Strong Convergence Theorem for the Split Equality Fixed Point Problem for Quasi-nonexpansive Mapping and Application

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Abstract—Motivated by the work of Zhao [9], [10], [11] and by reducing some of his conditions, we consider a split equality fixed point problem for quasi-nonexpansive mappings which includes split feasibility problem, split equality problem, split fixed point problem, etc. The strong convergence theorem of the proposed iterative scheme could be obtained, under some control conditions. Furthermore, we use S-mapping applied to our main result to prove strong convergence theorems.

Index Terms—Split equality fixed point problem, Split equality problem, Split feasibility problem, Fixed Point problem, Quasi-nonexpansive mapping.

I. Introduction

Let C and Q be the non-empty closed convex subsets of the Hilbert spaces H_1 and H_2 respectively, and $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem (SFP) is formulated as finding a point x^* with the property

$$x^* \in C \quad \text{and} \quad Ax^* \in Q. \tag{1}$$

The SFP in finite-dimensional spaces was firstly introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and medical image reconstruction [6]. The SFP has drawn attention from many researchers due to its applications in many branches of engineering and medical sciences. Many iterative algorithms have been suggested, ([7], [8], [12], [16], etc).

Assuming that SFP (1) is consistent (that is, (1) has a solution), it is easy to see that $x^* \in C$ is a solution of (1) if and only if it solves the following fixed point equation

$$x^* = P_C(I - \gamma A^*(I - P_Q))x^*, \tag{2}$$

where P_C and P_Q are the metric projections from H_1 onto C and from H_2 onto Q respectively, γ is a positive constant and A^* denotes by adjoint of A.

The popular algorithm used in approximating the solution of the SFP (1) is the CQ-algorithm, which was firstly proposed by Byrne [6]:

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q))x_n, \tag{3}$$

for all $n \in \mathbf{N}$, where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of the operator A^*A.

Recently, Moudafi [8] introduced the following split equality feasibility problem (SEFP) to find x^* and y^* with the property

$$x^* \in C, y^* \in Q \quad \text{s.t.} \quad Ax^* = By^*, \tag{4}$$

where H_1, H_2 and H_3 be real Hilbert spaces. $C \subset H_1$, $Q \subset H_2$ be two non-empty closed convex sets, $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded linear operators.
It is easy to see that the problem (4) could be reduced to the problem (1) where $H_3 \equiv H_2$ and $B \equiv I$ (I be the identity mappings on $H_2 \to H_2$).

In order to solve SEFP (4), Moudafi [8] introduced the following simultaneous iterative method:

$$\begin{cases} x_{n+1} = P_C(x_n - \gamma A^*(Ax_n - By_n)), \\ y_{n+1} = P_C(y_n + \beta B^*(Ax_{n+1} - By_n)), \quad \forall n \geq 0. \end{cases}$$

Under suitable conditions, he proved the weak convergence of sequence $\{(x_n, y_n)\}$ to a solution of (4) in Hilbert spaces.

Zhao [9] introduced the following algorithm for solving problem (4):

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \beta_n u_n + (1 - \beta_n)Su_n, \\ w_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = \beta_n w_n + (1 - \beta_n)Tw_n, \quad \forall n \geq 0, \end{cases}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators. Let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be quasi-nonexpansive mappings, A^* and B^* are the adjoints of A and B respectively, $\{\gamma_n\} \in (\varepsilon, \frac{2\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2} - \varepsilon)$ (for ε small enough). Under some conditions, the authors obtained the sequence $\{(x_n, y_n)\}$ converged weakly to (x^*, y^*) in (4).

Dong and He [10] introduced following projection algorithm for SEFP (4) where the stepsizes do not depend on the operator norms $\|A\|$ and $\|B\|$:

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = P_C u_n, \\ w_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = P_Q w_n, \quad \forall n \geq 0. \end{cases}$$

Subsequently, Moudafi [7] introduced the following split equality fixed point problem (SEFPP); let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be non-linear operators such that

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$F(U) \neq \emptyset$ and $F(T) \neq \emptyset$, where $F(U)$ and $F(T)$ denote the sets of fixed point of U and T respectively. In (4), if $C := F(U)$ and $Q := F(T)$, then SEFP (4) could be reduced to the SEFPP, to find x^* and y^* with the property

$$x^* \in F(U), y^* \in F(T) \quad \text{s.t.} \quad Ax^* = By^*, \tag{5}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators, which allows asymmetric and partial relations between x and y. This can further be used to cover many situations, such as decomposition methods for PDEs, applications in the game theory, in intensity-modulated radiation therapy(see [17]).

Very recently, Che and Li [11] proposed the following iterative algorithm for finding a solution of SEFPP (5):

$$\begin{cases} u_n = x_n - \gamma_n A^* \left(Ax_n - By_n\right), \\ x_{n+1} = \beta_n x_n + (1-\beta_n) T u_n, \\ v_n = y_n + \gamma_n B^* \left(Ax_n - By_n\right), \\ y_{n+1} = \beta_n y_n + \left(1-\beta_n\right) S w_n, \quad \forall n \geq 0, \end{cases} \tag{6}$$

and under suitable conditions, they also established the weak convergence of the scheme (6).

In this work, we established the following iterative algorithm to solve the split equality fixed point problem (SEFPP),

$$\begin{cases} u_n = x_n - \gamma_n A^* \left(Ax_n - By_n\right), \\ x_{n+1} = \alpha_n u + \left(1-\alpha_n\right) P_{C_1} \left(I - \lambda_n^1 (I - T_1)\right) u_n, \\ v_n = y_n + \gamma_n B^* \left(Ax_n - By_n\right), \\ y_{n+1} = \alpha_n v + \left(1-\alpha_n\right) P_{C_2} \left(I - \lambda_n^2 (I - T_2)\right) v_n, \end{cases}$$

where $T_1 : C_1 \to C_1$, $T_2 : C_2 \to C_2$ are two quasi-nonexpansive mappings. Under suitable conditions, we proved strong convergence theorems of the iterative scheme (10) to a solution of the split equality fixed point problem (5) in the real Hilbert spaces.

