Solving Multi-objective Fuzzy Matrix Games via Fuzzy Relation Approach

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***Abstract*—A generalized model for a multi-objective matrix game with fuzzy goals and fuzzy payoffs via fuzzy relation approach is presented in this paper. It is shown that solving such a game is equivalent to solving a pair of multi-objective non-linear optimization problems, and then the pair of multi-objective non-linear optimization problems can be reduced to two mutually dual multi-objective linear programming problems.**

***Index Terms*—Multi-objective game, Fuzzy relation, Multi-objective non-linear optimization.**

I. Introduction

A multi-objective zero-sum game is an extension of the standard two person zero-sum game. The two person zero-sum game is also referred to a matrix game because it can be expressed by a pair of payoff matrices. As conflicting interests appear not only between different decision makers, but also within each individual, the study of multi-objective fuzzy games is very important.

Blackwell [5] first applied the concepts of maxmin and minmax values to deal with the theory of multi-objective games as a generalization of the theory of scalar games. Zeleny [19] studied the multi-objective zero-sum games by aggregating multiple pay-offs into a single payoff via weighting coefficients approach. Ghose et al. [9] proposed the concepts of Pareto-optimal Security Strategies (POSS) for multi-objective two person zero-sum games and solved this games problems by the weightage average approach. Afterwards, Fernandez et al. [8] studied the same game model and proved the equivalence between POSS and efficient solutions of a pair of multi-objective programming problems.

Though single objective fuzzy matrix games have been extensively studied in [2], [3], [21], [22], the results on multi-objective scenario are rather scarce. These main contribution in this direction have been worked in [1], [18]. Sakawa et al. [18] studied fuzzy multi-objective games model by the concepts of maxmin value. Inspired by [8], [9], Aggarwal et al. [1] applied the concepts of POSS and security levels for players to study fuzzy multi-objective matrix game model and proved that solving such a game model can be obtained by solving a pair of multi-objective linear programming problems. In [12], B. Jiang and D. Qiu considered the expansion of linear inequality and some related theorems in the fuzzy case, and proposed an optimization criterion of fuzzy linear problems. Taking motivation from [1], [21] we present a new generalized model for a multi-objective matrix game with fuzzy goals and fuzzy payoffs by fuzzy relation approach.

This paper is divided into five sections. The background of this paper is introduced in Section 1. Section 2 introduces some basic definitions and recalls some results that regard to crisp multi-objective matrix games and the fuzzy relations. In Section 3, a new generalized model for a multi-objective matrix game with fuzzy goals and fuzzy payoffs via fuzzy relation approach is established. Section 4 presents a kind of multi-objective linear programming problems. In Section 5, the results of this paper are illustrated with a small numerical example.

II. Preliminaries

In this section, we begin to describe a crisp multi-objective matrix game in [1]. For this we recall some definitions.

Definition 1: [9] *(mixed strategy)*. A mixed strategy $x = (x_1, x_2, \cdots, x_m)^T$ for Player I is a probability distribution on the set I of his pure strategies, where x^T is the transposition of x. The set of mixed strategies for Player I is represented by

$$S^m = \{x = (x_1, x_2, \cdots, x_m)^T \in \mathbb{R}^m | \sum_{i=1}^{m} x_i = 1, \quad x_i \geq 0, i = 1, 2, \cdots, m.\} \tag{1}$$

Similarly, The set of mixed strategies for Player *II* is represented by

$$S^n = \{y = (y_1, y_2, \cdots, y_n)^T \in \mathbb{R}^n | \sum_{j=1}^{n} y_j = 1, \quad y_j \geq 0, j = 1, 2, \cdots, n.\} \tag{2}$$

where $\mathbb{R}^m$ and $\mathbb{R}^n$ are m- and n-dimensional Euclidean spaces. The sets S^m and S^n are the convex ploy-topes.

In multi-objective two-person matrix games, a multiple payoff matrix of the Player I and Player II are defined as follows [9]:

$$A^1 = \begin{pmatrix} a_{11}^1 & \cdots & a_{1n}^1 \\ \vdots & \ddots & \vdots \\ a_{m1}^1 & \cdots & a_{mn}^1 \end{pmatrix}, A^r = \begin{pmatrix} a_{11}^r & \cdots & a_{1n}^r \\ \vdots & \ddots & \vdots \\ a_{m1}^r & \cdots & a_{mn}^r \end{pmatrix} \tag{3}$$

where we assume that each of the two players has r objectives. Mixed strategies correspond to the rows and the columns of each matrix A^k $(k = 1, ..., r)$ for Player I and Player II, respectively. And it is a convention to assume that Player I is a maximizing player and Player II is a minimizing player.

