

Effect of Predator Mutual Interference on an Autonomous Leslie-Gower Predator-prey Model

Shengbin Yu

Abstract—This article studies an autonomous Leslie-Gower predator-prey model with predator mutual interference and shows that the unique positive equilibrium of the system is globally stable unconditionally by constructing a suitable Lyapunov function. Mathematic analysis also indicates that predator mutual interference has same effect on both prey and predator species which is quite different with conclusions on Lotka-Volterra type system. Numerical simulations have been presented to validate the analytical results.

Index Terms—Globally stable, Leslie-Gower model, Mutual interference, Positive equilibrium, Lyapunov function.

I. INTRODUCTION

IN the last few decades, one of important predator-prey models is Leslie-Gower model [1,2] where the “carrying capacity” of the predator’s environment is proportional to the number of prey:

$$\begin{cases} \frac{dH}{dt} = (r_1 - b_1H - a_1P)H, \\ \frac{dP}{dt} = \left(r_2 - a_2\frac{P}{H}\right)P. \end{cases} \quad (1)$$

In system (1), H and P stand for the density of the prey species and predator species at time t , respectively. Parameters b_1 and $r_i, a_i, i = 1, 2$ are all positive constants and have the following biological meanings: r_1 and r_2 are the intrinsic growth rates of prey H and predator P , respectively; b_1 is the strength of competition among individuals of species H ; a_1 is the maximum value of the per capita reduction rate of H due to P and a_2 has a similar meaning to a_1 . By using Lyapunov function method, Korobeinikov [3] obtained the following result on the global stability of system (1):

Theorem 1.1 ([3]). *The unique coexisting fixed point (H_{1*}, P_{1*}) of system (1) is globally stable, where*

$$H_{1*} = \frac{r_1 a_2}{a_1 r_2 + a_2 b_1}, \quad P_{1*} = \frac{r_1 r_2}{a_1 r_2 + a_2 b_1}.$$

Chen et al. [4] further considered system (1) with a prey refuge and showed that the prey refuge has no influence on the persistent property of both predator and prey species, but could influence the densities of both prey and predator species greatly. Mohammadi and Mahzoon [5] extended system (1) by incorporating partial prey protection and studied the existence, uniqueness and stability of positive equilibrium also the influence of the effects of strong prey becoming weak prey. For more works on system (1) or its analogs, one could refer to [6-36] and the references cited therein.

Furthermore, Wang [29] pointed out that a more typical system should consider the mutual interference which was introduced by Hassell [30] in 1971. During Hassell’s research of the capturing behavior between the host (a bee) and parasite (a butterfly), he discovered that the host or parasite had the tendency to leave each other when they met, which interfered the hosts capturing effects. It is clear that the mutual interference will be stronger when the densities of the host and parasite become larger. Therefore he introduced the concept of mutual interference constant m ($0 < m \leq 1$), see [28-33] for more details. Especially, Chen et al. [33] and Ma et al. [34] considered a Lotka-Volterra predator-prey model incorporating predator mutual interference with a constant number of refuge and a constant fraction refuge, respectively. Both Chen et al. [33] and Ma et al. [34] show that mutual interference of predator species plays an essential role on the persistent property of the system. Since the results obtained by Chen et al. [33] and Ma et al. [34] are based on Lotka-Volterra type system, what is the role of mutual interference in Leslie-Gower type system is an another interesting question. Stimulated by the above reasons, in this paper, we extend system (1) by incorporating predator mutual interference and obtain the following system:

$$\begin{cases} \frac{dH}{dt} = (r_1 - b_1H - a_1P^m)H, \\ \frac{dP}{dt} = \left(r_2 - a_2\frac{P}{H}\right)P, \end{cases} \quad (2)$$

where r_1, b_1, a_1, r_2, a_2 are positive constants and m is mutual interference constant. Obviously, system (1) investigated by Korobeinikov [3] is a special cases of (2) with $m = 1$, i.e. there is no mutual interference between predator species. So we only investigate system (2) in the case of $0 < m < 1$ combining with the following initial conditions for the biological meaning:

$$H(0) > 0, \quad P(0) > 0. \quad (3)$$

It is not difficult to obtain that the corresponding solution $(H(t), P(t))^T$ satisfies $H(t) > 0, P(t) > 0$ for all $t \geq 0$.

