The Generalized $L_p$-mixed Volume and the Generalized $L_p$-mixed Projection Body

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Abstract—In this paper, based on three classical concept of the mixed surface area measure, the mixed volumes and the mixed projection bodies, we introduce three new concepts of the generalized $L_p$-mixed surface area measure, the generalized $L_p$-mixed volumes and the generalized $L_p$-mixed projection bodies of convex bodies. In addition, some important inequalities, such as, the Aleksandrov-Fenchel inequality for the generalized $L_p$-mixed volumes, the Aleksandrov-Fenchel inequality for the generalized $L_p$-mixed projection bodies and the Brunn-Minkowski inequality for polar of $L_p$-mixed projection bodies are established, respectively. We also give a generalization of Pythagorean inequality and Loomis-Whitney inequality for $L_p$-mixed volumes, respectively.

Index Terms—Convex body, surface area measure; $L_p$-mixed surface area measure; $L_p$-mixed volume; projection body, $L_p$-mixed projection body.

I. INTRODUCTION

The mixed volume is a central part of the Brunn-Minkowski theory of convex bodies. The monograph by Schneider [80] introduced the mixed volume and closely related mixed area measures, establish their fundamental properties. In the early 1960s, Firey [25] defined the Minkowski-Firey $L_p$-additions of convex bodies for each $p \geq 1$ and also established the $L_p$-Brunn-Minkowski inequality. Based on the $L_p$-additions, Lutwak [51] defined the $L_p$-mixed volume of two convex bodies and established the famous $L_p$-Brunn-Minkowski mixed volume inequality. In the mid 1990s, study on the volume of Minkowski-Firey $L_p$-additions in [51] and [52] leads to an $L_p$-Brunn-Minkowski theory. The rapidly developing $L_p$-Brunn-Minkowski theory of convex bodies is a natural extension of the Brunn-Minkowski theory (see, e.g., [2], [3], [4], [6], [8], [14], [15], [16], [17], [18], [21], [22], [23], [30], [31], [32], [33], [35], [40], [41], [42], [43], [44], [51], [52], [56], [58], [59], [60], [61], [62], [63], [64], [65], [66], [67], [68], [69], [70], [72], [73], [74], [75], [76], [79], [83], [85], [87], [91]).

The study of projection bodies or zonoids has a long history [36]. An article [10] first considered this problem, since then, considerable attention has been paid to the projection bodies [5], [11], [13], [19], [28], [46], [47], [71], [78], [80], [86], [89]. The related applications appeared in [86], [7], [81], [88]. The projection bodies topic has been focus on the intense study [11], [12], [20], [29], [45], [46], [47], [48], [49], [50], [77], [82].

Let $\mathcal{K}_n^o$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$; $\mathcal{K}_n^o, \mathcal{K}_n^w$ denote the subset of $\mathcal{K}_n^o$ containing the origin in their interiors and the subset of $\mathcal{K}_n^o$ that contains the centered (centrally symmetric with respect to the origin) bodies, respectively.

For $K_1, \ldots, K_n \in \mathcal{K}_n^o$, Aleksandrov-Fenchel inequality [80] is

$$V(K_1, \ldots, K_n) \geq \prod_{j=1}^r V(K_{j_1}, \ldots, K_{j_r} \cup \{L\})$$

(1)

with equality if and only if $K$ and $L$ are homothetic.

In 1939, the Aleksandrov-Fenchel inequality and Brunn-Minkowski inequality for the mixed projection bodies have been established by Lutwak [45]. If $K, L \in \mathcal{K}_n^o$, then

$$V(\Pi(K + L)^{1/(n-1)}) \geq V(\Pi(K) \Gamma^{1/(n-1)}) + V(\Pi(L) \Gamma^{1/(n-1)})$$

(4)

with equality if and only if $K$ and $L$ are homothetic.

If $K_1, \ldots, K_n \in \mathcal{K}_n^o$, then

$$V(\Pi(K_i, \ldots, K_{n-1})) \geq \prod_{j=1}^r V(\Pi(K_{j_1}, \ldots, K_{j_r} \cup \{L\}))$$

(5)

In particular, taking $K_{n-i} = \cdots = K_{n-2} = B(i = 0, 1, \ldots, n-2)$ in (5), and denotes

$$\Pi_i(K_1, \ldots, K_{n-1-i}) = \Pi(K_1, \ldots, K_{n-1-i}, B, \ldots, B).$$

So, we have

$$V(\Pi_i(K_1, \ldots, K_{n-1-i})) \geq \prod_{j=1}^r V(\Pi_i(K_{j_1}, \ldots, K_{j_r} \cup \{L\} \cup \{B\}))$$

(6)

In 2004, Leng et al. [38] established the Aleksandrov-Fenchel inequality for the polar of projection bodies as follows

$$V(\Pi^*(K_1, \ldots, K_{n-1})) \geq \prod_{j=1}^r V(\Pi^*(K_{j_1}, \ldots, K_{j_r} \cup \{L\} \cup \{B\}))$$

(7)
with equality if $K_1, \ldots, K_{n-1}$ are homothetic. In particular, taking $K_{n-i} = \cdots = K_{n-1} = B$ ($i = 0, 1, \ldots, n-2$) in (7), and denotes
\[ \Pi_i^*(K_1, \ldots, K_{n-1-i}) = \Pi_i^*(K_1, \ldots, K_{n-1-i}, B, \ldots, B), \]
Therefore, we have
\[ V(\Pi^*_i(K_1, \ldots, K_{n-1-i}))^p \leq \prod_{j=1}^r V(\Pi^*_i(K_j, K_{r+1}, \ldots, K_{n-1-i})) \] (8)
with equality if $K_1, \ldots, K_{n-1}$ are homothetic $n$-balls.

Based on the concepts of classical mixed surface area, mixed volume and mixed projection body involving a plurality of convex bodies, the purpose of this paper first is to introduce three notions of the generalized $L_p$-mixed surface area measure, generalized $L_p$-mixed volume and generalized $L_p$-mixed projection bodies, respectively. In addition, we will establish the Aleksandrov-Fenchel inequality for the generalized $L_p$-mixed volume, Aleksandrov-Fenchel inequality for the generalized $L_p$-mixed projection bodies and Brunn-Minkowski inequality for polars of $L_p$-mixed projection bodies of convex bodies, respectively. Our findings further enrich the architecture of $L_p$-Brunn-Minkowski theory.

First at all, we introduce the abbreviation
\[ (K_1, \ldots, K_m) := (K_1[r_1], \ldots, K_m[r_m]). \]

The following is our main results.

**Theorem 1.** If $p \geq 1$ and $K_1, \ldots, K_n \in \mathcal{K}_n$, then
\[ V_p(K_1, \ldots, K_n)^p \geq \prod_{j=1}^r V(K_j[r], K_{r+1}, \ldots, K_{n-1})^p \times \prod_{j=1}^{n-1} V(K_1, \ldots, K_{n-1}, K_j)^{\frac{(n-1)p}{n \cdot 2}}. \]

**Theorem 2.** If $p \geq 1$ and $K_1, \ldots, K_n \in \mathcal{K}_n$, then
\[ V(\Pi_p(K_1, \ldots, K_{n-1})) \geq n^{1-p} \prod_{j=1}^r V(\Pi(K_j[r], K_{r+1}, \ldots, K_{n-1})) \times \prod_{j=1}^{n-1} V(K_1, \ldots, K_{n-1}, K_j)^{\frac{1-p}{n \cdot 2}}. \]

**Theorem 3.** If $p \geq 1$ and $K_1, \ldots, K_n \in \mathcal{K}_n$, then
\[ V(\Pi^*_p(K_1, \ldots, K_{n-1})) \leq n^{\frac{(n-1)p}{n+1}} \prod_{j=1}^r V(\Pi^*(K_j[r], K_{r+1}, \ldots, K_{n-1}))^{\frac{n}{n+1}} \times \prod_{j=1}^{n-1} V(K_1, \ldots, K_{n-1}, K_j)^{\frac{n-1}{n+1}}. \]

with equality if and only if $K_i$ is the line segment joining $-\lambda_i u$ and $\lambda_i u$, where $\lambda_i > 0$ ($i = 1, \ldots, n-1$).

**Theorem 4.** If $K, L, K_2, \ldots, K_{n-1} \in \mathcal{K}_n$, $\lambda, \mu \geq 0$ (not both zero), $p \geq 1$, $i \in \mathbb{R}$ and $C = (K_2, \ldots, K_{n-1})$, then
\[ 4V_p(\Pi^*(\lambda K + \mu L, C))^p \leq \lambda^p V_p(\Pi^*(K, C))^p + \mu^p V_p(\Pi^*(L, C))^p, \]
with equality if and only if $\Pi(K, C) = \Pi(L, C)$.

