Almost Periodic Solution, Local Asymptotical Stability and Uniform Persistence for A Harvesting Model of Plankton Allelopathy with Time Delays

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Abstract—**In this paper, we consider a class of delayed differential equation model of plankton allelopathy with harvesting terms. Firstly, the existence and uniqueness of positive almost periodic solution for the model are obtained by using the theory of exponential dichotomy and Banach fixed point theorem. Secondly, the local asymptotical stability of the model is studied. Finally, the uniform persistence of the above model is also considered. Examples with computer simulations are given to illustrate the feasibility and effectiveness of the main result.**

Index Terms—**Almost periodic solution; Biological model; Banach fixed point theorem; Local asymptotical stability; Uniform persistence.**

I. Introduction

RECENTLY, the effects of toxic substances have been incorporated into competitive systems, and many excellent results have been obtained (see [1-9] for example). Maynard Smith [1] introduced the effects of toxic substances into a two-species Lotka-Volterra competitive system by considering that each species produces a substance toxic to the other only when the other is present. The modified model takes the following form:

$$\begin{cases} \dot{x}_1(t) = x_1(t)\Big[r_1(t) \\ \qquad - \sum\limits_{j=1}^{2} a_{1j}(t)x_j(t) - b_1(t)x_1(t)x_2(t)\Big], \\ \dot{x}_2(t) = x_2(t)\Big[r_2(t) \\ \qquad - \sum\limits_{j=1}^{2} a_{2j}(t)x_j(t) - b_2(t)x_2(t)x_1(t)\Big], \end{cases} \tag{1.1}$$

where $x_i(t)$ denotes the population density of the ith species at time t for a common pool of resources. The terms $b_1x_1^2x_2$ and $b_2x_2^2x_1$ denote the effect of toxic substances.

In applications, if the various constituent components of the temporally nonuniform environment is with incommensurable (nonintegral multiples) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. If we consider the effects of the environmental factors, almost periodicity is sometimes more realistic and more general than periodicity. Therefore, more and more attention are paid to almost periodic dynamics behavior for nonlinear differential equations. For example, in recent years, the existence and uniqueness of almost periodic solution for some kinds of neural networks [10-19] have been widely studied by using contraction mapping principle. Unlike the neural networks, the solution of biological models is positive. So we could not study the existence and uniqueness of positive almost periodic solution for biological models as the same as the neural networks by using contraction mapping principle. In view of this, we introduce varying harvesting rate into a class of differential equation model of plankton allelopathy and investigate the existence and uniqueness of positive almost periodic solution by using contraction mapping principle.

In many earlier studies, it has been shown that harvesting has a strong impact on dynamic evolution of a population, e.g., see [20-24]. So the study of the population dynamics with harvesting is becoming a very important subject in mathematical bio-economics. The delayed model of plankton allelopathy with harvesting is generally described as :

$$\begin{cases} \dot{x}_1(t) = x_1(t)\Big[r_1(t) - \sum\limits_{j=1}^{2} a_{1j}(t)x_j(t-\alpha_j(t)) \\ \qquad -b_1(t)x_1(t-\mu_1(t))x_2(t-\nu_1(t))\Big] \\ \qquad -h_1(t), \\ \dot{x}_2(t) = x_2(t)\Big[r_2(t) - \sum\limits_{j=1}^{2} a_{2j}(t)x_j(t-\beta_j(t)) \\ \qquad -b_2(t)x_2(t-\nu_2(t))x_1(t-\mu_2(t))\Big] \\ \qquad -h_2(t), \end{cases} \tag{1.2}$$

where h_1 and h_2 represent harvesting terms. By using contraction mapping principle, the main purpose of this paper is to study the existence and uniqueness of positive almost periodic solution of system (1.2).