The organization of this paper is as follows. In Section 2, we discuss the existence and stability of positive equilibrium point of system (2). The influence of predator mutual interference is discussed in Section 3. Then, in Section 4, two examples with numerical simulations are given to illustrate the feasibility of the main results. Finally, we conclude in Section 5.

II. EXISTENCE AND STABILITY OF POSITIVE EQUILIBRIUM

In this part, we investigate the existence and stability of positive equilibrium point.

Manuscript received July 12, 2018; revised October 9, 2018.

Shengbin Yu is with the Department of Basic Teaching and Research, Yango University, Fuzhou, Fujian 350015, China. (yushengbin.8@163.com)

Theorem 2.1. System (2) admits a unique positive equilibrium point (H_*, P_*) .

Proof. Obviously, positive equilibrium point (H_*, P_*) of system (2) satisfies the following equalities:

$$r_1 - b_1 H_* - a_1 P_*^m = 0, \quad r_2 - a_2 \frac{P_*}{H_*} = 0. \quad (4)$$

That is H_* satisfies

$$a_2^m r_1 - a_1 r_2^m H_*^m - a_2^m b_1 H_* = 0. \quad (5)$$

Set $f(H) = a_2^m r_1 - a_1 r_2^m H^m - a_2^m b_1 H$, then

- (a) $f(0) = a_2^m r_1 > 0$;
- (b) $f\left(\frac{r_1}{b_1}\right) = -\frac{a_1 r_2^m r_1^m}{b_1^m} < 0$;
- (c) $\frac{df}{dH} = -m a_1 r_2^m H^{m-1} - a_2^m b_1 < 0$.

Thus, equation $f(H) = 0$ has at least one positive root in $(0, \frac{r_1}{b_1})$ by Zero point theorem. On the other hand, (c) shows that $f(H)$ is monotonically decreasing over $(0, +\infty)$, so $f(H) = 0$ has a unique positive root H_* in $(0, +\infty)$. Hence, positive equilibrium point (H_*, P_*) is unique and this completes the proof. ■

Theorem 2.2. The positive equilibrium point (H_*, P_*) of the system (2) is globally stable.

Proof. Inspired by the idea of Korobeinikov [3], Lu and Liu [35], in order to prove Theorem 2.2, we construct the following function

$$V(H, P) = \ln \frac{H}{H_*} + \frac{H_*}{H} - 1 + \frac{a_1 H_*}{a_2} \int_{P_*}^P \frac{\xi^m - P_*^m}{\xi^2} d\xi. \quad (6)$$

In the following, we will show that $V(H, P)$ is a Lyapunov function. One can easily verified that function $V(H, P)$ is zero at the equilibrium (H_*, P_*) and the fixed point (H_*, P_*) is the only extremum of $V(H, P)$ in the positive quadrant. Now, We come to show that $V(H, P)$ is positive for all positive values of (H, P) except (H_*, P_*) . Firstly, set $y = \frac{H}{H_*}$ and $g(y) = \ln y + \frac{1}{y} - 1$, then $y > 0$ and $\frac{dg}{dy} = \frac{y-1}{y^2}$. Hence, $g(y) \geq g(1) = 0$. Secondly, When $P > P_*$, for any $\xi \in (P_*, P)$, since $0 < m < 1$, we have $\xi^m > P_*^m$ and then $\int_{P_*}^P \frac{\xi^m - P_*^m}{\xi^2} d\xi > 0$. If $P < P_*$, one can similarly obtain $\int_{P_*}^P \frac{\xi^m - P_*^m}{\xi^2} d\xi > 0$. So $V(H, P) > 0$ for all positive values of (H, P) except (H_*, P_*) and (H_*, P_*) is the global minimum of $V(H, P)$. To sum up, $V(H, P)$ is a Lyapunov function.