Next, we use the methods to give a generalization of Pythagorean inequality for mixed volumes obtained by Firey [24].

**Theorem 5.** Let $p > 0$ and $K_1, \ldots, K_{n-1} \in \mathcal{K}_n$. Assume that $u_1, \ldots, u_m$ is a sequence of unit vectors in $\mathbb{R}^n$ and $c_1, \ldots, c_m$ be a sequence of positive numbers satisfying
\[ \sum_{i=1}^m c_i u_i \otimes u_i = I_n, \]
with equality if and only if $\sum_{i=1}^m c_i u_i \otimes u_i = I_n$.

The classical Loomis-Whitney inequality [39] shows the relation between the volume of a convex body and the geometric mean of its shadows. The Loomis-Whitney inequality is one of the fundamental inequalities in convex geometry and has been studied intensively. We generalize the Loomis-Whitney inequality to the following form of $L_p$-mixed volume associated with John basis.

**Theorem 6.** Suppose that $K_1, \ldots, K_{n-1} \in \mathcal{K}_n$, $\{u_i\}_{i=1}^m$ is a sequence of unit vectors in $\mathbb{R}^n$, and $\{c_i\}_{i=1}^m$ is a sequence of positive numbers such that $\sum_{i=1}^m c_i u_i = I_n$. Then for $p \geq 1$,
\[ \prod_{i=1}^m \left( \frac{n-2}{n} \prod_{j=1}^r v(K_j^{u_i}, [K_1], \ldots, [K_{n-2}])^{\frac{n-1}{n \cdot 2}} \times v(K_1^{u_i}, \ldots, K_1^{u_i})^{\frac{1}{n \cdot 2}} \right) c_i \geq \prod_{i=1}^m \left( \frac{n-1}{n} V([K_1], [K_{n-1}]) \right). \]

**Contents of the paper.** For our studies, we state some relevant knowledge for the convex geometric analysis in Section 2. In Section 3, we propose two new concepts for the generalized $L_p$-mixed volumes and generalized $L_p$-mixed quermassintegrals, discuss some of their related properties. Simultaneously, we introduce a new concept for the $L_p$-mixed projection bodies. Section 4, we prove Theorems 1-6 which stated in the beginning of this paper, respectively. As an application, we give a generalization of Pythagorean inequality for mixed volumes, which has been obtained by Firey [24]. In addition, we established a generalized inequality of Loomis-Whitney inequality.
II. BACKGROUND MATERIAL

The setting for this paper is \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) \((n \geq 2)\). Let \( u \) denotes unit vector, and \( B \) denotes unit ball centered at the origin, the surface of \( B \) is \( S^{n-1} \). For \( u \in S^{n-1} \), let \( E_u \) denote the hyperplane, through the origin, that is orthogonal to \( u \). We use \( K^u \) to denote the image of \( K \) under an orthogonal projection onto the hyperplane \( E_u \).

\[ [u] \text{ denotes the line segment joining } \{\lambda u : |\lambda| \leq \frac{1}{2}\}. \]

Let \( V(K) \) denote the \( n \)-dimensional volume of a body \( K \), and \( x \in K \). Then the \( n \)-dimensional volume of \( K \) is given by \( \int_S V(\phi K) \text{d}\mu_L \). Also, \( \phi \) denotes the transpose of \( \phi \), and \( \phi^{-1} \) denotes the inverse of the transpose of \( \phi \).

\[ \phi^t \text{ denotes the transpose of } \phi, \quad \phi^{-t} \text{ denotes the inverse of the transpose of } \phi. \]

A. Support function, radial function, polar of convex body and Minkowski linear combination

Let \( h(K, \cdot) : \mathbb{R}^n \to (0, \infty) \) denote the support function of \( K \in \mathcal{K}^n \), defined by \( h(K, x) = \max\{x \cdot y : y \in K\} \). If \( \phi \in \text{GL}(n) \), then for the support function of the image \( \phi K = \{\phi x : x \in K\} \), we easily have

\[ h_{\phi K}(x) = h_K(\phi^t x). \quad (16) \]

For \( K, L \in \mathcal{K}^n \), Hausdorff metric \( \delta \) of \( K \) and \( L \) is defined by

\[ \delta(K, L) = \sup\{|h_K(u) - h_L(u)| : u \in S^{n-1}\}. \]

For \( K \in \mathcal{K}^n \) and a nonnegative scalar \( \lambda \), \( \lambda K = \{\lambda x : x \in K\} \). For \( K_i \in \mathcal{K}^n \), \( \lambda_i \geq 0 \) \((i = 1, \ldots, r)\), Minkowski linear combination \( \sum_{i=1}^r \lambda_i K_i \in \mathcal{K}^n \) is defined by

\[ \sum_{i=1}^r \lambda_i K_i = \left\{ \sum_{i=1}^r \lambda_i x_i \in \mathcal{K}^n : x_i \in K_i, i=1, \ldots, r \right\}. \]

It is trivial to verify that

\[ h\left( \sum_{i=1}^r \lambda_i K_i, \cdot \right) = \sum_{i=1}^r \lambda_i h(K_i, \cdot). \quad (17) \]

For \( K, L \in \mathcal{K}_o^n \), \( p \geq 1 \), \( \lambda \), \( \mu \geq 0 \) (not both zero), the Firey \( L_p \)-combination \( \lambda \cdot K + \mu \cdot L \in \mathcal{K}_o^n \) is defined by (see [25])

\[ h((\lambda \cdot K + \mu \cdot L), \cdot)^p = \lambda h(K, \cdot)^p + \mu h(L, \cdot)^p. \quad (18) \]

B. Mixed volumes, \( L_p \)-mixed volumes, mixed surface area measure and \( L_p \)-mixed quermassintegrals

If \( K_1, \ldots, K_r \in \mathcal{K}^n \) and \( \lambda_1, \ldots, \lambda_r \) are nonnegative real numbers, then the volume of \( \lambda_1 K_1 + \cdots + \lambda_r K_r \) is a homogeneous polynomial in \( \lambda_1, \ldots, \lambda_r \) (see [80]).

\[ V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{i_1, \ldots, i_n} \lambda_{i_1} \cdots \lambda_{i_n} V_{i_1, \ldots, i_n}, \quad (20) \]

where the sum is taken over all \( n \)-tuples \((i_1, \ldots, i_n)\) of positive integers not exceeding \( r \). The coefficient \( V_{i_1, \ldots, i_n} \) depends only on the bodies \( K_{i_1}, \ldots, K_{i_n} \) and is uniquely determined by (20), it is called the mixed volume of \( K_{i_1}, \ldots, K_{i_n} \) and is written as \( V(K_{i_1}, \ldots, K_{i_n}) \).

Associated with \( K_1, \ldots, K_{n-1} \in \mathcal{K}^n \) Borel measure, \( S(K_1, \ldots, K_{n-1}) \), on \( S^{n-1} \), called the mixed surface area measure of \( K_1, \ldots, K_{n-1} \), which has the property that for each \( L \in \mathcal{K}^n \) (see [37]),

\[ V(K_1, \ldots, K_{n-1}, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u) dS(K_1, \ldots, K_{n-1}; u). \quad (21) \]

For \( \lambda, \mu > 0 \), if \( K_1 \) is replaced by \( \lambda K_1 + \mu L \), then we have

\[ S(\lambda K_1 + \mu L_1, \ldots, K_{n-1}; \cdot) = \lambda S(K_1, \ldots, K_{n-1}; \cdot) \]

\[ + \mu S(L_1, \ldots, K_{n-1}; \cdot). \quad (22) \]

An important fact [26] is

\[ \int_{S^{n-1}} u dS(K_1, \ldots, K_{n-1}; u) = 0. \]

We noted that the mixed area measure \( S(K_1, \ldots, K_{n-1}; \cdot) \) also satisfies the hypothesis of Minkowski’s existence theorem. Thus, for \( K_1, \ldots, K_{n-1} \in \mathcal{K}^n \), there exists a convex body denoted by \( [K_1, \ldots, K_{n-1}] \), whose area function is \( S([K_1, \ldots, K_{n-1}], \cdot) \), namely,

\[ S([K_1, \ldots, K_{n-1}], \cdot) = S(K_1, \ldots, K_{n-1}; \cdot), \]

where \([K_1, \ldots, K] = K\).

