(\alpha-1)(1-s)^{\alpha-1}]\omega(s)ds,\end{aligned}$$

we have

$$|(Tu - Tv)(t)| < \frac{L\|u-v\|}{(1-\beta)\Gamma(\alpha)\Gamma(\alpha-1)},$$

consequently,

$$\begin{aligned}&|(T^2u - T^2v)(t)| \\ &\quad = \textstyle\int_0^1 G(t,s)\,|f(s,(Tu)(s)) - f(s,(Tv)(s))|\,ds \\ &\quad < \frac{L\|u-v\|}{(1-\beta)\Gamma(\alpha)\Gamma(\alpha-1)}\int_0^1 G(t,s)\omega(s)ds \\ &\quad < \frac{L\|u-v\|}{((1-\beta)\Gamma(\alpha)\Gamma(\alpha-1))^2} \\ &\quad \Big(\textstyle\int_0^\xi [(1-\beta)\Gamma(\alpha-1)(1-s)^{\alpha-1} \\ &\quad + \beta(1+\gamma)\Gamma(\alpha)(\xi-s)^{\alpha-2} \\ &\quad + (1-\beta)\gamma\Gamma(\alpha)(\eta-s)^{\alpha-2}]\omega(s)ds \\ &\quad + \textstyle\int_\xi^\eta [(1-\beta)\Gamma(\alpha-1)(1-s)^{\alpha-1} \\ &\quad + (1-\beta)\gamma\Gamma(\alpha)(\eta-s)^{\alpha-2}]\omega(s)ds \\ &\quad + \textstyle\int_\eta^1 [(1-\beta)\Gamma(\alpha-1)(1-s)^{\alpha-1}]\omega(s)ds\Big) \\ &\quad = \frac{L^2\|u-v\|}{((1-\beta)\Gamma(\alpha)\Gamma(\alpha-1))^2}.\end{aligned}$$

By induction, we have

$$|(T^n u - T^n v)(t)| \le \frac{L^n\|u-v\|}{((1-\beta)\Gamma(\alpha)\Gamma(\alpha-1))^n}.$$

Taking into account (8), choose n sufficiently large, we have

$$\frac{L^n}{((1-\beta)\Gamma(\alpha)\Gamma(\alpha-1))^n} < \frac{1}{2},$$

and therefore,

$$\|T^n u - T^n v\| \le \frac{1}{2}\|u-v\|,$$

which gives the proof.

IV. Existence of positive solutions

In this section, we impose some growth conditions on f which allow us to apply some fixed point index lemmas to establish the existence results of positive solutions for problem (1), (2).

Define an operator $A: E \to E$ by

$$(A\psi)(t) = \int_0^1 G(t,s)\psi(s)ds. \tag{9}$$

Then A is a completely continuous linear operator and $A(P) \subset P$. By virtue of the Krein-Rutmann theorem, we have the following lemma.

Lemma 4.1 Suppose A is defined by (9), then the spectral radius $r(A) > 0$ and A has a positive eigenfunction ϕ_1 corresponding to its first eigenvalue $\lambda_1 = (r(A))^{-1}$.

Proof By Lemma 2.7, we have $G(t,s) > 0$ for $s, t \in (0,1)$. Choose $[t_1, t_2] \subset (0,1)$ and $\phi \in C[0,1]$ such that

$\phi(t) \geq 0,\ t \in [0,1],\ \phi(t) > 0,\ \forall t \in (t_1, t_2)$ and $\phi(t) = 0,\ \forall t \notin (t_1, t_2)$. Then for $t \in [t_1, t_2]$,

$$(A\phi)(t) = \int_0^1 G(t,s)\phi(s)ds \geq \int_{t_1}^{t_2} G(t,s)\phi(s)ds > 0,$$

so there exists a constant $c > 0$ such that $c(A\phi)(t) \geq \phi(t),\ t \in [0,1]$. In view of Lemma 2.5, we complete the proof.

Theorem 4.2 Suppose that $f(t,0) \not\equiv 0,\ t \in (0,1)$. Furthermore,

$$0 \leq \overline{\lim}_{u\to+\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u} < \left\{\frac{1}{\Gamma(\alpha+1)} + \frac{\beta(1+\gamma)\xi^{\alpha-1}}{(1-\beta)\Gamma(\alpha)} + \frac{\gamma\eta^{\alpha-1}}{\Gamma(\alpha)}\right\}^{-1}. \quad (10)$$

Then problem (1), (2) has at least one positive solution.