Using equalities (4) and calculating the derivatives of

$V(H, P)$ along the solution of system (2) lead to

$$\begin{aligned} \frac{dV}{dt} &= \frac{1}{H} \left(1 - \frac{H_*}{H}\right) \frac{dH}{dt} + \frac{a_1 H_* (P^m - P_*^m)}{a_2 P^2} \frac{dP}{dt} \\ &= \left(\frac{H - H_*}{H}\right) (b_1 H_* + a_1 P_*^m - b_1 H - a_1 P^m) \\ &\quad + \frac{a_1 H_* (P^m - P_*^m)}{a_2 P} \left(a_2 \frac{P_*}{H_*} - a_2 \frac{P}{H}\right) \\ &= -\frac{b_1 (H - H_*)^2}{H} + \frac{a_1 (H - H_*) (P_*^m - P^m)}{H} \\ &\quad + \frac{a_1 (P^m - P_*^m) (P_* - P)}{P} \\ &\quad + \frac{a_1 (P^m - P_*^m) (H - H_*)}{H} \\ &= -\frac{b_1 (H - H_*)^2}{H} + \frac{a_1 (P^m - P_*^m) (P_* - P)}{P} < 0 \end{aligned} \quad (7)$$

strictly for all $H, P > 0$ except the positive equilibrium (H_*, P_*) , where $\frac{dV}{dt} = 0$. Therefor (H_*, P_*) is globally stable which completes the proof of Theorem 2.2. ■

III. THE INFLUENCE OF PREDATOR MUTUAL INTERFERENCE

We will discuss this on two aspects:

(1) Predator mutual interference has no impact on the persistent property of the system.

Definition 3.1. System (2) is permanent if any positive solution $(H(t), P(t))^T$ of system (2) satisfies

$$\begin{aligned} m_1 &\leq \liminf_{t \rightarrow +\infty} H(t) \leq \limsup_{t \rightarrow +\infty} H(t) \leq M_1, \\ m_2 &\leq \liminf_{t \rightarrow +\infty} P(t) \leq \limsup_{t \rightarrow +\infty} P(t) \leq M_2, \end{aligned}$$

where $m_i, M_i, i = 1, 2$ are all positive constants.

Theorem 2.2 shows that

$$\lim_{t \rightarrow +\infty} H(t) = H_*, \quad \lim_{t \rightarrow +\infty} P(t) = P_*. \quad (8)$$

Since H_* and P_* depend only on the coefficient of system (2), thus (8) indicates that system (2) is always permanent and predator mutual interference has no impact on the persistent property of the system.

Chen et al. [33] and Ma et al. [34] pointed out that mutual interference of predator species plays an essential role on the persistent property of the Lotka-Volterra type system. Obviously our results on system (2) are quite different with theirs. We think this can be more attributed to that Leslie-Gower predator-prey systems do not follow the “mass conservation” principle while Lotka-Volterra ones are based on the principles of “mass conservation” and “decomposition of the dynamics of a population into birth and death processes” (see for instance [36]).

(2) Predator mutual interference has same effect on prey densities and predator densities.

According to (5), by implicit differentiation, one can get

$$\begin{aligned} r_1 a_2^m \ln a_2 - a_1 r_2^m \ln r_2 H_*^m - b_1 a_2^m \ln a_2 H_* - b_1 a_2^m \frac{dH_*}{dm} \\ - a_1 r_2^m H_*^m \left\{ \ln H_* + m \frac{1}{H_*} \frac{dH_*}{dm} \right\} = 0, \end{aligned}$$

or

$$\begin{aligned} \frac{dH_*}{dm} &= \frac{a_2^m \ln a_2 (r_1 - b_1 H_*) - a_1 r_2^m H_*^m \ln(r_2 H_*)}{b_1 a_2^m + m a_1 r_2^m H_*^{m-1}} \\ &= \frac{a_2^m (r_1 - b_1 H_*) \ln \frac{a_2}{r_2 H_*}}{b_1 a_2^m + m a_1 r_2^m H_*^{m-1}}. \end{aligned} \tag{9}$$