II. Preliminaries

Throughout this paper, we always assume that H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. Let C be a non-empty closed convex subset of H. Recall that a mapping T of C into itself is called quasi-nonexpansive if

$$\|Tx - y^*\| \leq \|x - y^*\|,$$

for all $x \in C$ and $y^* \in F(T)$. The set of all elements of fixed point of a mapping T is denoted by $F(T) = \big\{x \in C : Tx = x\big\}$. Goebel and Kirk [5] showed that $F(T)$ is closed and convex. For $\lambda \in [0, 1]$,

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda \|x\|^2 + (1-\lambda) \|y\|^2 - \lambda(1-\lambda) \|x - y\|^2$$

and

$$\|x + y\|^2 = \|x\|^2 + 2 \langle x, y\rangle + \|y\|^2$$

for all $x, y \in H$. Let P_C be the metric projection of H onto C i.e.,for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

Remark 2.1: It is well-known that metric projection P_C has the following properties:

1) P_C is firmly nonexpansive, i.e.,
$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \ \forall x, y \in H.$$
2) For each $x \in H$,
$$z = P_C(x) \Leftrightarrow \langle x - z, z - y\rangle \geq 0, \ \forall y \in C.$$

Lemma 2.2: [4] Let H be a real Hilbert space. Then

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Lemma 2.3: [2] Let $\{\mathcal{Q}_n\} \subset [0, +\infty], \{\upsilon_n\} \subset [0, 1]$ and $\{\eta_n\}$ be three real number sequences. Suppose that $\{\mathcal{Q}_n\}, \{\upsilon_n\}$ and $\{\eta_n\}$ satisfy the following three conditions:

(i) $\mathcal{Q}_{n+1} \leq (1 - \upsilon_n) \mathcal{Q}_n + \eta_n \upsilon_n$,

(ii) $\displaystyle\sum_{n=1}^{\infty} \upsilon_n = \infty$,

(iii) $\displaystyle\limsup_{n\to\infty} \eta_n \leq 0$ or $\displaystyle\sum_{n=1}^{\infty} |\eta_n \upsilon_n| < \infty$.

Then, $\displaystyle\lim_{n\to\infty} \mathcal{Q}_n = 0$.

Lemma 2.4: [4] Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$,

$$u = P_C(I - \lambda A) u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Lemma 2.5: Let C be a non-empty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Then $VI(C, I - T) = F(T)$.

Proof: It is easy to see that $F(T) \subseteq VI(C, I - T)$. Let $u \in VI(C, I - T)$, then we have

$$\langle v - u, (I - T)u\rangle \geq 0, \ \forall v \in C. \tag{7}$$

Let $v^* \in F(T)$, then we have

$$\|Tu - v^*\|^2 \leq \|u - v^*\|^2. \tag{8}$$

On the other hand

$$\begin{aligned} &\|Tu - v^*\|^2 \\ &= \|(u - v^*) - (I - T)u\|^2 \\ &= \|u - v^*\|^2 - 2\langle u - v^*, (I - T)u\rangle + \|(I - T)u\|^2. \end{aligned} \tag{9}$$

From (8) and (9), we have

$$\|u - v^*\|^2 - 2\langle u - v^*, (I - T)u\rangle + \|(I - T)u\|^2 \leq \|u - v^*\|^2.$$

From (7), we have

$$\|(I - T)u\|^2 \leq 2\langle u - v^*, (I - T)u\rangle.$$

It follows that $v^* \in F(T)$. Hence $VI(C, I - T) \subseteq F(T)$. ■

Remark 2.6: From Lemma 2.4 and 2.5, we have

$$F(T) = VI(C, I - T) = F(P_C(I - \lambda(I - T))),$$

for all $\lambda > 0$.

Lemma 2.7: [3] Let $\{t_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $t_{n_i} < t_{n_i+1}$ for all $i \in \mathbf{N}$. Then there exists a nondecreasing sequence $\{\tau(n)\} \subset \mathbf{N}$ such that $\tau(n) \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $n \in \mathbf{N}$;

$$t_{\tau(n)} \leq t_{\tau(n)+1} \quad , t_n \leq t_{\tau(n)+1}.$$

In fact

$$\tau(n) = \max\{k \leq n : t_k < t_{k+1}\}.$$

III. Main result

Theorem 3.1: For every $i = 1, 2, 3$, let H_i be a real Hilbert space and let C_1, C_2 be non-empty closed convex subset of H_1 and H_2, respectively. Let $T_i : C_i \to C_i$ be quasi-nonexpansive mapping for all $i = 1, 2$ and let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operator with adjoints A^* and B^*, respectively. Suppose that $\Omega = \{(x, y) \in C_1 \times C_2 \mid x \in F(T_1), y \in F(T_2) \text{ and } Ax = By\}$ is a non-empty set and let $\{x_n\}$, $\{y_n\}$ be sequences generated by $u, x_1 \in C_1; v, y_1 \in C_2$ and

$$\begin{cases} u_n = x_n - \gamma_n A^* \left(Ax_n - By_n\right), \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) P_{C_1} \left(I - \lambda_n^1 (I - T_1)\right) u_n, \\ v_n = y_n + \gamma_n B^* \left(Ax_n - By_n\right), \\ y_{n+1} = \alpha_n v + (1 - \alpha_n) P_{C_2} \left(I - \lambda_n^2 (I - T_2)\right) v_n, \end{cases} \tag{10}$$

for all $n \geq 1$,where $\{\alpha_n\} \subset [0, 1]$ with $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \lambda_n^i < \infty$ and $\lambda_n^i \in (0, 1)$ for all $i = 1, 2$ and $\gamma_n \in (a, b) \subset \left(\epsilon, \frac{2}{\lambda_A + \lambda_B} - \epsilon\right)$ for all $n \in \mathbf{N}$ and λ_A, λ_B are spectral radius of A^*A, B^*B respectively, ϵ is a small enough. Then the sequence $\{(x_n, y_n)\}$ converge strongly to $(\bar{x^*}, \bar{y^*}) \in \Omega$, where $\bar{x^*} = P_{F(T_1)} u$ and $\bar{y^*} = P_{F(T_2)} v$.