A direct consequence of (21) is following

\[ V([K_1, \ldots, K_{n-1}]; [n-1], K_{n-1}) = V(K_1, \ldots, K_{n-1}, K_{n-1}). \quad (23) \]

Since

\[ V([K_1, \ldots, K_{n-1}]) = V([K_1, \ldots, K_{n-1}]; [n-1], [K_1, \ldots, K_{n-1}]), \]

(23) implies that

\[ V([K_1, \ldots, K_{n-1}]) = V(K_1, \ldots, K_{n-1}, [K_1, \ldots, K_{n-1}]), \quad (24) \]

\[ V(K_1, \ldots, K_{n-1}, K_{n-1}) = V(K_1, \ldots, K_{n-1}, K_{n-1}). \quad (25) \]

If \( K_1 = \cdots = K_{n-1} = K \) and \( K_{n-1} = \cdots = K_{n-1} = B \), then \( S(K_1, \ldots, K_{n-1}; \cdot) \) is written as \( S(K, \cdot) \), \( V(K_1, \ldots, K_{n-1}, L) \) is written as \( W_i(K, L) \). If \( L = K \),

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\[ W_i(K, K) \text{ is written as } W_i(K) \text{ that is called } i \text{th quermassintegrals of convex body } K; \text{ i.e., } \\
W_i(K) = \frac{1}{n} \int_{S^{n-1}} h(K,u)dS_i(K,u). \] (26)

We recall that \( W_0(K) \) is \( V(K) \).

In [37], Lutwak proved that if \( K_1, \ldots, K_n \in \mathcal{K}^n \), and \( \phi \in GL(n) \), then
\[ V(\phi K_1, \ldots, \phi K_n) = |\det \phi|V(K_1, \ldots, K_n). \] (27)

Suppose \( K, L \in \mathcal{K}_o^n \), then for \( p \geq 1 \), the \( L_p \)-mixed volume, \( V_p(K,L) \), of \( K \) and \( L \) is defined by (see [51])
\[ \frac{n}{p} V_p(K,L) = \lim_{\varepsilon \to 0^+} \frac{V(K + p \varepsilon \cdot L) - V(K)}{\varepsilon}. \] (28)

For \( K \in \mathcal{K}_o^n \), there is a positive Borel measure, \( S_p(K, \cdot) \), on \( S^{n-1} \) such that (see [51])
\[ V_p(K,Q) = \frac{1}{n} \int_{S^{n-1}} h(Q,u)^p dS_p(K,u), \] (29)
for each \( Q \in \mathcal{K}_o^n \). The measure \( S_p(K, \cdot) \) is just the \( L_p \)-surface area measure of \( K \), which is absolutely continuous with respect to classical surface area measure \( S(K, \cdot) \), and process Radon-Nikodym derivative
\[ \frac{dS_p(K, \cdot)}{dS(K, \cdot)} = h^{1-p}(K, \cdot), \] (30)

it follows from (30) that \( S_1(K, \cdot) \) is just \( S(K, \cdot) \).

The \( L_p \)-Minkowski inequality was given by Lutwak [52]. If \( K, L \in \mathcal{K}_o^n \) and \( p > 1 \), then
\[ V_p(K,L) \geq V(K)^{\frac{n}{n-p}} V(L)^{\frac{p}{n-p}}, \] (31)
with equality for \( p = 1 \) if and only if \( K \) and \( L \) are homothetic, for \( p > 1 \) if and only if \( K \) and \( L \) are dilates.

For \( K \in \mathcal{K}_o^n, \varepsilon > 0 \) and real \( p \geq 1 \), the \( L_p \)-mixed quermassintegrals, \( W_{p,i}(K,L) \) \((i = 0, 1, \ldots, n-1)\), of \( K \) and \( L \) is defined (see [51])
\[ \frac{n-i}{p} W_{p,i}(K,L) = \lim_{\varepsilon \to 0^+} W_i(K + p \varepsilon \cdot L) - W_i(K). \] (32)

The \( W_{p,0}(K,L) \) is just \( L_p \)-mixed volume \( V_p(K,L) \), namely
\[ W_{p,0}(K,L) = V_p(K,L). \] In [51], Lutwak has shown that, for \( p \geq 1 \) and each \( K \in \mathcal{K}_o^n \), there exists a positive Borel measure \( S_{p,i}(K, \cdot) \) \((i = 0, 1, \ldots, n-1)\) on \( S^{n-1} \), such that the \( L_p \)-mixed quermassintegrals \( W_{p,i}(K,L) \) has the following integral representation
\[ W_{p,i}(K,L) = \frac{1}{n} \int_{S^{n-1}} h^p_{L}(v)dS_{p,i}(K,v), \] (33)
for all \( L \in \mathcal{K}_o^n \). It turns out that the measure \( S_{p,0}(K, \cdot) \) \((i = 0, 1, \ldots, n-1)\) on \( S^{n-1} \) is absolutely continuous with respect to \( S_i(K, \cdot) \), and has the Radon-Nikodym derivative
\[ \frac{dS_{p,i}(K, \cdot)}{dS(K, \cdot)} = h^{1-p}(K, \cdot), \] (34)

where \( S_i(K, \cdot) \) is a classical positive Borel measure on \( S^{n-1} \) (see [51]). Obviously, \( S_{p,0}(K, \cdot) = S_p(K, \cdot) \). The Formula (34) has shown that, for \( p \geq 1, i = 0, 1, \ldots, n-1 \), and each \( K \in \mathcal{K}_o^n \), there exists a positive Borel measure on \( S^{n-1} \), by (see [51])
\[ S_{p,i}(K, \omega) = \int_{\omega} h(K,u)^{1-p}dS_i(K,u), \] (35)
for each Borel \( \omega \subset S^{n-1} \).

C. Dual mixed volume

If \( K_i \in \mathcal{S}^n_o \) \((i = 1, \ldots, n)\), then the dual mixed volume of \( K_1, \ldots, K_n \) is defined by (see [53])
\[ \tilde{V}(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1,u) \cdots \rho(K_n,u)dS(u), \] (36)
where \( dS(u) \) denotes the area element of \( S^{n-1} \) at \( u \). Note that \( \tilde{V}(K,L) = V(K[n-1],L[i]) \). Thus, if \( i \) is any real, then \( \tilde{V}(K,L) \) is said the dual mixed volume of \( K, L \in \mathcal{S}^n_o \), and
\[ \tilde{V}_i(K,L) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^{n-i} \rho(L,u)^i dS(u). \] (37)

In (37), let \( L = B \) and we write \( \tilde{V}_i(K,B) = \tilde{W}_i(K) \), together with \( \rho(B,u) = 1 \) for all \( u \in S^{n-1} \), the definition of dual quermassintegrals can be stated that: \( K \in \mathcal{S}^n_o, i \in \mathbb{R} \), the dual quermassintegrals, \( \tilde{W}_i(K) \), of \( K \) is defined by (see [27])
\[ \tilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^{n-i} dS(u). \] (38)
We recall that the polar coordinate formula for volume of \( K \in \mathcal{K}^n \) is \( V(K) = \tilde{W}_0(K) \).

III. THE MAIN CONCEPTS AND THEIR RELATED PROPERTIES

A. Generalized \( L_p \)-mixed surface area and generalized \( L_p \)-mixed volumes

In this section, we first proposed the two concepts of the generalized \( L_p \)-mixed surface area and the generalized \( L_p \)-mixed volume. Motivated by (35), we introduce the following definitions.

Definition 7. For \( p > 0 \) and \( K_1, \ldots, K_{n-1} \in \mathcal{K}_o^n \), the Borel measure \( S_p(K_1, \ldots, K_{n-1}; \cdot) \) on \( S^{n-1} \) is defined by
\[ S_p(K_1, \ldots, K_{n-1}; \omega) = \int_{\omega} (h(K_1, u) \cdots h(K_{n-1}, u)) \frac{1}{\omega} \times dS(K_1, \ldots, K_{n-1}; \omega), \] (39)
for each Borel \( \omega \subset S^{n-1} \), where \( S(K_1, \ldots, K_{n-1}; \cdot) \) is the classical the mixed surface area measure of \( K_1, \ldots, K_{n-1} \).

From (39), we easily obtain that
\[ \frac{dS_p(K_1, \ldots, K_{n-1}; \cdot)}{dS(K_1, \ldots, K_{n-1}; \cdot)} = (h(K_1, \cdot) \cdots h(K_{n-1}, \cdot))^\frac{1}{\omega}. \] (40)

Associated with \( K_1, \ldots, K_{n-1} \in \mathcal{K}_o^n \) is a Borel measure, \( S_p(K_1, \ldots, K_{n-1}; \cdot) \), on \( S^{n-1} \), called the generalized \( L_p \)-mixed surface area measure of \( K_1, \ldots, K_{n-1} \).