Let $C(\mathbb{X},\mathbb{Y})$ and $C^1(\mathbb{X},\mathbb{Y})$ be the space of continuous functions and continuously differential functions which map $\mathbb{X}$ into $\mathbb{Y}$, respectively. Especially, $C(\mathbb{X}) := C(\mathbb{X},\mathbb{X})$, $C^1(\mathbb{X}) := C^1(\mathbb{X},\mathbb{X})$. For any bounded function $f \in C(\mathbb{R})$, $f^+ = \sup_{s\in\mathbb{R}} f(s)$, $f^- = \inf_{s\in\mathbb{R}} f(s)$.

We list some assumptions which will be used in this paper.

(H_1) r_i, a_{ij}, b_i and h_i are nonnegative almost periodic functions with $0 < h_i^- < r_i^+$, $i,j = 1,2$.

(H_2) There exist positive constants $\eta_i \in \left[\frac{r_i^+h_i^+}{r_i^-}, \frac{(r_i^+)^2h_i^+}{r_i^-h_i^-}\right)$ $(i=1,2)$ such that

$$\sup_{s\in\mathbb{R}}\Big\{-r_i(s) + \sum_{j=1}^{2} 2a_{ij}(s) + 3b_i(s)\Big\} < -\eta_i < 0,$$

where $i = 1,2$.

The organization of this paper is as follows. In Section 2,

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we give some basic definitions and necessary lemmas which will be used in later sections. In Section 3, by using Banach fixed point theorem, we obtain some sufficient conditions ensuring existence and uniqueness of almost periodic solution of system (1.2). In Sections 4-5, the local asymptotical stability and uniform persistence of the model are considered. Finally, examples with computer simulations are given to illustrate that the result of this paper is feasible.

II. Preliminaries

Now, let us state the following definitions and lemmas, which will be useful in proving our main result.

Definition 1. ([25, 26]) $x \in C(\mathbb{R}, \mathbb{R}^n)$ is called almost periodic, if for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\tau = \tau(\epsilon)$ in this interval such that $\|x(t+\tau) - x(t)\| < \epsilon$, $\forall t \in \mathbb{R}$. The collection of those functions is denoted by $AP(\mathbb{R}, \mathbb{R}^n)$.

Definition 2. ([25, 26]) Let $y \in C(\mathbb{R}, \mathbb{R}^n)$ and $P(t)$ be a $n \times n$ continuous matrix defined on $\mathbb{R}$. The linear system

$$\dot{y}(t) = P(t)y(t)$$

is said to be an exponential dichotomy on $\mathbb{R}$ if there exist constants $k, \lambda > 0$, projection S and the fundamental matrix $Y(t)$ satisfying

$$\|Y(t)SY^{-1}(s)\| \le ke^{-\lambda(t-s)}, \quad \forall t \ge s,$$
$$\|Y(t)(I-S)Y^{-1}(s)\| \le ke^{-\lambda(s-t)}, \quad \forall t \le s.$$

Lemma 1. ([25, 26]) *If the linear system $\dot{y}(t) = P(t)y(t)$ has an exponential dichotomy, then almost periodic system*

$$\dot{y}(t) = P(t)y(t) + g(t)$$

has a unique almost periodic solution $y(t)$ which can be expressed as follows:

$$y(t) = \int_{-\infty}^{t} Y(t)SY^{-1}(s)g(s)\,\mathrm{d}s - \int_{t}^{\infty} Y(t)(I-S)Y^{-1}(s)g(s)\,\mathrm{d}s.$$

Lemma 2. ([26, 27]) *Let $a, b \in AP(\mathbb{R}, \mathbb{R})$. If*

$$M(a) = \lim_{T\to\infty} \frac{1}{T}\int_0^T a(s)\,\mathrm{d}s \ne 0,$$

then $\dot{y}(t) = a(t)y(t) + b(t)$ exists a unique almost periodic solution $y(t)$ can be written as follows

$$y(t) = \begin{cases} \int_{-\infty}^{t} e^{\int_s^t a(u)\,\mathrm{d}u} b(s)\,\mathrm{d}s, & m(a) < 0, \\ -\int_{t}^{+\infty} e^{\int_s^t a(u)\,\mathrm{d}u} b(s)\,\mathrm{d}s, & m(a) > 0. \end{cases}$$