Proof: Let $(x^*, y^*) \in \Omega$, then $x^* \in F(T_1)$, $y^* \in F(T_2)$ and $Ax^* = By^*$. From Lemma 2.5, we have

$$\|A^1 x\|^2 \leq 2 < x - x^*, A^1 x >, \tag{11}$$

where $A^1 = I - T_1$ and for all $x \in C_1$. Similarly, we have

$$\|A^2 y\|^2 \leq 2 < y - y^*, A^2 y >, \tag{12}$$

where $A^2 = I - T_2$ and for all $y \in C_2$.
Since $x^* \in F(T_1), y^* \in F(T_2)$ and $Ax^* = By^*$.
By Remark 2.6, we have $x^* \in F\left(P_{C_1}\left(I - \lambda_n^1 A^1\right)\right)$ and $y^* \in F\left(P_{C_2}\left(I - \lambda_n^2 A^2\right)\right)$.
Since P_{C_1} is a nonexpansive mapping, we have

$$\begin{aligned} &\|P_{C_1}\left(I - \lambda_n^1 A^1\right) x - x^*\|^2 \\ =&\|P_{C_1}\left(I - \lambda_n^1 A^1\right) x - P_{C_1}\left(I - \lambda_n^1 A^1\right) x^*\|^2 \\ \leq&\|x - x^* - \lambda_n^1 (A^1 x - A^1 x^*)\|^2 \\ \leq&\|x - x^* - \lambda_n^1 A^1 x\|^2 \\ =&\|x - x^*\|^2 + (\lambda_n^1)^2 \|A^1 x\|^2 - 2\lambda_n^1 \langle x - x^*, A^1 x\rangle \\ \leq&\|x - x^*\|^2 + (\lambda_n^1)^2 \|A^1 x\|^2 - \lambda_n^1 \|A^1 x\|^2 \\ =&\|x - x^*\|^2 - (\lambda_n^1)(1 - \lambda_n^1)\|A^1 x\|^2 \\ \leq&\|x - x^*\|^2, \end{aligned}$$

for all $x \in C_1$. Similarly, we obtain

$$\|P_{C_2}\left(I - \lambda_n^2 A^2\right) y - y^*\|^2 \leq \|y - y^*\|^2,$$

for all $y \in C_2$.
From definition of $\{u_n\}$, we have

$$\begin{aligned} \|u_n - x^*\|^2 &= \|x_n - x^* - \gamma_n A^* \left(Ax_n - By_n\right)\|^2 \\ &= \|x_n - x^*\|^2 + \gamma_n^2 \|A^* \left(Ax_n - By_n\right)\|^2 \\ &\quad - 2\gamma_n \langle x_n - x^*, A^* \left(Ax_n - By_n\right)\rangle. \end{aligned} \tag{13}$$

Consider that

$$\begin{aligned} &\|A^* \left(Ax_n - By_n\right)\|^2 \\ =&\langle A^* \left(Ax_n - By_n\right), A^* \left(Ax_n - By_n\right)\rangle \\ =&\langle Ax_n - By_n, AA^* \left(Ax_n - By_n\right)\rangle \\ \leq&\lambda_A \|Ax_n - By_n\|^2 \end{aligned} \tag{14}$$

and

$$\begin{aligned} &- 2\langle x_n - x^*, A^* \left(Ax_n - By_n\right)\rangle \\ =&- 2\langle Ax_n - Ax^*, Ax_n - By_n\rangle \\ =&- \|Ax_n - Ax^*\|^2 - \|Ax_n - By_n\|^2 + \|Ax^* - By_n\|^2. \end{aligned} \tag{15}$$

Substitute (14) and (15) into (13), we have

$$\begin{aligned} &\|u_n - x^*\|^2 \\ \leq&\|x_n - x^*\|^2 + \gamma_n^2 \lambda_A \|Ax_n - By_n\|^2 - \gamma_n \|Ax_n - Ax^*\|^2 \\ &- \gamma_n \|Ax_n - By_n\|^2 + \gamma_n \|Ax^* - By_n\|^2 \\ =&\|x_n - x^*\|^2 - \gamma_n (1 - \lambda_A \gamma_n)\|Ax_n - By_n\|^2 \\ &- \gamma_n \|Ax_n - Ax^*\|^2 + \gamma_n \|Ax^* - By_n\|^2. \end{aligned} \tag{16}$$

By using the same method as (16), we have

$$\begin{aligned} \|v_n - y^*\|^2 \leq&\|y_n - y^*\|^2 - \gamma_n (1 - \lambda_B \gamma_n)\|Ax_n - By_n\|^2 \\ &- \gamma_n \|By_n - By^*\|^2 + \gamma_n \|By^* - Ax_n\|^2. \end{aligned} \tag{17}$$

From (16) and (17), we have

$$\begin{aligned} &\|u_n - x^*\|^2 + \|v_n - y^*\|^2 \\ \leq&\|x_n - x^*\|^2 + \|y_n - y^*\|^2 - \gamma_n (1 - \lambda_A \gamma_n)\|Ax_n - By_n\|^2 \\ &- \gamma_n (1 - \lambda_B \gamma_n)\|Ax_n - By_n\|^2 \\ &- \gamma_n \|Ax_n - Ax^*\|^2 + \gamma_n \|Ax^* - By_n\|^2 \\ &- \gamma_n \|By_n - By^*\|^2 + \gamma_n \|By^* - Ax_n\|^2 \\ =&\|x_n - x^*\|^2 + \|y_n - y^*\|^2 \\ &- \gamma_n \left(2 - \gamma_n \left(\lambda_A + \lambda_B\right)\right) \|Ax_n - By_n\|^2. \end{aligned} \tag{18}$$