Taking \( K_1 = \cdots = K_{n-1} = K \) in (40), then (40) reduces to (30), where \( S_p(K, \cdot) := S_p(K, \cdots, K; \cdot) \) and \( S(K, \cdot) := S(K, \cdots, K; \cdot) \).

Definition 8. For \( p > 0 \) and \( K_1, \ldots, K_{n-1} \in \mathcal{K}_o^n \), the generalized \( L_p \)-mixed volume, \( V_p(K_1, \ldots, K_{n-1}; L) \), is defined by
\[ V_p(K_1, \ldots, K_{n-1}; L) = \frac{1}{n} \int_{S^{n-1}} h(L,u)^p dS_p(K_1, \ldots, K_{n-1}; u), \] (41)
where the Borel measure \( S_p(K_1, \ldots, K_{n-1}; \cdot) \) depends only on the bodies \( K_1, \ldots, K_{n-1} \), and is uniquely determined by (39).
Some properties of $V_p(K_1, \ldots, K_n)$ are as follows.

(i) **Continuity** If $K_i \in K^n_0$ ($i = 1, \ldots, n$), then $V_p(K_1, \ldots, K_n)$ is continuous for $p$;

(ii) **Positive definite property** If $K_i \in K^n_0$ ($i = 1, \ldots, n$), then $V_p(K_1, \ldots, K_n) > 0$;

(iii) **Positive definite homogeneity** If $\lambda_i > 0, K_i \in K^n_0$ ($i = 1, \ldots, n$), then

\[
V_p(\lambda K_1, \ldots, \lambda K_n) = \frac{\lambda^n}{n!} V_p(K_1, \ldots, K_n);
\]

(iv) **$p$-additivity** If $K_i \in K^n_0$ ($i = 1, \ldots, n$), $L \in K^n_0$, and $\lambda, \mu \geq 0$ (not both zero), then

\[
V_p(K_1, \ldots, K_{n-1}, \lambda \cdot K_n + \mu \cdot L) = \lambda V_p(K_1, \ldots, K_{n-1}, K_n) + \mu V_p(K_1, \ldots, K_{n-1}, L);
\]

(v) **Monotonicity** If $K_i \in K^n_0$ ($i = 1, \ldots, n-1$), $K, L \in K^n_0$, then

\[
K \subset L \Rightarrow V_p(K_1, \ldots, K_{n-1}, K) \leq V_p(K_1, \ldots, K_{n-1}, L),
\]

with equality if and only if $h(K, u) = h(L, u)$ for all $u$ in the support of the measure $S(K_1, \ldots, K_{n-1})$.

**Remark 9.** The condition in (v) is in general not equivalent to $K = L$, since the support of $S(K_1, \ldots, K_{n-1})$ can be a proper closed subset of the unit sphere.

It will be helpful to introduce some additional notation.

For $x \in \mathbb{R}^n$, let $(x) = x/|x|$, whenever $x \neq 0$.

**Definition 10.** (see [56]) Given a measure $d\mu(u)$ on $S^{n-1}$, a real $p > 0$, and a $\phi \in GL(n)$, define the measure $d\mu^{(p)}(\phi u)$ on $S^{n-1}$ by

\[
\int_{S^{n-1}} f(u) d\mu^{(p)}(\phi u) = \int_{S^{n-1}} |\phi^{-1} u|^p f(\phi^{-1} u) d\mu(u),
\]

for each $f \in C(S^{n-1})$.

First note that for any convex bodies $K_1, \ldots, K_{n-1}$ and each $\phi \in GL(n)$ for the classical mixed surface area measure we have

\[
dS(\phi K_1, \ldots, \phi K_{n-1}; u) = |\det \phi| dS^{(1)}(K_1, \ldots, K_{n-1}; \phi^{-1} u).
\]

To see this note that for any convex bodies $K_1, \ldots, K_{n-1}$ it follow from Definition 10, the homogeneity of $h_Q$, (16) and (27) that

\[
\int_{S^{n-1}} h_Q(u) dS^{(1)}(K_1, \ldots, K_{n-1}; \phi^{-1} u) = \int_{S^{n-1}} |\phi^{-1} u|^p h_Q(\phi^{-1} u) dS(K_1, \ldots, K_{n-1}; u)
\]

\[
= \int_{S^{n-1}} h_Q(\phi^{-1} u) dS(K_1, \ldots, K_{n-1}; u) = \int_{S^{n-1}} h_{\phi^{-1} Q}(u) dS(K_1, \ldots, K_{n-1}; u)
\]

\[
= |\det \phi^{-1}| \int_{S^{n-1}} h_Q(u) dS(\phi K_1, \ldots, \phi K_{n-1}; u).
\]

**Proposition 11.** If $K_1, \ldots, K_{n-1} \in K^n_0$ and real $p > 0$, then for $\phi \in GL(n),

\[
dS\phi(\phi K_1, \ldots, \phi K_{n-1}; u) = |\det \phi| dS^{(p)}(K_1, \ldots, K_{n-1}; \phi^{-1} u).
\]

**Proof.** If $f \in C(S^{n-1})$, then from Definition 7, (16), (42), Definition 10, the homogeneity of $h_K$, (42) again, and Definition 10 again, we have

\[
\int_{S^{n-1}} f(u) dS_p(\phi K_1, \ldots, \phi K_{n-1}; u) = \int_{S^{n-1}} f(u) (h_K(\phi K_1) \cdots h_K(\phi K_{n-1}))^{\frac{1-p}{p}} \times dS(\phi K_1, \ldots, \phi K_{n-1}; u)
\]

\[
= |\det \phi| \int_{S^{n-1}} f(u) (h_K(\phi^{-1} u) \cdots h_K(\phi^{-1} u))^{\frac{1-p}{p}} \times dS(K_1, \ldots, K_{n-1}; u)
\]

\[
= |\det \phi| \int_{S^{n-1}} f(u) dS^{(p)}(K_1, \ldots, K_{n-1}; \phi^{-1} u).
\]

An immediate result of Proposition 11 is:

**Corollary 12.** If $K_1, \ldots, K_{n-1} \in K^n_0, \text{real } p > 0 \text{ and } \phi \in GL(n)$, then

\[
V_p(\phi K_1, \ldots, \phi K_{n-1}, L) = |\det \phi| V_p(K_1, \ldots, K_{n-1}, \phi^{-1} L).
\]

**Proof.** From Definition 8, Proposition 11, Definition 10, the homogeneity of the support function, (16), and finally Definition 8 again, we have

\[
n V_p(\phi K_1, \ldots, \phi K_{n-1}, L) = \int_{S^{n-1}} h(L, u)^p dS_p(\phi K_1, \ldots, \phi K_{n-1}; u)
\]

\[
= |\det \phi| \int_{S^{n-1}} h(L, u)^p dS^{(p)}(K_1, \ldots, K_{n-1}; \phi^{-1} u)
\]

\[
= |\det \phi| \int_{S^{n-1}} h(L, \phi^{-1} u)^p dS_p(K_1, \ldots, K_{n-1}; u)
\]

\[
= |\det \phi| \int_{S^{n-1}} h(\phi^{-1} L, \phi^{-1} u)^p dS_p(K_1, \ldots, K_{n-1}; u)
\]

\[
= n |\det \phi| V_p(K_1, \ldots, K_{n-1}, \phi^{-1} L).
\]

Corollary 12 shows that for $K_1, \ldots, K_n$ are convex bodies that contain the origin in their interiors, real $p > 0$, and $\phi \in GL(n),

\[
V_p(\phi K_1, \ldots, \phi K_n) = |\det \phi| V_p(K_1, \ldots, K_n).
\]
B. Generalized $i$th $L_p$-mixed surface area and generalized $i$th $L_p$-mixed quermassintegrals

**Definition 13.** Let $p > 0$, and $K_1 = \cdots = K_{n-1-i} \in \mathcal{K}_o^n$, $K_{n-i} = \cdots = K_{n-1} = B$ $(i = 0, 1, \cdots, n-2)$, define the Borel measure $S_{p,i}(K_1, \cdots, K_{n-1-i}; \omega)$ on $S^{n-1}$, by

$$
S_{p,i}(K_1, \cdots, K_{n-1-i}; \omega) = \int_{S^{n-1}} (h(K_1, u) \cdots h(K_{n-1-i}, u))^\frac{1}{n-p} \times dS_{i}(K_1, \cdots, K_{n-1-i}; u), \quad (45)
$$

for each Borel $\omega \subset S^{n-1}$, where we denote

$$
S_{i}(K_1, \cdots, K_{n-1-i}; \cdot) := S(K_1, \cdots, K_{n-1-i}, B[i]; \cdot),
S_{p,i}(K_1, \cdots, K_{n-1-i}; \cdot) := S_p(K_1, \cdots, K_{n-1-i}, B[i]; \cdot).
$$

From (45), it is easily to obtain that

$$
dS_{p,i}(K_1, \cdots, K_{n-1-i}; \cdot) = (h(K_1, \cdots h(K_{n-1-i}, \cdot))^{1-p} n-p. \quad (46)
$$

Taking $K_1 = \cdots = K_{n-1-i} = K$ in (46), then (46) reduces to (34).