A two-person zero-sum multi-objective matrix game (MOG) [1] is defined by

$$MOG = (S^m, S^n, A^k(1, 2, \cdots, r)),$$

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where S^m (respectively, S^n) is the strategy space for Player I (respectively, Player II).

Definition 2: [9] *(expected payoff of multi-objective matrix game).* When Player *I* chooses a mixed strategy $x \in S^m$ and Player *II* chooses a mixed strategy $y \in S^n$, a vector

$$\begin{aligned} E(x,y) = x^T Ay = [E_1(x,y), E_2(x,y), \cdots, E_r(x,y)] \\ = \left[x^T A^1 y, x^T A^2 y, \cdots, x^T A^r y\right] \end{aligned} \tag{4}$$

is called an expected payoff of Player *I*. As the multi-objective game (MOG) is zero-sum, the payoff for Player *II* is $-x^T Ay$.

Definition 3: A pair $(\bar{x}, \bar{y}) \in S^m \times S^n$ is called a solution of the multi-objective game (MOG) if

$$\bar{x}A^k y \geq \bar{V}^k, \ \forall y \in S^n,$$

$$xA^k \bar{y} \leq \bar{V}^k, \ \forall x \in S^m.$$

Here, $\bar{x}$ (respectively $\bar{y}$) is called the optimal strategy for Player I (respectively Player II) and $\bar{V}^k (k = 1, 2, \cdots, r)$ are called the valued of the multi-objective game (MOG).

Given a multi-objective game (MOG), its solution can be obtained by solving the following pair of multi-objective linear programming problems for Player I $(MOGLP)$ and Player II $(MOGLD)$, respectively.

$$(MOGLP) \qquad \max \quad (V^1, V^2, \cdots, V^r)$$

$$\begin{aligned} & such\ that \\ & \sum_{i=1}^{m} (a_{ij})^k x_i \geq V^k, (k = 1, 2, \cdots, r, j = 1, 2, \cdots, n), \\ & \qquad \forall x \in S^m, \forall y \in S^n, \end{aligned}$$

$$(MOGLD) \qquad \min \quad (W^1, W^2, \cdots, W^r)$$

$$\begin{aligned} & such\ that \\ & \sum_{j=1}^{n} (a_{ij})^k y_j \leq W^k, (k = 1, 2, \cdots, r, i = 1, 2, \cdots, m), \\ & \qquad \forall x \in S^m, \forall y \in S^n. \end{aligned}$$

Next we present some basic definitions and results related to fuzzy numbers and fuzzy relations.

A fuzzy set $\tilde{B}$ of $\mathbb{R}$ is characterized by a membership function $u_{\tilde{B}} : \mathbb{R} \to [0, 1]$ [25]. An α-level set of $\tilde{B}$ is given as $[\tilde{B}]_\alpha = \{x \in \mathbb{R} : u_{\tilde{B}}(x) \geq \alpha\}$ for each $\alpha \in (0, 1]$. And a strict α-level set of $\tilde{B}$ is given by $(\tilde{B})_\alpha = \{x \in \mathbb{R} : u_{\tilde{B}}(x) > \alpha\}$ for each $\alpha \in (0, 1]$. We define the set $[\tilde{B}]_0$ by $[\tilde{B}]_0 = \overline{\{x \in \mathbb{R} : u_{\tilde{B}}(x) > 0\}}$, where $\overline{B}$ denotes the closure of a crisp set B. A fuzzy set $\tilde{B}$ is said to be a fuzzy number if it satisfies the following conditions [7]:
(1) $\tilde{B}$ is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u_{\tilde{B}}(x_0) = 1$;
(2) $\tilde{B}$ is convex, i.e., $u_{\tilde{B}}(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{u_{\tilde{B}}(x_1), u_{\tilde{B}}(x_2)\}$, for all $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0, 1]$;
(3) $\tilde{B}$ is upper semi-continuous;
(4) $[\tilde{B}]_0$ is compact.

Let X and Y be any non-empty sets. Then a binary relation $\mathcal{P}$ between the elements of X and Y is a subset of the cartesian product $X \times Y$, i.e. $\mathcal{P} \subset X \times Y$. In fuzzy set theory, for this let $\mathscr{F}(X)$, $\mathscr{F}(Y)$, and $\mathscr{F}(X \times Y)$ be the sets of all fuzzy subsets of X, Y and $X \times Y$, respectively.