Lemma 3. *([28]) Assume that $(\mathbb{B}, \rho)$ is a complete metric space, $T : (\mathbb{B}, \rho) \to (\mathbb{B}, \rho)$ is a contraction mapping, i.e., there exists $\lambda \in (0, 1)$, such that*

$$\rho(Tx, Ty) \le \lambda\rho(x, y), \ \forall x, y \in \mathbb{B}.$$

Then T has a unique fixed point in $\mathbb{B}$.

III. Almost periodic solution

In this section, we study the existence and uniqueness of almost periodic solution of system (1.2) by using Banach fixed point theorem.

Let

$$k_i := \frac{h_i^-}{r_i^+}, \quad l_i := \frac{r_i^+ h_i^+}{r_i^- \eta_i}, \quad i = 1, 2.$$

By (H_2), it is easy to see that $k_i < l_i \le 1$, $i = 1, 2$. Set

$$\mathbb{B} = \Big\{ x = (x_1, x_2)^T \in AP(\mathbb{R}, \mathbb{R}^2) : k_i \le x_i(t) \le l_i, \ \forall t \in \mathbb{R}, i = 1, 2 \Big\}$$

with the distance $\rho(x, y)$ from x to y is defined by

$$\rho(x, y) = \max_{1\le i\le 2}\{\sup_{t\in\mathbb{R}} |x_i(t) - y_i(t)|\},$$

where $x(t) = (x_1(t), x_2(t))^T$, $y(t) = (y_1(t), y_2(t))^T \in \mathbb{B}$. Obviously, $(\mathbb{B}, \rho)$ is a complete metric space.

Theorem 1. *Assume that (H_1)-(H_2) hold, then system (1.2) has a unique almost periodic solution in $\mathbb{B}$.*

Proof: For $\forall\varphi \in \mathbb{B}$, we consider the almost periodic solution of nonlinear almost periodic differential system

$$\begin{cases} \dot{x}_1(t) = r_1(t)x_1(t) - \varphi_1(t)\Big[\sum_{j=1}^{2} a_{1j}(t)\varphi_j(t-\alpha_j(t)) \\ \qquad +b_1(t)\varphi_1(t-\mu_1(t))\varphi_2(t-\nu_1(t))\Big] - h_1(t), \\ \dot{x}_2(t) = r_2(t)x_2(t) - \varphi_2(t)\Big[\sum_{j=1}^{2} a_{2j}(t)\varphi_j(t-\beta_j(t)) \\ \qquad +b_2(t)\varphi_2(t-\nu_2(t))\varphi_1(t-\mu_2(t))\Big] - h_2(t). \end{cases} \tag{3.1}$$

Notice that $M(r_i) > 0$, $i = 1, 2$. Thus, by Lemma 2.2, we obtain that the system (3.1) has exactly one almost periodic solution:

$$x^\varphi(t) = (x_1^\varphi(t), x_2^\varphi(t))^T,$$

where

$$x_1^\varphi(t) = \int_t^{+\infty} e^{-\int_t^s r_1(u)\,\mathrm{d}u} \Big\{\varphi_1(s)\Big[\sum_{j=1}^{2} a_{1j}(s)\varphi_j(s-\alpha_j(s)) +b_1(s)\varphi_1(s-\mu_1(s))\varphi_2(s-\nu_1(s))\Big] + h_1(s)\Big\}\,\mathrm{d}s,$$

$$x_2^\varphi(t) = \int_t^{+\infty} e^{-\int_t^s r_2(u)\,\mathrm{d}u} \Big\{\varphi_2(s)\Big[\sum_{j=1}^{2} a_{2j}(s)\varphi_j(s-\beta_j(s)) +b_2(s)\varphi_2(s-\nu_2(s))\varphi_1(s-\mu_2(s))\Big] + h_1(s)\Big\}\,\mathrm{d}s.$$