Let $K_1 = \cdots = K_{n-1-i} = K$ and $K_{n-i} = \cdots = K_{n-1} = B$, and abbreviating the introduction

$$
W_{p,i}(K_1, \cdots, K_{n-1-i}, L) := V_p(K_1, \cdots, K_{n-1-i}, B[i], L).
$$

**Definition 14.** For $p > 0$, and $K_1, \cdots, K_{n-1-i}, L \in \mathcal{K}_o^n$ $(i = 0, 1, \cdots, n-2)$, we define the generalized $L_p$-mixed quermassintegrals, $W_{p,i}(K_1, \cdots, K_{n-1-i}, L)$, by

$$
W_{p,i}(K_1, \cdots, K_{n-1-i}, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p dS_p(K_1, \cdots, K_{n-1-i}; u), \quad (47)
$$

where the Borel measure $S_p(K_1, \cdots, K_{n-1-i}; \cdot)$ depends only on the bodies $K_1, \cdots, K_{n-1-i}$, and is uniquely determined by (45), it is the generalized $i$th $L_p$-mixed surface area measure of $K_1, \cdots, K_{n-1-i}$.

**Remark 15.** By Definition 14 with that (47), we can deduce that the Definition 8 with that (41) but not vice versa. Therefore, Definition 14 with that (47) extend some known ones in the sense of the Definition 8 with that (41).

C. Generalized $L_p$-mixed projection bodies

In this section, we first introduce the concept of generalized $L_p$-mixed projection body.

**Definition 16.** If $p > 0$, and $K_i \in \mathcal{K}_o^n$ $(i = 1, \cdots, n-1)$, then for $u \in S^{n-1}$, the generalized $L_p$-mixed projection body $\Pi_p(K_1, \cdots, K_{n-1})$, of $K_i(i = 1, \cdots, n-1)$ is defined by

$$
h(\Pi_p(K_1, \cdots, K_{n-1}), u)^p = \frac{1}{2p} \int_{S^{n-1}} |u \cdot v|^p dS_p(K_1, \cdots, K_{n-1}; v), \quad (48)
$$

where $S_p(K_1, \cdots, K_{n-1-i})$ depends only on the bodies $K_1, \cdots, K_{n-1}$, and is uniquely determined by (40).

When $p = 1$, then (48) reduces to the following definition of mixed projection bodies introduced by Lutwak [57]:

$$
h(\Pi(K_1, \cdots, K_{n-1}), u) = \frac{1}{2} \int_{S^{n-1}} |u \cdot v| dS(K_1, \cdots, K_{n-1}; v). \quad (49)
$$

This is equivalent to

$$
v(K_1, \cdots, K_{n-1}) = nV(K_1, \cdots, K_{n-1}, [u]). \quad (50)
$$

Additional, it follow from (41) and (49) that

$$
h(\Pi_p(K_1, \cdots, K_{n-1}), u)^p = nV_p(K_1, \cdots, K_{n-1}, [u]). \quad (51)
$$

**Remark 17.** If $p \neq 1$, it follows that

$$
v_p(K_1, \cdots, K_{n-1}) \neq nV_p(K_1, \cdots, K_{n-1}, [u]).
$$

In fact, a function of $K_1$, the right-hand side is (under dilatation) homogeneous of degree $\frac{1}{n-p}$, while the left-hand side is homogeneous of degree $\frac{1}{n-2}$. Further, a function of $K_{n-1}$, the right-hand side is homogeneous of degree $\frac{1}{n-2}$, while the left-hand side is homogeneous of degree $p$. However, from (40), (41) and Hölder’s inequality (58) in the back, we have that for $p \geq 1$

$$
v_p(K_1, \cdots, K_{n-1}) \geq nV(K_1, \cdots, K_{n-1}, [u])^p \times \prod_{j=1}^{n-2} V(K_j, K_{n-1}, [u]) \frac{1}{n-p}, \quad (52)
$$

for $0 < p \leq 1$, the inequality (52) is reversed. Equality holds in either if and only if $p = 1$.

We use $\Pi_p^i(K_1, \cdots, K_{n-1})$ to denote the polar body of $\Pi_p(K_1, \cdots, K_{n-1})$ called the polar of generalized $L_p$-mixed projection body of $K_1, \cdots, K_{n-1}$.

D. Generalized $i$th $L_p$-mixed projection bodies

**Definition 18.** If $p > 0$, $K_1, \cdots, K_{n-1-i} \in \mathcal{K}_o^n$ $(i = 0, 1, \cdots, n-2)$, then the generalized $i$th $L_p$-mixed projection body of $K_j$ $(j = 1, \cdots, n-1-i)$ is defined by $\Pi_{p,i}(K_1, \cdots, K_{n-1-i})$, and whose support function is given, for $u \in S^{n-1}$, by

$$
h(\Pi_{p,i}(K_1, \cdots, K_{n-1-i}), u)^p = \frac{1}{2p} \int_{S^{n-1}} |u \cdot v|^p \times dS_{p,i}(K_1, \cdots, K_{n-1-i}; v), \quad (53)
$$

where $S_{p,i}(K_1, \cdots, K_{n-1-i}; \cdot)$ is uniquely determined by (46).

From Definition 14 and Definition 18, it follows that

$$
h(\Pi_{p,i}(K_1, \cdots, K_{n-1-i}), u)^p = nW_{p,i}(K_1, \cdots, K_{n-1-i}, [u]). \quad (54)
$$

If $K_1 = \cdots = K_{n-2-i} = K$ and $K_{n-1-i} = L$, then $\Pi_{p,i}(K_1, \cdots, K_{n-1-i})$ will be written as $\Pi_{p,i}(K, L)$. If $L = B$, then $\Pi_{p,i}(K, B)$ is called the $i$th $L_p$-mixed projection body of $K$ and denoted by $\Pi_{p,i}K$. We write $\Pi_{p,0}K$ as $K$. It easily to see that

$$
h(\Pi_{p,i}(K, L), u)^p = \frac{1}{2p} \int_{S^{n-1}} |u \cdot v|^p dS_{p,i}(K, L, v) \quad (55)
$$

and

$$
h(\Pi_{p,i}(K, B), u)^p = \frac{1}{2p} \int_{S^{n-1}} |u \cdot v|^p dS_{p,i}(K, v). \quad (56)
$$

The following property will be used later. If $K, L \in \mathcal{K}_o^n, K_2, \cdots, K_{n-1} \in \mathcal{K}_o^n$ and $C = (K_2, \cdots, K_{n-1})$, then

$$
\Pi(\lambda K + \mu L, C) = \lambda \Pi(K, C) + \mu \Pi(L, C). \quad (57)
$$

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IV. THE MAIN RESULTS AND THEIR PROOFS

A. The Aleksandrov-Fenchel inequality for the generalized $L_p$-mixed volume of convex bodies

In this section, we prove the Aleksandrov-Fenchel inequality for the generalized $L_p$-mixed volume of convex bodies stated in the beginning of this paper.

**Proof of Theorem 1.** For $p = 1$, Theorem 1 is just inequality (1) stated in the beginning of this paper, its proof was completed by Schneider (see [80], p.401).

For $p > 1$, we use Hölder’s inequality (see [34], p.140) to complete the proof.

Suppose that $f_i ∈ L^p_{α_i}(E), 1 < α_i < ∞ (i = 0, 1, ⋯ , m)$ are nonnegative functions, $α_i$ satisfies

$$\frac{1}{α_1} + \cdots + \frac{1}{α_m} = 1.$$ 

Then $\prod_{i=1}^m f_i ∈ L^p_{α_1}(E)$, and

$$\int_E \prod_{i=1}^m f_i(x)\omega(x) dμ(x) \leq \prod_{i=1}^m ||f_i||_{α_i, ω} \cdot \int_E \prod_{i=1}^m \left( f_i(x) \right)\alpha_i\omega(x) dμ(x) \cdot \frac{1}{α_i},$$

(58)

with equality if and only if there exist positive constants $\lambda_1, \cdots, \lambda_m$ such that $\lambda_1 f_1(x) \alpha_1 = \cdots = \lambda_m f_m(x) \alpha_m$ for $x ∈ E$.