Definition 4: [11]*(valued relation).* A fuzzy subset $\mathcal{P} \subset \mathscr{F}(X \times Y)$ is called a valued relation on $X \times Y$ i.e., $\mathcal{P} : X \times Y \to [0, 1]$.

Definition 5: [11] *(fuzzy relation).* A valued relation $\mathcal{P}$ on $\mathscr{F}(X \times Y)$ is called a fuzzy relation on $X \times Y$ and it is denoted by $\tilde{\mathcal{P}}$, i.e., $\tilde{\mathcal{P}} : \mathscr{F}(X \times Y) \to [0, 1]$.

Definition 6: [11] *(fuzzy extension of a valued relation $\mathcal{P}$).* Let $\mathcal{P}$ be a valued relation on $X \times Y$. A fuzzy relation $\tilde{\mathcal{Q}}$ on $X \times Y$ with $u_{\tilde{\mathcal{Q}}}(x, y) = u_{\mathcal{P}}(x, y)\ \forall x \in X, \forall y \in Y$ is called a fuzzy extension of the relation $\mathcal{P}$.

Definition 7: [11] *(dual fuzzy extension of a valued relation $\mathcal{P}$).* Let $\mathcal{P}$ be a valued relation on X. Let $u_{c\mathcal{P}}$ be the membership function of the valued relation $c\mathcal{P}$ given by

$$u_{c\mathcal{P}}(x, y) = 1 - u_{\mathcal{P}}(x, y)\ \forall x, y \in X. \tag{5}$$

Let $\tilde{\mathcal{Q}}$ be a fuzzy extension of valued relation $c\mathcal{P}$. Then a fuzzy relation $\tilde{\mathcal{Q}}^d$ on X defined by

$$u_{\tilde{\mathcal{Q}}^d}(\tilde{B}, \tilde{D}) = 1 - u_{\tilde{\mathcal{Q}}}(\tilde{D}, \tilde{B})\ \forall \tilde{B}, \tilde{D} \in \mathscr{F}(X) \tag{6}$$

is called the dual fuzzy extension of the valued relation $\mathcal{P}$.

For a valued relation $\mathcal{P}$ there might be many fuzzy extensions, Inuiguchi et al [11] have used the well known concept of triangular norm (t-norm) T and its dual triangular conorm (t-conorm) S to extend a given valued relation to its fuzzy extension and dual fuzzy extension, respectively.

Definition 8: [11] Let $\mathcal{P}$ be a valued relation on X, T be a t-norm and S be its dual t-conorm. The T-fuzzy extension and T-dual fuzzy extension of $\mathcal{P}$, denoted by fuzzy relations $\mathcal{P}^T$ and $\mathcal{P}_S$, respectively, are defined by

$$u_{\tilde{\mathcal{P}}^T}(\tilde{B}, \tilde{D}) = \sup_{x,y \in X} \{T(u_{\mathcal{P}}(x, y), T(u_{\tilde{B}}(x), u_{\tilde{D}}(y)))\}, \tag{7}$$

$$u_{\tilde{\mathcal{P}}_S}(\tilde{B}, \tilde{D}) = \inf_{x,y \in X} \{S(S(1 - u_{\tilde{B}}(x), 1 - u_{\tilde{D}}(y)), u_{\mathcal{P}}(x, y))\}. \tag{8}$$

$\forall \tilde{B}, \tilde{D} \in \mathscr{F}(X)$.

Definition 9: [3] *(semistrictly quasiconcave function).* Let $X \subset \mathbb{R}$ and $f : \mathbb{R} \to [0, 1]$. Then f is called semistrictly quasiconcave function if it satisfies the following conditions:
(1) $f(\lambda x + (1 - \lambda)y) \geq min\{f(x), f(y)\}$, $\forall x, y \in X$, $\lambda \in [0, 1]$ with $\lambda x + (1 - \lambda)y \in X$;
(2) $f(\lambda x + (1 - \lambda)y) > min\{f(x), f(y)\}$, $\forall x, y \in X$, $\lambda \in [0, 1]$ with $f(x) \neq f(y)$, $\lambda x + (1 - \lambda)y \in X$, $f(\lambda x + (1 - \lambda)y) > 0$.

Definition 10: [3] *(fuzzy quantity).* Let $\tilde{S}$ be a fuzzy subset of $\mathbb{R}$. Then $\tilde{S}$ is called fuzzy quantity if $\tilde{S}$ is normal, compact, and has a semistrictly quasiconcave membership function.