From the definition of $\{x_n\}$, we have

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ =&\|\alpha_n u + (1 - \alpha_n) P_{C_1}\left(I - \lambda_n^1 (I - T_1)\right) u_n - x^*\|^2 \\ \leq&\alpha_n \|u - x^*\|^2 + (1 - \alpha_n)\|P_{C_1}\left(I - \lambda_n^1 (I - T_1)\right) u_n - x^*\|^2 \\ \leq&\alpha_n \|u - x^*\|^2 + (1 - \alpha_n)\|u_n - x^*\|^2. \end{aligned} \tag{19}$$

By using the same method as (19), we have

$$\|y_{n+1} - y^*\|^2 \leq \alpha_n \|v - y^*\|^2 + (1 - \alpha_n)\|v_n - y^*\|^2. \tag{20}$$

From (18), (19) and (20), we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2\\
\le&\alpha_n\|u-x^*\|^2+(1-\alpha_n)\|u_n-x^*\|^2\\
&+\alpha_n\|v-y^*\|^2+(1-\alpha_n)\|v_n-y^*\|^2\\
=&\alpha_n(\|u-x^*\|^2+\|v-y^*\|^2)\\
&+(1-\alpha_n)(\|u_n-x^*\|^2+\|v_n-y^*\|^2)\\
\le&\alpha_n(\|u-x^*\|^2+\|v-y^*\|^2)\\
&+(1-\alpha_n)(\|x_n-x^*\|^2+\|y_n-y^*\|^2\\
&-\gamma_n\left(2-\gamma_n\left(\lambda_A+\lambda_B\right)\right)\|Ax_n-By_n\|^2) \qquad (21)\\
\le&\alpha_n(\|u-x^*\|^2+\|v-y^*\|^2)\\
&+(1-\alpha_n)(\|x_n-x^*\|^2+\|y_n-y^*\|^2)\\
\le&\max\{\|u-x^*\|^2+\|v-y^*\|^2,\|x_1-x^*\|^2+\|y_1-y^*\|^2\}.
\end{aligned}$$

From mathematical induction, we have $\{x_n\}$ and $\{y_n\}$ are bounded. Furthermore, $\{u_n\}$ and $\{v_n\}$ are bounded. From (21), we have

$$\begin{aligned}
&\gamma_n(1-\alpha_n)\left(2-\gamma_n\left(\lambda_A+\lambda_B\right)\right)\|Ax_n-By_n\|^2\\
\le&\alpha_n(\|u-x^*\|^2+\|v-y^*\|^2)+C_n-C_{n+1}, \qquad (22)
\end{aligned}$$

where $C_n=\|x_n-x^*\|^2+\|y_n-y^*\|^2$, for all $x^*\in F(T_1)$, $y^*\in F(T_2)$ and $n\in\mathbf{N}$.
From (22), we separate the proof into two cases.
<u>*Case*1.</u> Suppose that $C_{n+1}\le C_n$ for all $n\ge n_0$ (for n_0 large enough). Since the sequence $\{C_n\}$ is bounded, we get $\lim_{n\to\infty}C_n=c$, for some $c\in\mathbf{R}$.
From (22) and properties of γ_n and α_n, we obtain

$$\lim_{n\to\infty}\|Ax_n-By_n\|=0. \qquad (23)$$

From the definition of $\{u_n\}$ and $\{v_n\}$, we have

$$\|u_n-x_n\|=\gamma_n\|A^*\left(Ax_n-By_n\right)\| \qquad (24)$$

and

$$\|v_n-y_n\|=\gamma_n\|B^*\left(Ax_n-By_n\right)\|. \qquad (25)$$

From (23), (24) and (25), we have

$$\lim_{n\to\infty}\|u_n-x_n\|=\lim_{n\to\infty}\|v_n-y_n\|=0. \qquad (26)$$

By using properties of P_{C_1}, we have

$$\begin{aligned}
&\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
\le&\|\left(I-\lambda_n^1(I-T_1)\right)u_n-\left(I-\lambda_n^1(I-T_1)\right)x^*\|^2\\
=&\|u_n-x^*-\lambda_n^1(I-T_1)(u_n-x^*)\|^2\\
=&\|u_n-x^*\|^2-2\lambda_n^1\langle u_n-x^*,(I-T_1)u_n\rangle\\
&+(\lambda_n^1)^2\|(I-T_1)u_n\|^2\\
\le&\|u_n-x^*\|^2-\lambda_n^1(1-\lambda_n^1)\|(I-T_1)u_n\|^2. \qquad (27)
\end{aligned}$$

By using the same method as (27), we have

$$\begin{aligned}
&\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y^*\|^2\\
\le&\|v_n-y^*\|^2-\lambda_n^2(1-\lambda_n^2)\|(I-T_2)v_n\|^2. \qquad (28)
\end{aligned}$$

From (18), (27) and (28), we have

$$\begin{aligned}
&\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&+\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y^*\|^2\\
\le&\|u_n-x^*\|^2+\|v_n-y^*\|^2\\
&-\lambda_n^1(1-\lambda_n^1)\|(I-T_1)u_n\|^2\\
&-\lambda_n^2(1-\lambda_n^2)\|(I-T_2)v_n\|^2\\
\le&\|x_n-x^*\|^2+\|y_n-y^*\|^2\\
&-\gamma_n\left(2-\gamma_n\left(\lambda_A+\lambda_B\right)\right)\|Ax_n-By_n\|^2\\
&-\lambda_n^1(1-\lambda_n^1)\|(I-T_1)u_n\|^2\\
&-\lambda_n^2(1-\lambda_n^2)\|(I-T_2)v_n\|^2. \qquad (29)
\end{aligned}$$