It follows from (9) and the second equation of (4) that

$$\frac{dP_*}{dm} = \frac{r_2}{a_2} \frac{dH_*}{dm} = \frac{r_2 a_2^m (r_1 - b_1 H_*) \ln \frac{a_2}{r_2 H_*}}{b_1 a_2^{m+1} + m a_1 a_2 r_2^m H_*^{m-1}} \tag{10}$$

Since $H_* \in (0, \frac{r_1}{b_1})$ or $r_1 > b_1 H_*$, then equations (9) and (10) together with the second equation of (4) lead to the following result:

Proposition 3.1. $\frac{dH_*}{dm} > 0$ and $\frac{dP_*}{dm} > 0$ if $H_* < \frac{a_2}{r_2}$ or $P_* < 1$, while $\frac{dH_*}{dm} < 0$ and $\frac{dP_*}{dm} < 0$ if $H_* > \frac{a_2}{r_2}$ or $P_* > 1$.

Remark 3.1. Proposition 3.1 shows that mutual interference of predator species has same effect on both prey and predator species. When prey species or predator is at low density ($H_* < \frac{a_2}{r_2}$ or $P_* < 1$), mutual interference of predator species has positive effect on both two species; and has negative effect on both two species when prey species or predator is at high density ($H_* > \frac{a_2}{r_2}$ or $P_* > 1$).

IV. THE INFLUENCE OF PREDATOR MUTUAL INTERFERENCE

In this section, we will give the following two examples to verify the feasibilities of our results. Since $\frac{dP_*}{dm} = \frac{r_2}{a_2} \frac{dH_*}{dm}$ i.e. P_* and H_* have same monotonicity with respect to m , we only use m and H_* instead of P_* for convenient.

Example 4.1. Consider the following system:

$$\begin{cases} \frac{dH}{dt} = (3 - H - 2P^m)H, \\ \frac{dP}{dt} = (1 - \frac{P}{2H})P. \end{cases} \tag{11}$$

In this case, we have $\frac{a_2}{r_2} = 0.5$ and $f(H_*) = 3(\frac{1}{2})^m - 2H_*^m - (\frac{1}{2})^m H_* = 0$. According to Theorem 2.1 and Theorem 2.2, system (11) has a unique equilibrium which is globally stable (see Fig. 1). Moreover, Fig. 1 also shows that H_* is greater than 0.5, so it follows from Proposition 3.1 that H_* strictly decreasing for $m \in (0, 1)$ which is supported by Fig. 2.

Example 4.2. Consider the following system:

$$\begin{cases} \frac{dH}{dt} = (3 - H - 2P^m)H, \\ \frac{dP}{dt} = (2 - \frac{4P}{H})P. \end{cases} \tag{12}$$

In this case, we have $\frac{a_2}{r_2} = 2$ and $f(H_*) = 3 \times 4^m - 2^{m+1}H_*^m - 4^m H_* = 0$. According to Theorem 2.1 and Theorem 2.2, system (12) has a unique equilibrium which is globally stable (see Fig. 3). Moreover, Fig. 3 also shows that H_* is less than 2, so it follows from Proposition 3.1 that H_* strictly increasing for $m \in (0, 1)$. Numerical simulation confirms our result (see Fig. 4).

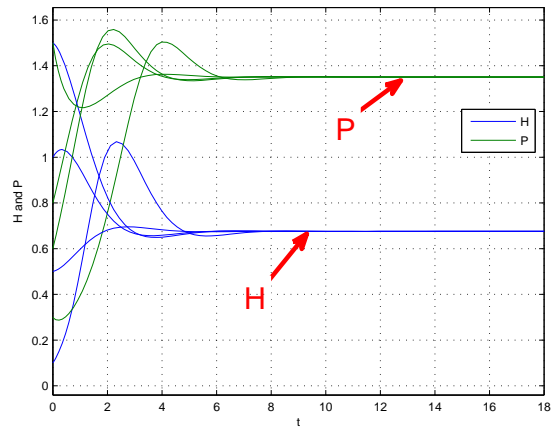


Fig. 1. Dynamic behavior of system (11) with $m = 1/2$ and the initial condition $(H(0), P(0)) = (0.1, 0.3)^T, (1, 0.8)^T, (1.5, 0.6)^T,$ and $(0.5, 1.5)^T$, respectively.