Let $\mathscr{F}_0(\mathbb{R})$ be the set of all fuzzy quantities on $\mathbb{R}$. It is important to note that all crisp numbers as well as all commonly used fuzzy numbers, like ,triangular fuzzy numbers, trapezoidal fuzzy numbers, bell shaped fuzzy numbers are in $\mathscr{F}_0(\mathbb{R})$. An important property of fuzzy quantity is that if $\tilde{B}$ is fuzzy quantity then

$$B^L(\alpha) = \inf\{x | x \in [\tilde{B}]_\alpha\} = \inf[\tilde{B}]_\alpha = \inf(\tilde{B})_\alpha, \tag{9}$$

$$B^R(\alpha) = \sup\{x | x \in [\tilde{B}]_\alpha\} = \sup[\tilde{B}]_\alpha = \sup(\tilde{B})_\alpha, \tag{10}$$

where $B^L(\alpha)$ and $B^R(\alpha)$ are called the left part and the right part of fuzzy quantity $\tilde{B}$, respectively [15].

Lemma 1: [3] Let $\tilde{a}_j$ be fuzzy quantity and $x_j \geq 0$ $(j = 1, 2, \cdots, n)$. Then, $\sum_{j=1}^{n} \tilde{a}_j x_j$ is also a fuzzy quantity. Here $\sum_{j=1}^{n} \tilde{a}_j x_j$ is understood to be a fuzzy set on $\mathbb{R}$ whose membership function is defined as per Zadeh extension principle.

III. A GENERALIZED MODEL FOR A MULTI-OBJECTIVE FUZZY MATRIX GAME

Let S^m, S^n be as introduced in section 2 and $\tilde{A}^k$ $(k = 1, 2, \cdots, r)$ be the payoff matrixes with entries as fuzzy numbers. Let $\tilde{V}_0^k$ (respectively, $\tilde{W}_0^k$) be the aspiration levels as fuzzy numbers of Player I (respectively, Player II) corresponding to kth payoffs. The multi-objective matrix game with fuzzy goals and fuzzy payoffs, denoted by $MOFGR$, is defined as

$$MOFGR = (S^m, S^n, \tilde{A}^k, \tilde{V}_0^k, \tilde{\mathcal{P}}^T, \tilde{W}_0^k, \tilde{\mathcal{P}}_S, (k = 1, 2, \cdots, r)),$$

where the fuzzy relation $\tilde{\mathcal{P}}^T$ and $\tilde{\mathcal{P}}_S$ are the fuzzified versions of symbols $\geq$ and $\leq$ respectively. And $\tilde{\mathcal{P}}^T$ and $\tilde{\mathcal{P}}_S$ are dual fuzzy relations to each other. Now, we have the following definition to define the solution of $MOFGR$ using these fuzzy relations.

Definition 11: (solution of $MOFGR$). $(\bar{x}, \bar{y}) \in S^m \times S^n$ is called a solution of the multi-objective fuzzy matrix games $(MOFGR)$ if

$$\tilde{V}_0^k \ \tilde{\mathcal{P}}^T \, ((\bar{x})^T \tilde{A}^k y), \quad \forall y \in S^n,$$

$$(x^T \tilde{A}^k \bar{y}) \, \tilde{\mathcal{P}}_S \ \tilde{W}_0^k, \quad \forall x \in S^m,$$

where $\bar{x}$ is called an optimal strategy for Player *I* and $\bar{y}$ is called an optimal strategy for Player *II* .

By using Definition 11, we construct the following pair of the multi-objective fuzzy optimization problems for Player I $(MOGFP1)$ and Player II $(MOGFD1)$, respectively.

$(MOGFP1)$ Find $x \in S^m$ such that

$$\tilde{V}_0^k \ \tilde{\mathcal{P}}^T \, (x^T \tilde{A}^k y), \quad \forall y \in S^n,$$

and

$(MOGFD1)$ Find $y \in S^n$ such that

$$(x^T \tilde{A}^k y) \, \tilde{\mathcal{P}}_S \ \tilde{W}_0^k, \quad \forall x \in S^m, (k = 1, 2, \cdots, r).$$