From the definition of $\{x_n\}$ and $\{y_n\}$, we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2\\
\le&\alpha_n\|u-x^*\|^2\\
&+(1-\alpha_n)\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&+(1-\alpha_n)\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y^*\|^2\\
&+\alpha_n\|v-y^*\|^2\\
=&\alpha_n\big(\|u-x^*\|^2+\|v-y^*\|^2\big)\\
&+(1-\alpha_n)\big(\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&+\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y^*\|^2\big)\\
\le&\alpha_n\big(\|u-x^*\|^2+\|v-y^*\|^2\big)\\
&+(1-\alpha_n)\big(\|x_n-x^*\|^2+\|y_n-y^*\|^2\\
&-\gamma_n\left(2-\gamma_n\left(\lambda_A+\lambda_B\right)\right)\|Ax_n-By_n\|^2\\
&-\lambda_n^1(1-\lambda_n^1)\|(I-T_1)u_n\|^2\\
&-\lambda_n^2(1-\lambda_n^2)\|(I-T_2)v_n\|^2\big). \qquad (30)
\end{aligned}$$

It implies that

$$\begin{aligned}
&(1-\alpha_n)\big(\lambda_n^1(1-\lambda_n^1)\|(I-T_1)u_n\|^2\\
&+\lambda_n^2(1-\lambda_n^2)\|(I-T_2)v_n\|^2\big)\\
\le&C_n-C_{n+1}+\alpha_n\big(\|u-x^*\|^2+\|v-y^*\|^2\big). \qquad (31)
\end{aligned}$$

From (31) and $\lim_{n\to\infty}C_n=c$, we have

$$\lim_{n\to\infty}\|(I-T_1)u_n\|=\lim_{n\to\infty}\|(I-T_2)v_n\|=0. \qquad (32)$$

By using properties of P_{C_1}, we have

$$\begin{aligned}
&\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&\le \langle \left(I-\lambda_n^1(I-T_1)\right)u_n-\left(I-\lambda_n^1(I-T_1)\right)x^*\\
&\quad ,P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\rangle\\
&=\frac{1}{2}\big(\|\left(I-\lambda_n^1(I-T_1)\right)u_n-\left(I-\lambda_n^1(I-T_1)\right)x^*\|^2\\
&\quad+\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&\quad-\|\left(I-\lambda_n^1(I-T_1)\right)u_n-\left(I-\lambda_n^1(I-T_1)\right)x^*\\
&\quad-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n+x^*\|^2\big)\\
&\le\frac{1}{2}\big(\|u_n-x^*\|^2+\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&\quad-\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\\
&\quad-\lambda_n^1\big((I-T_1)u_n-(I-T_1)x^*\big)\|^2\big)\\
&=\frac{1}{2}\big(\|u_n-x^*\|^2+\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&\quad-\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\|^2\\
&\quad-\left(\lambda_n^1\right)^2\|(I-T_1)u_n-(I-T_1)x^*\|^2\\
&\quad+2\lambda_n^1\langle u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\\
&\quad ,(I-T_1)u_n-(I-T_1)x^*\rangle\big).
\end{aligned}\tag{33}$$

From (33), we have

$$\begin{aligned}
&\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&\le\|u_n-x^*\|^2-\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\|^2\\
&\quad-\left(\lambda_n^1\right)^2\|(I-T_1)u_n-(I-T_1)x^*\|^2\\
&\quad+2\lambda_n^1\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\|\\
&\quad\cdot\|(I-T_1)u_n-(I-T_1)x^*\|.
\end{aligned}\tag{34}$$

Similarly, we have

$$\begin{aligned}
&\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y^*\|^2\\
&\le\|v_n-y^*\|^2-\|v_n-P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n\|^2\\
&\quad-\left(\lambda_n^2\right)^2\|(I-T_2)v_n-(I-T_2)y^*\|^2\\
&\quad+2\lambda_n^2\|v_n-P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n\|\\
&\quad\cdot\|(I-T_2)v_n-(I-T_2)y^*\|.
\end{aligned}\tag{35}$$

From the definition of $\{x_n\}$, $\{y_n\}$, (34) and (35), we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2\\
&\le\alpha_n\big(\|u-x^*\|^2+\|v-y^*\|^2\big)\\
&\quad+(1-\alpha_n)\big(\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x^*\|^2\\
&\quad+\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y^*\|^2\big)\\
&\le\alpha_n\big(\|u-x^*\|^2+\|v-y^*\|^2\big)+\|x_n-x^*\|^2+\|y_n-y^*\|^2\\
&\quad-(1-\alpha_n)\big(\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\|^2\\
&\quad+\|v_n-P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n\|^2\big)\\
&\quad+2\lambda_n^1\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\|\\
&\quad\cdot\|(I-T_1)u_n-(I-T_1)x^*\|\\
&\quad+2\lambda_n^2\|v_n-P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n\|\\
&\quad\cdot\|(I-T_2)v_n-(I-T_2)y^*\|.
\end{aligned}$$

It implies that

$$\begin{aligned}
&(1-\alpha_n)\big(\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\|^2\\
&\quad+\|v_n-P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n\|^2\big)\\
&\le\alpha_n\big(\|u-x^*\|^2+\|v-y^*\|^2\big)\\
&\quad+2\lambda_n^1\|u_n-P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n\|\\
&\quad\cdot\|(I-T_1)u_n-(I-T_1)x^*\|\\
&\quad+2\lambda_n^2\|v_n-P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n\|\\
&\quad\cdot\|(I-T_2)v_n-(I-T_2)y^*\|\\
&\quad+C_n-C_{n+1}.
\end{aligned}$$

From (32) and $\lim_{n\to\infty}C_n=c$, we have

$$\lim_{n\to\infty}\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-u_n\| \tag{36}$$

$$=\lim_{n\to\infty}\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-v_n\|=0. \tag{37}$$

From (26) and (36), we obtain

$$\lim_{n\to\infty}\|P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x_n\| \tag{38}$$

$$=\lim_{n\to\infty}\|P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y_n\|=0. \tag{39}$$

Since

$$\begin{aligned}
&x_{n+1}-x_n\\
&=\alpha_n\big(u-x_n\big)+(1-\alpha_n)\left(P_{C_1}\left(I-\lambda_n^1(I-T_1)\right)u_n-x_n\right),
\end{aligned}$$

$$\begin{aligned}
&y_{n+1}-y_n\\
&=\alpha_n\big(v-y_n\big)+(1-\alpha_n)\left(P_{C_2}\left(I-\lambda_n^2(I-T_2)\right)v_n-y_n\right),
\end{aligned}$$

and (38), we get

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}\|y_{n+1}-y_n\|=0. \tag{40}$$

Since $W_w(x_n)$ and $W_w(y_n)$ are non-empty sets, then there exists $\hat{x}\in C_1,\hat{y}\in C_2$ such that $\hat{x}\in W_w(x_n)$ and $\hat{y}\in W_w(y_n)$.