Now, we give a mapping T defined on $\mathbb{B}$ by setting

$$T(\varphi) = (T_1(\varphi), T_2(\varphi))^T = (x_1^\varphi, x_2^\varphi)^T, \quad \forall\varphi \in \mathbb{B}.$$

First, we prove that the mapping T is a self-mapping from $\mathbb{B}$ to $\mathbb{B}$. In fact, $\forall\varphi \in \mathbb{B}$, in view of definition of T, we have

$$T_1(\varphi)(t) \ge \int_t^{+\infty} e^{-\int_t^s r_1(u)\,\mathrm{d}u} h_1(s)\,\mathrm{d}s \ge k_1, \ \forall t \in \mathbb{R}.$$

Similarly, $T_2(\varphi)(t) \geq \frac{h_2^-}{r_2^+} = k_2$, $\forall t \in \mathbb{R}$. On the other hand, it follows that

$$\begin{aligned}
& T_1(\varphi)(t) \\
= & \int_t^{+\infty} e^{-\int_t^s r_1(u)\,\mathrm{d}u} \Big\{ \varphi_1(s) \Big[\sum_{j=1}^2 a_{1j}(s)\varphi_j(s-\alpha_j(s)) \\
& + b_1(s)\varphi_1(s-\mu_1(s))\varphi_2(s-\nu_1(s)) \Big] + h_1(s) \Big\} \,\mathrm{d}s \\
\leq & \int_t^{+\infty} e^{-\int_t^s r_1(u)\,\mathrm{d}u} \\
& \Big\{ l_1 \Big[\sum_{j=1}^2 a_{1j}(s) l_j + b_1(s) l_1 l_2 \Big] + h_1(s) \Big\} \,\mathrm{d}s \\
\leq & \int_t^{+\infty} e^{-\int_t^s r_1(u)\,\mathrm{d}u} \\
& \Big\{ l_1 \Big[\sum_{j=1}^2 a_{1j}(s) + b_1(s) \Big] + h_1(s) \Big\} \,\mathrm{d}s \\
\leq & \int_t^{+\infty} \Big[r_1(s) e^{-\int_t^s r_1(u)\,\mathrm{d}u} - \eta_1 e^{-r_1^+(s-t)} \Big] l_1 \,\mathrm{d}s + \frac{h_1^+}{r_1^-} \\
\leq & \ (1 - \frac{\eta_1}{r_1^+}) l_1 + \frac{h_1^+}{r_1^-} \\
= & \ l_1, \quad \forall t \in \mathbb{R}.
\end{aligned}$$

Similarly, $T_2(\varphi)(t) \leq l_2$, $\forall t \in \mathbb{R}$. So T is a self-mapping from $\mathbb{B}$ to $\mathbb{B}$.

Next, we show that $T : \mathbb{B} \to \mathbb{B}$ is a contraction mapping. In fact, for $\forall \varphi, \psi \in \mathbb{B}$, we have