Proof By (10), there exist $b > 0,\ 0 < N < \left(\frac{1}{\Gamma(\alpha+1)} + \frac{\beta(1+\gamma)\xi^{\alpha-1}}{(1-\beta)\Gamma(\alpha)} + \frac{\gamma\eta^{\alpha-1}}{\Gamma(\alpha)}\right)^{-1}$, such that

$$0 \leq f(t,u) < Nu + b,\ t \in [0,1],\ u \in [0,+\infty).$$

Let

$$B_h = \{u \in P | \|u - b\int_0^1 G(t,s)ds\| \leq h\}$$

be a convex, bounded and closed subset of the Banach Space E. For $u \in B_h$, we have

$$\begin{aligned}\|u\| &\leq b\|\int_0^1 G(t,s)ds\| + h \\ &\leq h + b\left(\frac{1}{\Gamma(\alpha+1)} + \frac{\beta(1+\gamma)\xi^{\alpha-1}}{(1-\beta)\Gamma(\alpha)} + \frac{\gamma\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\end{aligned}$$

and

$$\begin{aligned}&|Tu(t) - b\int_0^1 G(t,s)ds| \\ &\leq \int_0^1 G(t,s)\,|f(s,u(s)) - b|\,ds \\ &\leq N\|u\|\int_0^1 G(t,s)ds \\ &\leq N\|u\|\left(\frac{1}{\Gamma(\alpha+1)} + \frac{\beta(1+\gamma)\xi^{\alpha-1}}{(1-\beta)\Gamma(\alpha)} + \frac{\gamma\eta^{\alpha-1}}{\Gamma(\alpha)}\right) \\ &\leq N\left(\frac{1}{\Gamma(\alpha+1)} + \frac{\beta(1+\gamma)\xi^{\alpha-1}}{(1-\beta)\Gamma(\alpha)} + \frac{\gamma\eta^{\alpha-1}}{\Gamma(\alpha)}\right) \\ &\left\{h + b\left(\frac{1}{\Gamma(\alpha+1)} + \frac{\beta(1+\gamma)\xi^{\alpha-1}}{(1-\beta)\Gamma(\alpha)} + \frac{\gamma\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right. \\ &< (Nh + b) \\ &\left(\frac{1}{\Gamma(\alpha+1)} + \frac{\beta(1+\gamma)\xi^{\alpha-1}}{(1-\beta)\Gamma(\alpha)} + \frac{\gamma\eta^{\alpha-1}}{\Gamma(\alpha)}\right) \\ &< h\end{aligned}$$

as long as

$$h > \frac{b}{\left(\frac{1}{\Gamma(\alpha+1)} + \frac{\beta(1+\gamma)\xi^{\alpha-1}}{(1-\beta)\Gamma(\alpha)} + \frac{\gamma\eta^{\alpha-1}}{\Gamma(\alpha)}\right)^{-1} - N},$$

so we have $T(B_h) \subset B_h$. Using Schauder fixed point theorem, T has at least one fixed point in B_h, which is a positive solution of problem (1), (2).

Theorem 4.3 Assume that

$$\underline{\lim}_{u\to 0+} \min_{t\in[0,1]} \frac{f(t,u)}{u} > \lambda_1, \quad (11)$$

$$\overline{\lim}_{u\to +\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u} < \lambda_1, \quad (12)$$

which λ_1 is the first eigenvalue of the operator defined by (9). Then problem (1), (2) has at least one positive solution.

Proof By (11), there exists $l_1 > 0$ such that

$$f(t,u) \geq \lambda_1 u,\quad 0 \leq t \leq 1,\quad 0 \leq u \leq r_1.$$

Let ψ^* be the positive eigenfunction of A corresponding to λ_1, thus $\lambda_1 A\psi^* = \psi^*$. For each $\psi \in \overline{B_{l_1}} \cap P$, one has

$$(T\psi)(t) \geq \lambda_1 \int_0^1 G(t,s)\psi(s)ds = \lambda_1(A\psi)(t), \qquad t \in [0,1]. \quad (13)$$

Suppose that T has no fixed point on $\partial B_{l_1} \cap P$, now we prove that

$$\psi - T\psi \neq \mu\psi^*,\quad \forall\psi \in \partial B_{l_1} \cap P,\quad \mu \geq 0. \quad (14)$$