We may assume, there exists subsequences $\{x_{n_k}\}$, $\{y_{n_k}\}$ of $\{x_n\}$, $\{y_n\}$ such that

$$x_{n_k} \rightharpoonup \hat{x} \quad \text{as} \quad k \to \infty. \tag{41}$$

and

$$y_{n_k} \rightharpoonup \hat{y} \quad \text{as} \quad k \to \infty. \tag{42}$$

Next, we will show that $(\hat{x}, \hat{y}) \in \Omega$.
From (26), (41) and (42), we obtain $u_{n_k} \rightharpoonup \hat{x}$ and $v_{n_k} \rightharpoonup \hat{y}$ as $k \to \infty$.
Assume that $\hat{x} \notin F(T_1)$.
Since $F(T_1) = F(P_{C_1}\left(I - \lambda^1_{n_k}(I - T_1)\right))$, we have $\hat{x} \neq P_{C_1}\left(I - \lambda^1_{n_k}(I - T_1)\right)\hat{x}$. From Opial's condition, $\lim\limits_{k\to\infty} \lambda^1_{n_k} = 0$ and condition i), we have

$$\begin{aligned}
&\liminf_{k\to\infty} \|u_{n_k} - \hat{x}\| \\
&< \liminf_{k\to\infty} \|u_{n_k} - P_{C_1}\left(I - \lambda^1_{n_k}(I - T_1)\right)\hat{x}\| \\
&\leq \liminf_{k\to\infty} \big(\|u_{n_k} - P_{C_1}\left(I - \lambda^1_{n_k}(I - T_1)\right)u_{n_k}\| \\
&\quad + \|P_{C_1}\left(I - \lambda^1_{n_k}(I - T_1)\right)u_{n_k} \\
&\quad - P_{C_1}\left(I - \lambda^1_{n_k}(I - T_1)\right)\hat{x}\|\big) \\
&\leq \liminf_{k\to\infty} \big(\|u_{n_k} - P_{C_1}\left(I - \lambda^1_{n_k}(I - T_1)\right)u_{n_k}\| \\
&\quad + \|u_{n_k} - \hat{x}\| + \lambda^1_{n_k}\|(I - T_1)u_{n_k} - (I - T_1)\hat{x}\|\big) \\
&= \liminf_{k\to\infty} \|u_{n_k} - \hat{x}\|.
\end{aligned}$$

This is a contradiction. Thus $\hat{x} \in F(T_1)$.
From $v_{n_k} \rightharpoonup \hat{y}$ as $k \to \infty$ and using the same method as $\hat{x} \in F(T_1)$, we have $\hat{y} \in F(T_2)$.
Since $A\hat{x} - B\hat{y} \in W_w(Ax_n - By_n)$ and weakly lower semi-continuous of norm, we get

$$\|A\hat{x} - B\hat{y}\| \leq \liminf_{k\to\infty} \|Ax_{n_k} - By_{n_k}\| = 0.$$

Then $A\hat{x} = B\hat{y}$. Hence $(\hat{x}, \hat{y}) \in \Omega$.
Consider that

$$\begin{aligned}
&\limsup_{n\to\infty} \langle u - \hat{x^*}, x_n - \hat{x^*}\rangle \\
&= \limsup_{k\to\infty} \langle u - \hat{x^*}, x_{n_k} - \hat{x^*}\rangle \\
&= \langle u - \hat{x^*}, \hat{x} - \hat{x^*}\rangle \\
&\leq 0,
\end{aligned}$$

where $\hat{x^*} = P_{F(T_1)}u$ and

$$\begin{aligned}
&\limsup_{n\to\infty} \langle v - \hat{y^*}, y_n - \hat{y^*}\rangle \\
&= \limsup_{k\to\infty} \langle v - \hat{y^*}, y_{n_k} - \hat{y^*}\rangle \\
&= \langle v - \hat{y^*}, \hat{y} - \hat{y^*}\rangle \\
&\leq 0,
\end{aligned}$$

where $\hat{y^*} = P_{F(T_2)}v$.
Next, we show that a sequence $\{(x_n, y_n)\}$ converges strongly to $(\hat{x^*}, \hat{y^*}) \in \Omega$, where $\hat{x^*} = P_{F(T_1)}u$ and $\hat{y^*} = P_{F(T_2)}v$.
From the definitions of $\{x_n\}$ and $\{y_n\}$, we have

$$\|x_{n+1} - \hat{x^*}\|^2 \leq (1 - \alpha_n)\|x_n - \hat{x^*}\|^2 + 2\alpha_n\langle u - \hat{x^*}, x_{n+1} - \hat{x^*}\rangle$$

and

$$\|y_{n+1} - \hat{y^*}\|^2 \leq (1 - \alpha_n)\|y_n - \hat{y^*}\|^2 + 2\alpha_n\langle v - \hat{y^*}, y_{n+1} - \hat{y^*}\rangle.$$

Then

$$\begin{aligned}
&\|x_{n+1} - \hat{x^*}\|^2 + \|y_{n+1} - \hat{y^*}\|^2 \\
&\leq (1 - \alpha_n)\big(\|x_n - \hat{x^*}\|^2 + y_n - \hat{y^*}\|^2\big) \\
&\quad + 2\alpha_n\big(\langle u - \hat{x^*}, x_{n+1} - \hat{x^*}\rangle + \langle v - \hat{y^*}, y_{n+1} - \hat{y^*}\rangle\big),
\end{aligned}$$

or

$$C_{n+1} \leq (1 - \alpha_n)C_n + 2\alpha_n\varrho_n, \tag{43}$$

where $\varrho_n = \langle u - \hat{x^*}, x_{n+1} - \hat{x^*}\rangle + \langle v - \hat{y^*}, y_{n+1} - \hat{y^*}\rangle$, for all $n \in \mathbf{N}$.
From Lemma 2.3, thus