$$\begin{aligned}
& |T_1(\varphi) - T_1(\psi)|_0 \\
:= & \sup_{t\in\mathbb{R}} |T_1(\varphi)(t) - T_1(\psi)(t)| \\
= & \sup_{t\in\mathbb{R}} \Big| \int_t^{+\infty} e^{-\int_t^s r_1(u)\,\mathrm{d}u} \\
& \Big\{ \varphi_1(s) \Big[\sum_{j=1}^2 a_{1j}(s)\varphi_j(s-\alpha_j(s)) \\
& + b_1(s)\varphi_1(s-\mu_1(s))\varphi_2(s-\nu_1(s)) \Big] + h_1(s) \Big\} \,\mathrm{d}s \\
& - \int_t^{+\infty} e^{-\int_t^s r_1(u)\,\mathrm{d}u} \\
& \Big\{ \psi_1(s) \Big[\sum_{j=1}^2 a_{1j}(s)\psi_j(s-\alpha_j(s)) \\
& + b_1(s)\psi_1(s-\mu_1(s))\psi_2(s-\nu_1(s)) \Big] + h_1(s) \Big\} \,\mathrm{d}s \Big| \\
\leq & \sup_{t\in\mathbb{R}} \int_t^{+\infty} e^{-\int_t^s r_1(u)\,\mathrm{d}u} \\
& \times \Big\{ \sum_{j=1}^2 a_{1j}(s) \Big| \varphi_1(s)\varphi_j(s-\alpha_j(s)) \\
& - \psi_1(s)\psi_j(s-\alpha_j(s)) \Big| \,\mathrm{d}s \\
& + b_1(s) \Big| \varphi_1(s)\varphi_1(s-\mu_1(s))\varphi_2(s-\nu_1(s)) \\
& - \psi_1(s)\psi_1(s-\mu_1(s))\psi_2(s-\nu_1(s)) \Big| \Big\} \,\mathrm{d}s \\
\leq & \sup_{t\in\mathbb{R}} \int_t^{+\infty} e^{-\int_t^s r_1(u)\,\mathrm{d}u} \\
& \Big\{ \sum_{j=1}^2 2a_{1j}(s) + 3b_1(s) \Big\} \,\mathrm{d}s \rho(\varphi, \psi) \\
\leq & \sup_{t\in\mathbb{R}} \int_t^{+\infty} e^{-\int_t^s r_1(u)\,\mathrm{d}u} [r_1(s) - \eta_1] \,\mathrm{d}s \rho(\varphi, \psi) \\
\leq & \sup_{t\in\mathbb{R}} \Big[\int_t^{+\infty} r_1(s) e^{-\int_t^s r_1(u)\,\mathrm{d}u} \,\mathrm{d}s \\
& - \int_t^{+\infty} e^{r_1^+(s-t)} \eta_1 \,\mathrm{d}s \Big] \rho(\varphi, \psi) \\
\leq & \ (1 - \frac{\eta_1}{r_1^+}) \rho(\varphi, \psi). \qquad (3.2)
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
|T_1(\varphi) - T_1(\psi)|_0 \ := & \ \sup_{t\in\mathbb{R}} |T_2(\varphi)(t) - T_2(\psi)(t)| \\
\leq & \ (1 - \frac{\eta_2}{r_2^+}) \rho(\varphi, \psi). \qquad (3.3)
\end{aligned}$$

It follows from (3.2)-(3.3) that

$$\begin{aligned}
\rho(T(\varphi), T(\psi)) \ \leq & \ \max\Big\{ 1 - \frac{\eta_1}{r_1^+}, 1 - \frac{\eta_2}{r_2^+} \Big\} \rho(\varphi, \psi) \\
= & \ \lambda \rho(\varphi, \psi),
\end{aligned}$$

where $\lambda = \max\Big\{ 1 - \frac{\eta_1}{r_1^+}, 1 - \frac{\eta_2}{r_2^+} \Big\} \in [0, 1)$, which implies that the mapping $T : \mathbb{B} \to \mathbb{B}$ is a contraction mapping. Therefore, the mapping T possesses a unique fixed point

$$x^* = (x_1^*, x_2^*)^T \in \mathbb{B}, \quad Tx^* = x^*.$$

So system (1.2) has a unique almost periodic solution. This completes the proof. ■

IV. Local asymptotical stability

In this section, we will construct some suitable Lyapunov functions to study the local asymptotical stability of system (1.2).

Theorem 2. *Assume that $\alpha_i = \beta_i = \mu_i = \nu_i = 0$, $i = 1, 2$,*

(H_3) $\Theta = r^- - A - B > 0$, where

$$r^- := \min_{1\leq i\leq 2} r_i^-,$$

$$A := \max_{1\leq i\leq 2} [a_{1i}^+ + a_{2i}^+],$$

$$B := (b_1^+ + b_2^+) \max_{1\leq i\leq 2} l_i.$$

Then system (1.2) is locally asymptotically stable.