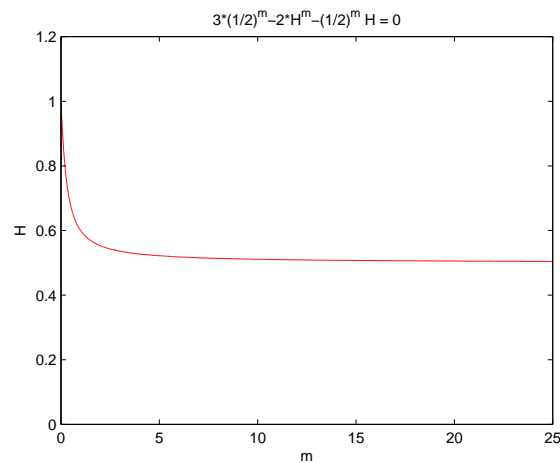


Fig. 2. Numeric simulations of $H_*(m)$ for system (11), where $m \in (0, 1)$.

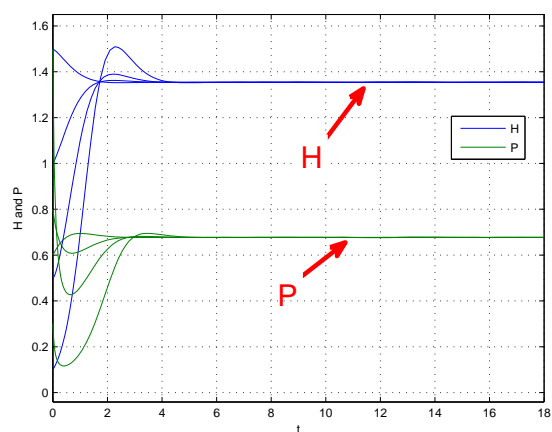


Fig. 3. Dynamic behavior of system (12) with $m = 1/2$ and the initial condition $(H(0), P(0)) = (0.1, 0.3)^T, (1, 0.8)^T, (1.5, 0.6)^T,$ and $(0.5, 1.5)^T$, respectively.

V. CONCLUSION

In this paper, we consider an autonomous Leslie-Gower predator-prey model with predator mutual interference and

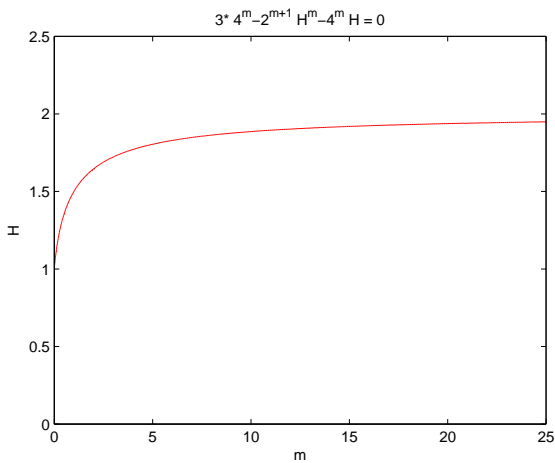


Fig. 4. Numeric simulations of $H_*(m)$ for system (12), where $m \in (0, 1)$.

show that predator mutual interference has no influence on the persistent property of system and has same effect on both prey and predator species: when prey species or predator is at low density ($H_* < \frac{a_2}{r_2}$ or $P_* < 1$), mutual interference of predator species has positive effect on both two species; and has negative effect on both two species when prey species or predator is at high density ($H_* > \frac{a_2}{r_2}$ or $P_* > 1$). However, based on Lotka-Volterra type system with predator mutual interference, Chen et al. [33] and Ma et al. [34] discovered that “mutual interference of predator species plays an essential role on the persistent property of the Lotka-Volterra predator-prey dynamics” and “the mutual interference of predator species has positive effect on the prey species if predator species is at low density; and has negative effect on the prey species if predator species is at high density. Also, if predator and prey species are at low density, then the mutual interference of predator species has positive effect on the density of predator species.” (see Remark 3.1-3.2 in [34]). So, our results are quite different with the above ones. It is well known that a suitable predator-prey model should consider the functional response of the predator species, it seems interesting to investigate the dynamic behaviors of the Leslie-Gower predator-prey model with some kind of functional response and the mutual interferences of predator species, and this can be studied in the future.