If $0 < α_1 < 1$ and $α_2 < 0, \cdots, α_m < 0$, then inequality (58) is reverse (the conditions of the reverse inequality of (58) is given by the author of this article).

For $p > 1$, the reverse Hölder’s inequality, together with (41), (40) and (21), yields

$$V_p(K_1, \cdots, K_n) \geq \frac{1}{n} \int_{S^n-1} h(K_n, u) dS_p(K_1, \cdots, K_{n-1}; u) \frac{1}{1-p} \int_{S^n-1} h(K_n, u)^p dS(K_1, \cdots, K_{n-1}; u) \frac{1}{p} \times V(K_1, \cdots, K_{n-1}, K_j)^\frac{1-p}{p}.$$ 

(60)

Using the Aleksandrov-Fenchel inequality (1), we have

$$V_p(K_1, \cdots, K_n) \geq \prod_{i=1}^n V(K_j[r], K_{r+1}, \cdots, K_n)^\frac{r}{r-p} \times \prod_{j=1}^{n-1} V(K_1, \cdots, K_{n-1}, K_j)^{1-p}.$$ 

Similarly, we can prove the reverse inequality.

Taking $r = n - 1$ in (9), we obtain

**Corollary 19.** If $p ≥ 1, K_i ∈ K^n_0 (i = 1, \cdots, n)$, then

$$V_p(K_1, \cdots, K_n)^{n-1} ≥ \prod_{j=1}^{n-1} V(K_j[n-1], K_n)^p \times V(K_1, \cdots, K_{n-1}, K_j)^{1-p}. \quad (59)$$

Taking $r = n$ in (9), we obtain

**Corollary 20.** If $p ≥ 1, K_i ∈ K^n_0 (i = 1, \cdots, n)$, then

$$V_p(K_1, \cdots, K_n)^n ≥ \prod_{i=1}^n V(K_i)^p \times \prod_{j=1}^{n-1} V(K_1, \cdots, K_{n-1}, K_j)^{(1-p)p/n}. \quad (60)$$

**Remark 21.** In particular, when $p = 1$ in (60), the result has proved by Lutwak [57]: If $K_i ∈ K^n_0 (i = 1, \cdots, n)$, then

$$V(K_1, \cdots, K_n)^n ≥ V(K_1) \cdots V(K_n), \quad (61)$$

with equality if and only if $K_1, \cdots, K_n$ are homothetic.

Using the same argument as in Theorem 1, we immediately get the following theorem.

**Theorem 22.** If $p ≥ 1, K_1, \cdots, K_{n-1}, K_n (i = 0, 1, \cdots, n - 1), K_n ∈ K^n_0$, then

$$W_{p,i}(K_1, \cdots, K_{n-1}, K_n)^i \geq \prod_{j=1}^{n-1} W_i(K_j[r], K_{r+1}, \cdots, K_{n-1}, K_n)^p \times \prod_{j=1}^{n-1} W_i(K_1, \cdots, K_{n-1}, K_j)^{1-p}.$$ 

(62)

Taking $r = n - 1 - i$ in (62), we obtain that

**Corollary 23.** If $p ≥ 1, K_1, \cdots, K_{n-1}, K_n (i = 0, 1, \cdots, n - 1) ∈ K^n_0$, then

$$W_{p,i}(K_1, \cdots, K_{n-1}, K_n)^{n-1-i} \geq \prod_{j=1}^{n-1} W_i(K_j[n - 1 - i], K_n)^p \times W_i(K_1, \cdots, K_{n-1}, K_j)^{1-p}. \quad (63)$$

**Remark 24.** Taking $r = n - 1, K_1 = \cdots = K_{n-1} = K$ and $K_n = L$ in (62), we can get the Minkowski inequality proved by Lutwak [51]: If $K, L ∈ K^n_0, p ≥ 1$, then for $i = 0, 1, \cdots, n - 2$,

$$W_{p,i}(K, L)^{n-i} ≥ W_i(K)^{n-i-p} W_i(L)^p,$$ 

with equality for $p > 1$ if and only if $K$ and $L$ are dilates, for $p = 1$ if and only if $K$ and $L$ are homothetic.

B. The Aleksandrov-Fenchel inequality for generalized $L_p$-mixed projection of convex bodies

In this section, we prove the Aleksandrov-Fenchel inequality for $L_p$-mixed projection bodies of convex bodies stated in the beginning of this paper.

**Lemma 25.** If $p ≥ 1$ and $K_1, \cdots, K_{n-1} ∈ K^n_0$, for $i = 0, 1, \cdots, n - 1$, then

$$W_i(Π_p(K_1, \cdots, K_{n-1}))^{n-i} \geq n^{1-p} W_i(Π(K_1, \cdots, K_{n-1}))^{n-i} \times \prod_{j=1}^{n-1} V(K_1, \cdots, K_{n-1}, K_j)^{1-p}.$$ 

(65)
Proof. We only prove the first inequality with \( p \geq 1 \). From Definition 1, Definition 6, the reverse of Hölder’s inequality (58), (49) and (21), it follows that

\[
\begin{align*}
& h(\Pi_p(K_1, \ldots, K_n), u)^p \\
& = \frac{1}{p} \int_{S^{n-1}} |u \cdot v|^p (h_K(v1) \cdots h_{K_n}(v))^{\frac{1-p}{p}} \times dS(K_1, \ldots, K_n; v) \\
& \geq n^{1-p} \left( \frac{1}{n} \int_{S^{n-1}} h_K(v) dS(K_1, \ldots, K_n; v) \right)^p \\
& \times \left( \prod_{j=1}^{n-1} \left( \frac{1}{n} \int_{S^{n-1}} h_{K_j}(v) dS(K_1, \ldots, K_n; v) \right) \right)^{\frac{1-p}{p}} \\
& = n^{1-p} h(\Pi(K_1, \ldots, K_n), u)^p \\
& \times \prod_{j=1}^{n-1} V(K_1, \ldots, K_n, K_j)^{\frac{1-p}{n-1}}.
\end{align*}
\]

Namely,

\[
h(\Pi_p(K_1, \ldots, K_n), u)^p \\
\geq n^{1-p} h(\Pi(K_1, \ldots, K_n), u)^p \\
\times \prod_{j=1}^{n-1} V(K_1, \ldots, K_n, K_j)^{\frac{1-p}{n-1}}.
\]

For each \( Q \in \mathcal{K}_n \), integrating both sides of (66) for \( dS_p(Q, u) \) in \( u \in S^{n-1} \), and by (33), we obtain

\[
\begin{align*}
W_p(\Pi_p(K_1, \ldots, K_n)) \\
\geq n^{1-p} W_p(\Pi(K_1, \ldots, K_n)) \\
\times \prod_{j=1}^{n-1} V(K_1, \ldots, K_n, K_j)^{\frac{1-p}{n-1}}.
\end{align*}
\]

Taking \( Q = \Pi_p(K_1, \ldots, K_n) \), we have

\[
\begin{align*}
W_p(\Pi_p(K_1, \ldots, K_n)) \\
\geq n^{1-p} W_p(\Pi_p(K_1, \ldots, K_n), \Pi(K_1, \ldots, K_n)) \\
\times \prod_{j=1}^{n-1} V(K_1, \ldots, K_n, K_j)^{\frac{1-p}{n-1}}.
\end{align*}
\]

Using inequality (64) in (67), we have (65).

Taking \( i = 0 \) in (65) and using inequality (5), this has Theorem 2. Taking \( r = n-1 \) in (10), we obtain

Corollary 26. Let \( K_1, \ldots, K_{n-1} \in \mathcal{K}_n \). If \( p \geq 1 \), then

\[
\begin{align*}
V(\Pi_p(K_1, \ldots, K_n))^{\frac{1}{p}} & \geq n^{1-p} \prod_{j=1}^{n-1} V(\Pi K_j)^{\frac{1-p}{n-1}} \\
& \times \prod_{j=1}^{n-1} V(K_1, \ldots, K_{n-1}, K_j)^{\frac{1-p}{n-1}}.
\end{align*}
\]

When \( p = 1 \) in (10), we obtain the Brunn-Minkowski inequality (5) for mixed projection bodies established by Lutwak [45]. From (53), (46), the reverse of Hölder’s inequality (58), (31) and (6), the same argument can get

Theorem 27. If \( p \geq 1 \), \( K_1, \ldots, K_{n-1-i} \in \mathcal{K}_n \) (\( i = 0, 1, \ldots, n-2 \)), then

\[
\begin{align*}
V(\Pi_p(K_1, \ldots, K_{n-1-i}), u)^p \\
\geq n^{1-p} \prod_{j=1}^{n-1-i} V(\Pi(K_j)^{r}[r], K_{r+1}, \ldots, K_{n-1-i})^{\frac{1-p}{n-i}} \\
\times \prod_{j=1}^{n-i} W_i(K_1, \ldots, K_{n-1-i}, K_j)^{\frac{1-p}{n-1}}.
\end{align*}
\]

Now we established the generalized Aleksandrov-Fenchel inequality for the polar of \( L_p \)-mixed projection bodies.