Proof: Assume that $x(t) = (x_1(t), x_2(t))^T \in \mathbb{B}$ and $x^*(t) = (x_1^*(t), x_2^*(t))^T \in \mathbb{B}$ are any two solutions of system (1.2). In view of system (1.2), for $t \in \mathbb{R}^+$, we have

$$\begin{aligned}
& (x_1(t) - x_1^*(t))' \\
= & \ r_1(t) \big[x_1(t) - x_1^*(t) \big] \\
& - \sum_{j=1}^2 a_{1j}(t) \big[x_j(t) - x_j^*(t) \big] \\
& - b_1(t) x_2(t) \big[x_1(t) - x_1^*(t) \big] \\
& - b_1(t) x_1^*(t) \big[x_2(t) - x_2^*(t) \big],
\end{aligned}$$

$$\begin{aligned}&(x_2(t)-x_2^*(t))'\\ &= r_2(t)\big[x_2(t)-x_2^*(t)\big]\\ &\quad -\sum_{j=1}^{2} a_{2j}(t)\big[x_j(t)-x_j^*(t)\big]\\ &\quad -b_2(t)x_2(t)\big[x_1(t)-x_1^*(t)\big]\\ &\quad -b_2(t)x_1^*(t)\big[x_2(t)-x_2^*(t)\big].\end{aligned}$$

Let

$$V(t)=\sum_{i=1}^{2}|x_i(t)-x_i^*(t)|.$$

Hence we can obtain from (H_3) that

$$\begin{aligned}D^+V(t) &= D^+\sum_{i=1}^{2}|x_i(t)-x_i^*(t)|\\ &\geq \sum_{i=1}^{2} r_i^-|x_i(t)-x_i^*(t)|\\ &\quad -\sum_{j=1}^{n}[a_{1j}^+ + a_{2j}^+]|x_j(t)-x_j^*(t)|\\ &\quad -(b_1^+ + b_2^+)\max_{1\leq i\leq 2} l_i\sum_{j=1}^{n}|x_j(t)-x_j^*(t)|\\ &\geq (r^- - A - B)V(t) = \Theta V(t).\end{aligned}$$

Integrating the last inequality from T_0 to t leads to

$$V(T_0)+\Theta\sum_{i=1}^{2}\int_{T_0}^{t}|x_i(s)-x_i^*(s)|\,\Delta s\leq V(t)<+\infty,$$

that is,

$$\sum_{i=1}^{2}\int_{T_0}^{+\infty}|x_i(s)-x_i^*(s)|\,\Delta s<+\infty,$$

which implies that

$$\sum_{i=1}^{2}\lim_{s\to+\infty}|x_i(s)-x_i^*(s)|=0.$$

Thus, system (1.2) is locally asymptotically stable. This completes the proof. ■

Theorem 3. *Assume that (H_1)-(H_3) hold. Then the unique almost periodic solution of system* (1.2) *is locally asymptotically stable.*

V. Uniform persistence

Our object in this section is to prove the uniform persistence of system (1.2).

Theorem 4. *Assume that $h_i(t)=\alpha_1(t)=\beta_1(t)\equiv 0$ in system* (1.2)*, $\forall t\in\mathbb{R}$, $i=1,2$. Suppose further that*

(H_3) $r_1^- > a_{12}^+M_2 + b_1^+M_1M_2$, $r_2^- > a_{21}^+M_1 + b_2^+M_1M_2$.