ACKNOWLEDGMENT

The research was supported by the Natural Science Foundation of Fujian Province (2019J01089), Program for New Century Excellent Talents in Fujian Province University (2018) and Program for Outstanding Youth Scientific Research Talents Cultivation in Fujian Province University (2016).

REFERENCES

[1] P. Leslie, “Some further notes on the use of matrices in population mathematics,” *Biometrika*, vol. 35, pp. 213-245, 1948.
 [2] P. Leslie, “A stochastic model for studying the properties of certain biological systems by numerical methods,” *Biometrika*, vol. 45, pp. 16-31, 1958.
 [3] A. Korobeinikov, “A Lyapunov function for Leslie-Gower predator-prey models,” *Appl. Math. Lett.*, vol. 14, pp. 697-699, 2001.
 [4] F. Chen, L. Chen and X. Xie, “On a Leslie-Gower predator-prey model incorporating a prey refuge,” *Nonlinear Anal. Real World Appl.*, vol. 10, pp. 2905-2908, 2009.

[5] H. Mohammadi and M. Mahzoon, “Effect of weak prey in Leslie-Gower predator-prey model,” *Appl. Math. Comput.*, vol. 224, pp. 196-204, 2013.
 [6] M. Aziz-Alaoui and M. Okiye, “Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling-type II schemes,” *Appl. Math. Lett.*, vol. 16, pp. 1069-1075, 2003.
 [7] S. Yu, “Global asymptotic stability of a predator-prey model with modified Leslie-Gower and Holling-type II schemes,” *Discrete Dyn. Nat. Soc.*, vol. 2012, Article ID 208167, 8 pages.
 [8] S. Yuan and Y. Song, “Stability and Hopf bifurcations in a delayed Leslie-Gower predator-prey system,” *J. Math. Anal. Appl.*, vol. 355, pp. 82-100, 2009.
 [9] Y. Li and C. Li, “Stability and Hopf bifurcation analysis on a delayed Leslie-Gower predator-prey system incorporating a prey refuge,” *Appl. Math. Comput.*, vol. 219, pp. 4576-4589, 2013.
 [10] T. Kar and A. Ghorai, “Dynamic behaviour of a delayed predator-prey model with harvesting,” *Appl. Math. Comput.*, vol. 217, pp. 9085-9104, 2011.
 [11] S. Yu, “Global stability of a modified Leslie-Gower model with Beddington-DeAngelis functional response,” *Adv. Difference Equ.*, 2014, 2014:84
 [12] Q. Yue, “Dynamics of a modified Leslie-Gower predator-prey model with Holling-type II schemes and a prey refuge,” *SpringerPlus*, vol. 5, pp. 1-12, 2016.
 [13] S. Yu and F. Chen, “Dynamic behaviors of a competitive system with Beddington-DeAngelis functional response,” *Discrete Dyn. Nat. Soc.*, vol. 2019, Article ID 4592054, 12 pages.
 [14] G. Liu, Y. Luo and L. Shu, “Asymptotic synchronization of complex dynamical networks with time-varying delays on time scales,” *Engineering Letters*, vol. 26, pp.210-215, 2018.
 [15] H. Zhou and M. Liu, “Analysis of a stochastic predator-prey model in polluted environments,” *IAENG International Journal of Applied Mathematics*, vol. 46, pp. 445-456, 2016.
 [16] C. Kang, H. Miao and X. Chen, “Global stability analysis for a delayed HIV infection model with general incidence rate and cell immunity,” *Engineering Letters*, vol. 24, pp. 392-398, 2016.
 [17] K. Yang, Z. Miao, F. Chen and X. Xie, “Influence of single feedback control variable on an autonomous Holling-II type cooperative system,” *J. Math. Anal. Appl.*, vol.435, pp. 874-888, 2016.
 [18] M. Hu and L. Wang, “Almost periodic solution for a Nabla BAM neural networks on time scales,” *Engineering Letters*, vol. 25, no.3, pp. 290-295, 2017.