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty}\big(\|x_n - \hat{x^*}\|^2 + \|y_n - \hat{y^*}\|^2\big) = 0.$$

Therefore (x_n, y_n) converges strongly to $(\hat{x^*}, \hat{y^*})$.
Since $A\hat{x^*} - B\hat{y^*} \in W_w(Ax_n - By_n)$ and weakly lower semi-continuous of norm, we get

$$\|A\hat{x^*} - B\hat{y^*}\| \leq \liminf_{k\to\infty} \|Ax_{n_k} - By_{n_k}\| = 0.$$

Then $A\hat{x^*} = B\hat{y^*}$. Hence $(\hat{x^*}, \hat{y^*}) \in \Omega$.
Case2. Suppose that C_n is not monotone sequence, then there exists an integer n_0 such that $C_{n_0} \leq C_{n_0+1}$.
Define the integer sequence $\tau(n)$ for all $n \geq n_0$ as follows,

$$\tau(n) = \max\{k \leq n : C_k < C_{k+1}\}.$$

It is clear that $\tau(n)$ is a nondecreasing with $\lim\limits_{n\to\infty} \tau(n) = \infty$ and $C_{\tau(n)} < C_{\tau(n)+1}$.
From (43), we have

$$C_{\tau(n)+1} \leq (1 - \alpha_{\tau(n)})C_{\tau(n)} + 2\alpha_{\tau(n)}\varrho_{\tau(n)}.$$

From Lemma 2.3, thus

$$\lim_{n\to\infty} C_{\tau(n)} = 0.$$

Applying (40), we have

$$\lim_{n\to\infty} C_{\tau(n)+1} = 0.$$

By Lemma (2.7), we have

$$C_n \leq \max\{C_n, C_{\tau(n)}\} \leq C_{\tau(n)+1}.$$

From above inequality and $\lim\limits_{n\to\infty} C_{\tau(n)+1} = 0$, we obtain

$$\lim_{n\to\infty}\big(\|x_n - \hat{x^*}\|^2 + \|y_n - \hat{y^*}\|^2\big) = \lim_{n\to\infty} C_n = 0.$$

That implies $\{(x_n, y_n)\}$ converges strongly to $\big(\hat{x^*}, \hat{y^*}\big)$.
By using the same methods as case 1, we have $\big(\hat{x^*}, \hat{y^*}\big) \in \Omega$, where $\hat{x^*} = P_{F(T_1)}u$ and $\hat{y^*} = P_{F(T_2)}v$.
This is complete the proof. ■

Corollary 3.2: For every $i = 1, 2, 3$, let H_i be a real Hilbert space and let C_1, C_2 be non-empty closed convex subset of H_1 and H_2, respectively. Let $T_i : C_i \to C_i$ be quasi-nonexpansive mapping for all $i = 1, 2$ and let $A : H_1 \to H_3$ be bounded linear operator with adjoints A^*, respectively. Suppose that $\Omega = \{(x, y) \in C_1 \times C_2 : x \in F(T_1), y \in F(T_2)$ and $Ax = y\}$ is a non-empty set and let $\{x_n\}$, $\{y_n\}$ be sequences generated by $u, x_1 \in C_1; v, y_1 \in C_2$ and

$$\begin{cases}
u_n = x_n - \gamma_n A^*\left(Ax_n - y_n\right), \\
x_{n+1} = \alpha_n u + (1 - \alpha_n)\, P_{C_1}\left(I - \lambda^1_n(I - T_1)\right)u_n, \\
v_n = (1 - \gamma_n)y_n + \gamma_n Ax_n, \\
y_{n+1} = \alpha_n v + (1 - \alpha_n)\, P_{C_2}\left(I - \lambda^2_n(I - T_2)\right)v_n,
\end{cases} \tag{44}$$

for all $n \geq 1$ where $\{\alpha_n\} \subset [0,1]$ with $\lim_{n\to\infty} \alpha_n = 0$,$\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \lambda_n^i < \infty$ and $\lambda_n^i \in (0,1)$ for all $i = 1,2$ and $\gamma_n \in (a,b) \subset \left(\epsilon, \frac{2}{\lambda_A} - \epsilon\right)$ for all $n \in \mathbf{N}$ and λ_A be spectral radius of A^*A, ϵ is a small enough. Then the sequence $\{(x_n, y_n)\}$ converge strongly to $(\bar{x^*}, \bar{y^*}) \in \Omega$, where $\bar{x^*} = P_{F(T_1)}u$ and $\bar{y^*} = P_{F(T_2)}v$.

Proof: By using Theorem 3.1 and taking $B \equiv I$, we obtain the conclusion. ■

IV. Application

A mapping $T : C \to C$ is called nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \ \forall x, y \in C.$$

Such mapping is defined by Kohsaka and Takahashi [13]. The following lemma will be used to prove in the application.

Lemma 4.1: [13] Let H be a Hilbert space, let C be a non-empty closed convex subset of H, and let S be a nonspreading mapping of C into itself. Then $F(S)$ is closed and convex.

In 2009, Kangtunyakarn and Suantai [14] introduced the S-mapping generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ as follows.

Definition 4.1: Let C be a non-empty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of (nonexpansive) mappings of C into itself. For each $j = 1,2,...,N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \to C$ as follows;

$$\begin{aligned}
&U_0 = I,\\
&U_1 = \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I,\\
&U_2 = \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I,\\
&U_3 = \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I,\\
&\quad .\\
&\quad .\\
&\quad .\\
&U_{N-1} = \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I,\\
&S = U_N = \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}$$

This mapping is called an S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$.