Lemma 28. If \( p \geq 1 \), \( K_1, \ldots, K_{n-1} \in \mathcal{K}_n \) and \( i \in \mathbb{R} \), then

\[
\begin{align*}
W_i(\Pi_p(K_1, \ldots, K_{n-1})) \\
\leq n^{p}(n-1-i) V(\Pi(K_1, \ldots, K_{n-1}), u)^{\frac{n-i}{n-1}} \\
\times \prod_{j=1}^{n-i} V(K_1, \ldots, K_{n-1}, K_j)^{\frac{p(n-1-i)}{n-i}},
\end{align*}
\]

with equality if and only if \( K_j \) is the line segment \( \lambda_j[u] \), where \( \lambda_j > 0 \) \((j = 1, \ldots, n-1)\).

Proof. From (66) we have

\[
\begin{align*}
\rho(\Pi_p(K_1, \ldots, K_{n-1}), u)^{n-i} \\
\leq n^{p(p-1)}(n-1-i) \rho(\Pi(K_1, \ldots, K_{n-1}), u)^{n-i} \\
\times \prod_{j=1}^{n-i} V(K_1, \ldots, K_{n-1}, K_j)^{\frac{p(n-1-i)}{n-i}},
\end{align*}
\]

For each \( Q \in \mathcal{K}_n \), integrate both sides of (71) for \( u \in S^{n-1} \), and by (38) and the formula of the dual quermassintegrals (38), we immediately obtain (70).

From the condition of equality in Hölder inequality, we know that equality holds in inequality (70) if and only if \( K_j \) is the line segment \( \lambda_j[u] \), where \( \lambda_j > 0 \) \((j = 1, \ldots, n-1)\).

Taking \( i = 0 \) in (70) and using inequality (7), we can obtain Theorem 3. Taking \( r = n-2 \) in (11), we obtain

Corollary 29. If \( p \geq 1 \) and \( K_1, \ldots, K_{n-1} \in \mathcal{K}_n \), then

\[
\begin{align*}
V(\Pi_p(K_1, \ldots, K_n))^{\frac{1}{p}} \\
\leq n^{1-p} \prod_{j=1}^{n-1} V(\Pi K_j)^{\frac{1}{n-1}} \\
\times \prod_{j=1}^{n-1} V(K_1, \ldots, K_{n-1}, K_j)^{\frac{p(n-1)}{n-1}},
\end{align*}
\]

with equality if and only if \( K_j \) is the line segment \( \lambda_j[u] \), where \( \lambda_j > 0 \) \((i = 1, \ldots, n-1)\).

Taking \( p = 1 \) in (11), we obtain the Aleksandrov-Fenchel inequality (7) for the polars of mixed projection bodies established by Leng et al. [38].

From Definition 18, (46), the reverse of Hölder’s inequality (58), (38), (8) and (21), and similar to the proof of Theorem 3, we have

Theorem 30. If \( p \geq 1 \) and \( K_1, \ldots, K_{n-1-i} \in \mathcal{K}_n \) (\( i = 0, 1, \ldots, n-2 \)), then

\[
\begin{align*}
V(\Pi_p(K_1, \ldots, K_{n-1-i}), u)^p \\
\geq n^{1-p} \prod_{j=1}^{n-1-i} V(\Pi(K_j)^{r}[r], K_{r+1}, \ldots, K_{n-1-i})^{\frac{1-p}{n-i}} \\
\times \prod_{j=1}^{n-i} W_i(K_1, \ldots, K_{n-1-i}, K_j)^{\frac{1-p}{n-1}}.
\end{align*}
\]
Taking \( K_1 = \cdots = K_{n-i} = K \) and \( K_{n-i+1} = \cdots = K_n = L \) in (34), we write \( \tilde{V}_{p,i}(K, L) = \tilde{V}_{p,i}(K[n-i], L[i]) \). If \( i \) is any real, then \( \tilde{V}_{p,i}(K, L) \) is said the \( i \)th \( L_p \)-quasi dual mixed volume of \( K, L \in S^n_0 \) (see [93]), and

\[
\tilde{V}_{p,i}(K, L) = \omega(n) \left( \frac{1}{n \omega_n} \int_{S^{n-1}} \rho(K, u)^{p(n-i)} \rho(L, u)^p dS(u) \right)^{\frac{1}{p}}.
\]  

(79)

In (79), let \( L = B \) and we write \( \tilde{V}_{p,i}(K, B) = \tilde{V}_{p,i}(K) \). Thereby, for \( K \in S^n_0, i \in \mathbb{R} \), the \( i \)th \( L_p \)-quasi dual quermassintegrals, \( \tilde{V}_{p,i}(K) \), of \( K \) is defined by

\[
\tilde{V}_{p,i}(K) = \omega(n) \left( \frac{1}{n \omega_n} \int_{S^{n-1}} \rho(K, u)^{p(n-i)} dS(u) \right)^{\frac{1}{p}}.
\]  

(80)

Lutwak proved the result [54]: If \( K \in \mathbb{R}^n \), then \( B_{2n}(K) \leq V(K^n) \), with equality if and only if \( K \) is symmetric with respect to \( p \). We first give a generalization of this inequality.

**Lemma 31.** If \( K \in \mathbb{R}^n \), \( p > 0 \), \( i < n \), then

\[
B_{p,2n-i}(K) \leq \tilde{V}_{p,i}(K^*),
\]  

(81)

with equality if and only if \( K \) is symmetric with respect to the origin.

**Proof.** From (77) and (75), we have

\[
B_{p,2n-i}(K) = \frac{1}{n \omega_n} \left( \frac{1}{n \omega_n} \int_{S^{n-1}} b(K, u)^{p(n-i)} dS(u) \right)^{\frac{1}{p(n-i)}} \leq \omega(n) \left( \frac{1}{n \omega_n} \int_{S^{n-1}} b(K, u)^{p(n-i)} dS(u) \right)^{\frac{1}{p(n-i)}}
\]  

(77)

and the \( L_p \)-mixed width integrals of index \( i \) of \( K \) is defined by

\[
B_{p,i}(K) = \frac{1}{n \omega_n} \int_{S^{n-1}} b(K, u)^{p(n-i)} dS(u),
\]  

(76)

with equality if and only if \( K \). is centered.

From (82), we know that inequality (81) is proved.

**Lemma 32.** If \( K, L \in K^n_0 \), \( p > 0 \) and \( i < n \), then

\[
4\tilde{V}_{p,i}(K + L^*) \leq \tilde{V}_{p,i}(K^*) + \tilde{V}_{p,i}(L^*),
\]  

(83)

with equality if and only if \( K = L \).
Proof. For $K, L \in \mathbb{R}_n^n$, $K + L \in \mathbb{R}_n^n$, it follow from (77) and (81) that (83) is equivalent to the following inequality

$$4B_{p,2n-1}(K + L)^{\frac{1}{p(n-1)}} \leq B_{p,2n-1}(K)^{\frac{1}{p(n-1)}} + B_{p,2n-1}(L)^{\frac{1}{p(n-1)}}.$$ 

In fact, from Minkowski inequality, (77) and (75), we have

$$\frac{1}{nomega_n} \int_{S^{n-1}} \left( \frac{1}{b(K,u)} \right)^{p(n-1)} \omega_d(u) \frac{1}{p(n-1)} \times dS(u) \leq \omega_n \frac{1}{nomega_n} \int_{S^{n-1}} \left( \frac{1}{b(K,u)} + \frac{1}{b(L,u)} \right)^{p(n-1)} \times dS(u) \geq 4\frac{1}{nomega_n} \int_{S^{n-1}} \left( \frac{1}{b(K,u)} + \frac{1}{b(L,u)} \right)^{p(n-1)} \times dS(u) = 4B_{p,2n-1}(K + L)^{\frac{1}{p(n-1)},}$$

with equality if and only if $b(K,u) = b(L,u), u \in S^{n-1}$. Since $K$ and $L$ are centered, it follows that $K = L$.

Noting that $\Pi(K), C$ and $\Pi(L), C$ are centered, from (57) and (83), we infer Theorem 4.