If (14) is not true, there exist $\psi_1 \in \partial B_{l_1} \cap P$ and $\mu_0 > 0$ such that $\psi_1 - T\psi_1 = \mu_0\psi^*$, so

$$\psi_1 = T\psi_1 + \mu_0\psi^* \geq \mu_0\psi^*. \quad (15)$$

Set $\mu^* = \sup\{\mu | \psi_1 \geq \mu\psi^*\}$, clearly $0 < \mu_0 \leq \mu^* < +\infty$, and $\psi_1 \geq \mu^*\psi^*$ and therefore, $\lambda_1 A\psi_1 \geq \mu^*\lambda_1 A\psi^* = \mu^*\psi^*$. Therefore, by (13) and (15),

$$\psi_1 = T\psi_1 + \mu_0\psi^* \geq \lambda_1 A\psi_1 + \mu_0\psi^* \geq \mu^*\psi^* + \mu_0\psi^*,$$

which contradicts the definition of μ. Hence, (14) holds. Lemma 2.3 implies that

$$i(T, B_{l_1} \cap P, P) = 0. \quad (16)$$

On the other hand, by (12), there exists $0 < \sigma < 1$ and $l_2 > l_1$ such that

$$f(t,u) \leq \sigma\lambda_1 u,\quad 0 \leq t \leq 1,\quad u \geq l_2.$$

Let $A_1\psi = \sigma\lambda_1 A\psi,\ \psi \in C[0,1]$. Then $A_1 : C[0,1] \to C[0,1]$ is a bounded linear operator and $A_1(P) \subset P$. Let

$$\overline{N} = \sup_{u\in\overline{B_{l_2}}\cap P,\ s\in[0,1]} f(s,u(s)) \max_{t\in[0,1]} \int_0^1 G(t,s)ds,$$

clearly $0 < \overline{N} < +\infty$. Let

$$S = \{\psi \in P | \psi = \mu A\psi,\quad 0 \leq \mu \leq 1\}.$$

In the following, we prove that S is bounded. For any $\psi \in S$, set $\overline{\psi(t)} = \min\{\psi(t), l_2\}$ and $E(\psi) = \{t \in [0,1] | \psi(t) > l_2\}$,then

$$\begin{aligned}\psi(t) = \mu A\psi(t) &\leq (T\psi)(t) \leq \int_0^1 G(t,s)f(s,\psi(s))ds \\ &= \int_{E(\psi)} G(t,s)f(s,\psi(s))ds \\ &+ \int_{[0,1]\setminus E(\psi)} G(t,s)f(s,\overline{\psi(s)})ds \\ &\leq \int_0^1 G(t,s)\sigma\lambda_1\psi(s)ds + \int_0^1 G(t,s)f(s,\overline{\psi(s)})ds \\ &\leq (A_1\psi)(t) + \overline{N}, \qquad t \in [0,1].\end{aligned}$$

Thus $((I - A_1)\psi)(t) \leq \overline{N},\ t \in [0,1]$. Since λ_1 is the first eigenvalue of A and $0 < \sigma < 1$, so $(r(A_1))^{-1} > 1$. Therefore, $(I - A_1)^{-1}$ exists and

$$(I - A_1)^{-1} = I + A_1 + A_1^2 + \cdots + A_1^n + \cdots.$$

It follows from $A_1(P) \subset P$ that $(I - A_1)^{-1} \subset P$. So, $\psi(t) \leq (I - A_1)^{-1}\overline{N},\ t \in [0,1]$. Therefore, S is bounded. Choose $l_3 > \max\{l_2,\ \|(I - A_1)^{-1}\overline{N}\|\}$. By Lemma 2.4, we have

$$i(T, B_{l_3} \cap P, P) = 1. \quad (17)$$

In view of (16),(17), one has

$$
\begin{aligned}
&i(T,(B_{l_3}\cap P)\backslash(\overline{B_{l_1}}\cap P),P)\\
&\quad = i(T,B_{l_3}\cap P,P)-i(T,B_{l_1}\cap P,P)=1.
\end{aligned}
$$

Then T has at least one fixed point on $(B_{l_3}\cap P)\backslash(\overline{B_{l_1}}\cap P)$, which means that problem (1), (2) has at least one positive solution.

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