Then for any positive solution $(x_1,x_2)^T$ of system (1.2) *satisfies*

$$N_i\leq x_i(t)\leq M_i,\quad i=1,2,$$

where N_i and M_i are defined as those in (5.2)-(4.5)*, respectively, $i=1,2$. That is, system* (1.2) *is uniformly persistent.*

Proof: By the first equation of system (1.2) that

$$\dot{x}_1(t)\leq x_1(t)\left[r_1^+ - a_{11}^- x_1(t)\right]. \tag{5.1}$$

By Lemmas 2.3 and 2.4 in [29], we have from (5.1) that

$$x_1(t)\leq\frac{r_1^+}{a_{11}^-}:=M_1. \tag{5.2}$$

Further, from the second equation of system (1.2) that

$$\dot{x}_2(t)\leq x_2(t)\left[r_2^+ - a_{22}^- x_2(t)\right].$$

By Lemmas 2.3 and 2.4 in [29], we have

$$x_2(t)\leq\frac{r_2^+}{a_{22}^-}:=M_2. \tag{5.3}$$

In view of the first equation of system (1.2), it follows that

$$\dot{x}_1(t)\geq x_1(t)\Big[r_1^- - a_{12}^+M_2 - b_1^+M_1M_2 - a_{11}^+x_1(t)\Big],$$

which implies that

$$x_1(t)\geq\frac{r_1^- - a_{12}^+M_2 - b_1^+M_1M_2}{a_{11}^+}:=N_1. \tag{5.4}$$

Similar to the argument as that in (5.4), we obtain from the second equation of system (1.2) that

$$x_2(t)\geq\frac{r_2^- - a_{21}^+M_1 - b_2^+M_1M_2}{a_{22}^+}:=N_2. \tag{5.5}$$

The proof is completed. ■

VI. An example and numerical simulations

Example 1. Consider the following differential equation model of plankton allelopathy with harvesting terms:

$$\begin{cases}\dot{x}_1(t) = x_1(t)\Big[1-\sum\limits_{j=1}^{2}a_{1j}(t)x_j(t-1)\\ \qquad\qquad -b_1(t)x_1(t)x_2(t-\sin^2(\sqrt{2}t))\Big]-0.1,\\ \dot{x}_2(t) = x_2(t)\Big[1-\sum\limits_{j=1}^{2}a_{2j}(t)x_j(t-1)\\ \qquad\qquad -b_2(t)x_2(t)x_1(t-\cos^2(\sqrt{3}t))\Big]-0.1,\end{cases} \tag{6.1}$$

where $b_1(s)=b_2(s)=0.1\sin^2(\sqrt{3}s)$ and

$$\begin{pmatrix}a_{11}(s) & a_{12}(s)\\ a_{21}(s) & a_{22}(s)\end{pmatrix}=0.1\begin{pmatrix}\sin^2(\sqrt{2}s) & \cos^2(\sqrt{3}s)\\ \cos^2(\sqrt{5}s) & \cos^2(\sqrt{7}s)\end{pmatrix},$$

$\forall s\in\mathbb{R}$. Then system (6.1) has a unique almost periodic solution.

Proof: Corresponding to system (1.2), $a_{ij}^+=0.1$, $b_i^+=0.1$, $r_i^-=1$ and $h_i^+=h_i^-=0.1$, $i,j=1,2$. Taking $\eta_1=\eta_2=0.2$. By a easy calculation, we obtain

$$\sup_{s\in\mathbb{R}}\Big\{-r_i(s)+\sum_{j=1}^{2}2a_{ij}(s)+3b_i(s)\Big\}<-0.3<-\eta_i<0,$$

where $i=1,2$, which implies that (H_2) in Theorem 1 holds. It is easy to verify that (H_1) in Theorem 1 is satisfied and the result follows from Theorem 1 (see Figures 1-3). This completes the proof. ■

Example 2. Consider system (5.1), corresponding to system (1.2), $l_i = 0.5$, $r^- = 1$, $A = 0.4$, $B = 0.1$ $i = 1, 2$. Therefore, $\Theta = 1 - 0.4 - 0.1 = 0.5 > 0$, which implies that (H_3) in Theorem 2 holds. By Theorem 2, system (6.1) is locally asymptotically stable (see Figures 4-5). Observing Figures 1-5, system (6.1) is uniformly persistent.