Lemma 4.2: [15] Let C be a non-empty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1,2,...,N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, α_1^j, $\alpha_3^j \in (0,1)$ for all $j = 1,2,...,N-1$ and $\alpha_1^N \in (0,1]$, $\alpha_3^N \in [0,1)$ $\alpha_2^j \in [0,1)$ for all $j = 1,2,...,N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$ and S is a quasi-nonexpansive mapping.

By using these results, we obtain the following theorem.

Theorem 4.3: For every $i = 1,2,3$, let H_i be a real Hilbert space and let C_1, C_2 be non-empty closed convex subset of H_1 and H_2, respectively. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C_1 into C_1 with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1,2,...,N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, α_1^j, $\alpha_3^j \in (0,1)$ for all $j = 1,2,...,N-1$ and $\alpha_1^N \in (0,1]$, $\alpha_3^N \in [0,1)$ $\alpha_2^j \in [0,1)$ for all $j = 1,2,...,N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Let $\{\bar{T}_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C_2 into C_2 with $\bigcap_{i=1}^N F(\bar{T}_i) \neq \emptyset$, and let $\beta_j = (\beta_1^j, \beta_2^j, \beta_3^j) \in I \times I \times I$, $j = 1,2,...,N$, where $I = [0,1]$, $\beta_1^j + \beta_2^j + \beta_3^j = 1$, β_1^j, $\beta_3^j \in (0,1)$ for all $j = 1,2,...,N-1$ and $\beta_1^N \in (0,1]$, $\beta_3^N \in [0,1)$ $\beta_2^j \in [0,1)$ for all $j = 1,2,...,N$. Let $\bar{S}$ be the mapping generated by $\bar{T}_1, \bar{T}_2, ..., \bar{T}_N$ and $\beta_1, \beta_2, ..., \beta_N$. Let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operator with adjoints A^* and B^*, respectively. Suppose that $\Omega = \{(x,y) \in C_1 \times C_2 : x \in \bigcap_{i=1}^N F(T_i), y \in \bigcap_{i=1}^N F(\bar{T}_i)$ and $Ax = By\}$ is a non-empty set and let $\{x_n\}$, $\{y_n\}$ be sequences generated by $u, x_1 \in C_1; v, y_1 \in C_2$ and

$$\begin{cases}
u_n = x_n - \gamma_n A^* \left(Ax_n - By_n\right),\\
x_{n+1} = \alpha_n u + (1-\alpha_n) P_{C_1}\left(I - \lambda_n^1 (I - S)\right) u_n,\\
v_n = y_n + \gamma_n B^* \left(Ax_n - By_n\right),\\
y_{n+1} = \alpha_n v + (1-\alpha_n) P_{C_2}\left(I - \lambda_n^2 (I - \bar{S})\right) v_n,
\end{cases} \tag{45}$$

for all $n \geq 1$ where $\{\alpha_n\} \subset [0,1]$ with $\lim_{n\to\infty} \alpha_n = 0$,$\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \lambda_n^i < \infty$ and $\lambda_n^i \in (0,1)$ for all $i = 1,2$ and $\gamma_n \in (a,b) \subset \left(\epsilon, \frac{2}{\lambda_A + \lambda_B} - \epsilon\right)$ for all $n \in \mathbf{N}$ and λ_A, λ_B are spectral radius of A^*A, B^*B respectively, ϵ is a small enough. Then the sequence $\{(x_n, y_n)\}$ converge strongly to $(\bar{x^*}, \bar{y^*}) \in \Omega$, where $\bar{x^*} = P_{F(S)}u$ and $\bar{y^*} = P_{F(\bar{S})}v$.

Proof: By using Theorem 3.1 and 4.2, we obtain the conclusion. ■

Moreover, if we put $F(T_1) = C_1$ and $F(T_2) = C_2$ in Theorem 3.1, we obtain the SEFPP reduced to the SEFP.

Theorem 4.4: For every $i = 1,2,3$, let H_i be a real Hilbert space and let C_1, C_2 be non-empty closed convex subset of H_1 and H_2, respectively. Let $T_i : C_i \to C_i$ be quasi-nonexpansive mapping for all $i = 1,2$ and let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operator with adjoints A^* and B^*, respectively. Suppose that $\Omega = \{(x,y) \in C_1 \times C_2 : Ax = By\}$ is a non-empty set and let $\{x_n\}$, $\{y_n\}$ be sequences generated by $u, x_1 \in C_1; v, y_1 \in C_2$ and

$$\begin{cases}
u_n = x_n - \gamma_n A^* \left(Ax_n - By_n\right),\\
x_{n+1} = \alpha_n u + (1-\alpha_n) P_{C_1} u_n,\\
v_n = y_n + \gamma_n B^* \left(Ax_n - By_n\right),\\
y_{n+1} = \alpha_n v + (1-\alpha_n) P_{C_2} v_n,
\end{cases} \tag{46}$$

for all $n \geq 1$,where $\{\alpha_n\} \subset [0,1]$ with $\lim_{n\to\infty} \alpha_n = 0$,$\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \lambda_n^i < \infty$ and $\lambda_n^i \in (0,1)$ for all $i = 1,2$ and $\gamma_n \in (a,b) \subset \left(\epsilon, \frac{2}{\lambda_A + \lambda_B} - \epsilon\right)$ for all $n \in \mathbf{N}$ and λ_A, λ_B are spectral radius of A^*A, B^*B respectively, ϵ is a small enough. Then the sequence $\{(x_n, y_n)\}$ converge strongly to $(\bar{x^*}, \bar{y^*}) \in \Omega$, where $\bar{x^*} = P_{C_1}u$ and $\bar{y^*} = P_{C_2}v$.

Proof: By using Theorem 3.1, we put $F(T_1) = C_1$ and $F(T_2) = C_2$, we obtain the conclusion. ■

V. Conclusion

We have proposed an algorithm for solving a new split equality fixed point problem for quasi-nonexpansive mapping, and proved its converges in the Hilbert spaces. In Application, we used S-mapping applied to our main result to prove the strong convergence theorems.

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