 [19] Y. Liao, “Dynamics of two-species harvesting model of almost periodic facultative mutualism with discrete and distributed delays,” *Engineering Letters*, vol. 26, no.1, pp.7-13, 2018.
 [20] L. Wang and P. Xie, “Permanence and extinction of delayed stage-structured predator-prey system on time scales,” *Engineering Letters*, vol. 25, no.2, pp. 147-151, 2017.
 [21] F. Chen, X. Xie, Z. Miao and L. Pu, “Extinction in two species nonautonomous nonlinear competitive systems,” *Appl. Math. Comput.*, vol. 274, pp. 119-124, 2016.
 [22] H. Alfifi and T. Marchant, “Feedback control for a diffusive delay Logistic equation: semi-analytical solutions,” *IAENG International Journal of Applied Mathematics*, vol. 48, pp. 317-323, 2018.
 [23] L. Yang, Y. Yang, Y. Li and T. Zhang, “Almost periodic solution for a Lotka-Volterra recurrent neural networks with harvesting terms on time scales,” *Engineering Letters*, vol. 24, pp. 455-460, 2016.
 [24] L. Li and B. Du, “Global asymptotical stability in a stochastic predator-prey system with variable delays,” *IAENG International Journal of Applied Mathematics*, vol. 46, pp. 241-246, 2016.
 [25] L. Pang, Y. Liao and T. Zhang, “Almost periodic solutions for Lasota-Ważewska model with multiple delays,” *IAENG International Journal of Applied Mathematics*, vol. 46, pp. 311-316, 2016.
 [26] Y. Li, L. Xu and T. Zhang, “Dynamics of almost periodic mutualism model with time delays,” *IAENG International Journal of Applied Mathematics*, vol. 48, pp. 168-176, 2018.
 [27] H. Qu, X. Gan and T. Zhang, “The properties of stochastic mutualism model with time-lagged delays,” *IAENG International Journal of Applied Mathematics*, vol. 46, pp. 194-201, 2016.
 [28] L. Xu and Y. Liao, “On an almost periodic Gilpin-Ayala competition model of phytoplankton allelopathy,” *IAENG International Journal of Applied Mathematics*, vol. 47, pp.182-190, 2017.
 [29] K. Wang, “Permanence and global asymptotical stability of a predator-prey model with mutual interference,” *Nonlinear Anal. Real World Appl.*, vol. 12, pp. 1062-1071, 2011.
 [30] M. Hassel, “Density dependence in single-species population,” *J. Anim. Ecol.*, vol. 44, pp. 283-295, 1975.
 [31] K. Wang, “Existence and global asymptotic stability of positive periodic solution for a predator-prey system with mutual interference,” *Nonlinear Anal. Real World Appl.*, vol. 10, pp. 2774-2783, 2009.

- [32] S. Yu and F. Chen, "Almost periodic solution of a modified Leslie-Gower predator-prey model with Holling-type II schemes and mutual interference," *Int. J. Biomath.*, vol. 7, 1450028 (15 pages), 2014.
- [33] L. Chen, F. Chen and Y. Wang, "Influence of predator mutual interference and prey refuge on Lotka-Volterra predator-prey dynamics," *Commun. Nonlinear. Sci. Numer. Simulat.*, vol. 18, pp. 3174-3180, 2013.
- [34] Z. Ma, F. Chen, C. Wu and W. Chen, "Dynamic behaviors of a Lotka-Volterra predator-prey model incorporating a prey refuge and predator mutual interference," *Appl. Math. Comput.*, vol. 219, pp.7945-7953, 2013.
- [35] Z. Lu and X. Liu, "Analysis of a predator-prey model with modified Holling-Tanner functional response and time delay," *Nonlinear Anal. Real World Appl.*, vol. 9, pp.641-650, 2008.
- [36] N. Ali and M. Jazar, "Global dynamics of a modified Leslie-Gower predator-prey model with Crowley-Martin functional responses," *J. Appl. Math. Comput.*, vol. 43, pp. 271-293, 2013.