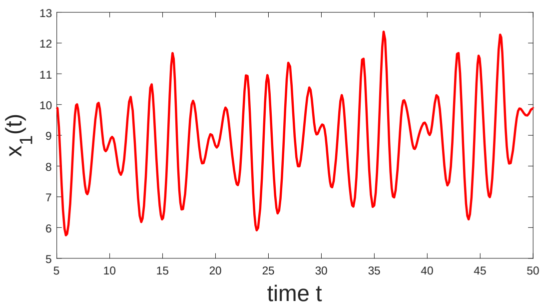

Fig. 1 Almost periodic oscillation of state variable x_1 system (6.1)

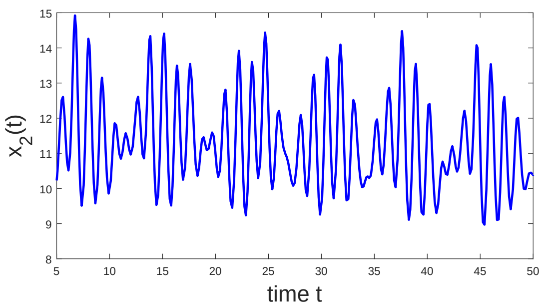

Fig. 2 Almost periodic oscillation of state variable x_2 system (6.1)

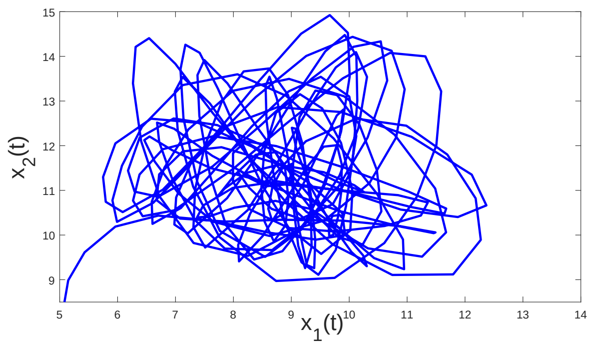

Fig. 3 Phase response of state variables x_1, x_2 of system (6.1)

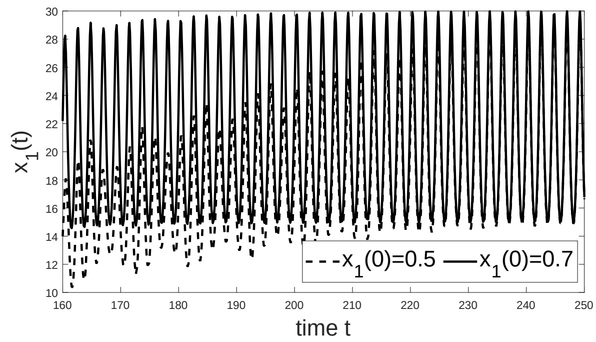

Fig. 4 Uniform asymptotical stability of state variable x_1 of system (6.1)

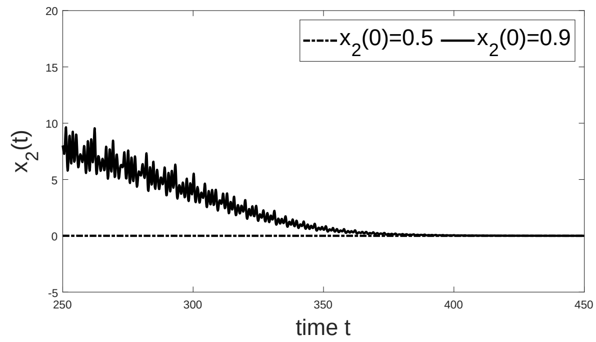

Fig. 5 Uniform asymptotical stability of state variable x_2 of system (6.1)

VII. Conclusion

In this paper, some sufficient conditions are established for the existence, uniqueness and local asymptotical stability of almost periodic solution for a harvesting model of plankton allelopathy with time delays. Further, the uniform persistence of the above model is also considered. The method used in this paper may be used to study many other ecological models.

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