D. A generalization of Pythagorean inequality for mixed volumes

Pythagorean inequalities were given by Firey [24]:

$$V(K_1, \cdots, K_{n-1}, [e])^2 \leq \sum_{i=1}^{n} V(K_1, \cdots, K_{n-1}, [e_i])^2,$$ (84)

where $\{e_1, \cdots, e_n\}$ is an orthogonal basis in $\mathbb{R}^n$ and $e$ is an arbitrary unit vector. Now, we generalize inequality (84) to John basis. Namely, we complete the proof of Theorem 5 stated in the beginning of this paper.

Proof of Theorem 5. From the support function of $\Pi_p(K_1, \cdots, K_{n-1})$ and (51), we get

$$h(u)^p = nV_p(K_1, \cdots, K_{n-1}, [u]).$$

Together with [3]

$$u = \sum_{i=1}^{m} c_i (u \cdot u_i) u_i,$$

we have

$$nV_p(K_1, \cdots, K_{n-1}, [u]) = h \left( \sum_{i=1}^{m} c_i (u \cdot u_i) u_i \right)^p = \left( \sum_{i=1}^{m} c_i h(u \cdot u_i) \right)^p = \left( \sum_{i=1}^{m} c_i |u \cdot u_i| b(\text{sgn}(u \cdot u_i) u_i) \right)^p = \left( n \sum_{i=1}^{m} c_i |u \cdot u_i| V_p(K_1, \cdots, K_{n-1}, [u_i]) \right)^p.$$ (85)

Together Cauchy inequality with $||x||^2 = \sum_{i=1}^{m} c_i |x \cdot u_i|^2$, we have

$$\sum_{i=1}^{m} c_i |u \cdot u_i| V_p(K_1, \cdots, K_{n-1}, [u_i]) \frac{1}{p} \leq \left( \sum_{i=1}^{m} c_i |u \cdot u_i|^2 \right)^{\frac{1}{p}} \times \left( \sum_{i=1}^{m} c_i V_p(K_1, \cdots, K_{n-1}, [u_i]) \frac{1}{p} \right)^{\frac{1}{p}}.$$ (86)

From (85) and (86), we have

$$nV_p(K_1, \cdots, K_{n-1}, [u]) \leq n \left( \sum_{i=1}^{m} c_i |u \cdot u_i|^2 \right)^{\frac{1}{p}} \times \left( \sum_{i=1}^{m} c_i V_p(K_1, \cdots, K_{n-1}, [u_i]) \right)^{\frac{1}{p}}.$$ (87)

From (87), the proof of inequality (14) is completed.

Remark 33. The inequality in (13) implies $\sum_{i=1}^{m} c_i = n$. Clearly, the sequence $\{a_1, \cdots, a_m\}$ is just like a standard orthogonal basis such that for any $x \in \mathbb{R}^n$,

$$||x||^2 = \sum_{i=1}^{m} c_i |a_i \cdot x|^2.$$ (88)

Moreover, let $c_i = \frac{n}{m}$. Then $\{a_1, \cdots, a_m\}$ is called star-coordinates by Kawashima [36]. It is also easy to prove that its inertial ellipsoid is a ball (see [90], [92]).

Taking $p = 1$ in Theorem 5, then inequality (14) reduces to Leng’s result [38]. Taking $K_1 = \cdots = K_{n-1} = K$, $K_{n-r} = \cdots = K_{n-1} = B$ in (14), it follows that

Corollary 34. If $p > 0, K \in \mathbb{R}_0^n$, then

$$w_p(K_u)^{\frac{1}{p}} \leq \sum_{i=1}^{m} c_i w_{p,i}(K_{u_i})^{\frac{1}{p}}.$$ (89)

In particular, let $r = 1$ to (89), we have Corollary 35. If $p > 0, K \in \mathbb{R}_0^n$, then

$$S_p(K_u)^{\frac{1}{p}} \leq \sum_{i=1}^{m} c_i S_p(K_{u_i})^{\frac{1}{p}}.$$ (90)

If $\{u_i\}_{i=1}^{m}$ is a standard orthogonal basis, then we can prove the generalized results of the obtained results by Firey [24].

E. Generalized Loomis-Whitney inequality

We require the following result on the zonotope. In fact, a zonotope is a Minkowski combination of line segments, and see [80]. A body in $\mathbb{R}^n$ being the limit (with respect to the Hausdorff metric) of zonotope is called a zonoid.
Lemma 36. Suppose that \( \{u_i\}_{i=1}^m \) is a sequence of unit vectors in \( \mathbb{R}^n \), and \( \{c_i\}_{i=1}^m \) is a sequence of positive numbers such that \( \sum_{i=1}^m c_i u_i \otimes u_i = I_n \). If \( \lambda_1, \ldots, \lambda_m \) are the sequence real numbers and \( Z = \sum_{i=1}^m \lambda_i |u_i| \), then

\[
V(Z) \geq \prod_{i=1}^m \left( \frac{\lambda_i}{c_i} \right).
\]

(91)

Proof of Theorem 6. Let \( \lambda_1, \ldots, \lambda_m \) are the sequence real numbers and \( Z = \sum_{i=1}^m \lambda_i |u_i| \). It follow from (52) and the property of mixed volume that

\[
\sum_{i=1}^m \lambda_i v_p(K_1^{u_i}, \ldots, K_{n-1}^{u_i}) \cdot \prod_{j=1}^{m-2} V(K_j, K_1, \ldots, K_{n-2}, [u_i]) \frac{p-1}{p(n-2)}
\geq n^{\frac{1}{p}} \sum_{i=1}^m \lambda_i V(K_1, \ldots, K_{n-1}, [u_i])
\geq n^{\frac{1}{p}} \prod_{i=1}^m \left( \frac{\lambda_i}{c_i} \right).
\]

(92)

Together (23), Minkowski inequality (3) with Lemma 8, we have

\[
V(K_1, K_2, \ldots, K_{n-1}, Z) = V([K_1, \ldots, K_{n-1}][n-1], Z)
\geq V([K_1, K_2, \ldots, K_{n-2}][n-2], [u_i]) \frac{n}{p} \prod_{j=1}^{m-2} V(Z) \frac{n}{p}
\geq V([K_1, \ldots, K_{n-1}]) \prod_{i=1}^m \left( \frac{\lambda_i}{c_i} \right).
\]

(93)

Together (92) with (93), we have

\[
\sum_{i=1}^m \lambda_i v_p(K_1^{u_i}, \ldots, K_{n-1}^{u_i}) \cdot \prod_{j=1}^{m-2} V(K_j, K_1, \ldots, K_{n-2}, [u_i]) \frac{p-1}{p(n-2)}
\geq n^{\frac{1}{p}} \prod_{i=1}^m \left( \frac{\lambda_i}{c_i} \right).
\]

(94)

Let

\[
A = v_p(K_1^{u_i}, \ldots, K_{n-1}^{u_i}) \cdot \prod_{j=1}^{m-2} V(K_j, K_1, \ldots, K_{n-2}, [u_i]) \frac{p-1}{p(n-2)},
\]

and note that \( \sum_{i=1}^m c_i = n \), we obtain Theorem 6.

Taking \( p = 1 \) in (15), we have

Corollary 37. Suppose that \( K_1, \ldots, K_{n-1} \in K^n, \{u_i\}_{i=1}^m \) is a sequence of unit vectors in \( \mathbb{R}^n \), and \( \{c_i\}_{i=1}^m \) is a sequence of positive numbers such that \( \sum_{i=1}^m c_i u_i \otimes u_i = I_n \). Then

\[
\prod_{i=1}^m v(K_1^{u_i}, \ldots, K_{n-1}^{u_i}) \geq V([K_1, \ldots, K_{n-1}]) \prod_{i=1}^m \left( \frac{\lambda_i}{c_i} \right).
\]

(95)

Inequality (95) is established by Si and Leng [84].

In particular, let \( K_1 = \cdots = K_{n-1} = K \), and note that \( [K_1, \ldots, K_{n-1}] = K \), then inequality (95) reduces to the following Ball’s Loomis-Whitney inequality for John basis [5].

Corollary 38. Suppose that \( K \in K^n, \{u_i\}_{i=1}^m \) is a sequence of unit vectors in \( \mathbb{R}^n \), and \( \{c_i\}_{i=1}^m \) is a sequence of positive numbers such that \( \sum_{i=1}^m c_i u_i \otimes u_i = I_n \). Then

\[
\prod_{i=1}^m c_i v(K_1^{u_i}, \ldots, K_{n-1}^{u_i}) \geq V(K)^{n-1}.
\]

(96)

REFERENCES