According to Definition 8, we define membership functions for the fuzzy constraints $\tilde{V}_0^k \ \tilde{\mathcal{P}}^T \, (x^T \tilde{A}^k y)$ and $(x^T \tilde{A}^k y) \, \tilde{\mathcal{P}}_S \ \tilde{W}_0^k$ respectively as follows

$$u_{\tilde{\mathcal{P}}^T}(\tilde{V}_0^k \ , x^T \tilde{A}^k y) = \sup_{s, s' \in \mathbb{R}} \{T(u_{\mathcal{P}}(s, s'), T(u_{x^T \tilde{A}^k y}(s), u_{\tilde{V}_0^k}(s')))\},$$

$$u_{\tilde{\mathcal{P}}_S}(x^T \tilde{A}^k y \ , \tilde{W}_0^k) = \inf_{s, s'' \in \mathbb{R}} \{S(S(1 - u_{x^T \tilde{A}^k y}(s), 1 - u_{\tilde{W}_0^k}(s'')), u_{\mathcal{P}}(s'', s))\}.$$

Inspired by [4], problems $(MOGFP1)$ and $(MOGFD1)$ become the following pair of multi-objective optimization problems for Player I and Player II

$$\max_{x \in S^m} \min_{y \in S^n} \min_k \{u_{\tilde{\mathcal{P}}^T}(\tilde{V}_0^k \ , x^T \tilde{A}^k y)\},$$

$$\max_{y \in S^n} \min_{x \in S^m} \min_k \{u_{\tilde{\mathcal{P}}_S}(x^T \tilde{A}^k y \ , \tilde{W}_0^k)\}.$$

The above pair of multi-objective optimization problems can be obtained through the following crisp pair of multi-objective optimization problems for Player I $(MOGCP1)$ and Player II $(MOGCD1)$

$(MOGCP1)$ max λ such that

$$\lambda \leq u_{\tilde{\mathcal{P}}^T}(\tilde{V}_0^k \ , x^T \tilde{A}^k y), (k = 1, 2, \cdots, r),$$
$$0 \leq \lambda \leq 1,$$
$$\forall x \in S^m, \forall y \in S^n.$$

$(MOGCD1)$ max η such that

$$\eta \leq u_{\tilde{\mathcal{P}}_S}(x^T \tilde{A}^k y \ , \tilde{W}_0^k), (k = 1, 2, \cdots, r),$$
$$0 \leq \eta \leq 1,$$
$$\forall x \in S^m, \forall y \in S^n.$$

$(MOGCP1)$ and $(MOGCD1)$ are the crisp formulations of the multi-objective fuzzy optimization problems for Player I $(MOGFP1)$ and Player II $(MOGFD1)$. Therefore, these problems belong to the class of multi-objective non-linear optimization problems. ł In the following, we present certain special cases of the multi-objective fuzzy optimization problem $(MOGFP1)$ and $(MOGFD1)$.

IV. A MODEL FOR MULTI-OBJECTIVE FUZZY MATRIX GAMES

Let $X = \mathbb{R}$, $\tilde{B}, \tilde{D} \in \mathscr{F}(X)$, and the t-norm T and t-conorm S be min and max operators, respectively. The fuzzy relation $\tilde{\mathcal{P}}$ be $\lesssim$. Consequently, the fuzzy relation $\tilde{\mathcal{P}}^T$ and $\tilde{\mathcal{P}}_S$ are respectively denoted by $\lesssim^{min}$ and$\lesssim_{max}$. Here we note that $\lesssim^{min}$ and $\lesssim_{max}$ are dual to each other.

Consider a game in which the aspiration levels of Player I and Player II are crisp real numbers V_0^k and $W_0^k (k = 1, 2, \cdots r)$, and the payoff matrixes $A^k (k = 1, 2, \cdots r)$ are crisp matrixes having entries as real numbers. Hence, the multi-objective matrix game with fuzzy goals and fuzzy payoffs $(MOFGR)$ reduces to a matrix game model $(MOFG)$ [1]

$$(S^m, S^n, A^k, V_0^k, \lesssim^{min}, W_0^k, \lesssim_{max}, (k = 1, 2, \cdots r)).$$

Here, it is assumed that Player I and Player II is optimizing with regard to the ordering $\lesssim^{min}$ and$\lesssim_{max}$, respectively.

The problems of the multi-objective matrix game with fuzzy goals $(MOFG)$ have been discussed [1], and its results become a special case of the multi-objective matrix game with fuzzy goals and fuzzy payoffs model $(MOFGR)$.

In the discussion to follow, we assume that the aspiration levels of the two players, $\tilde{V}_0^k$ and $\tilde{W}_0^k$ $(k = 1, 2, \cdots, r)$, and the entries of the payoff matrixes $\tilde{A}^k$ $(k = 1, 2, \cdots, r)$ are fuzzy quantities. Therefore, the multi-objective matrix game with fuzzy goals and fuzzy payoffs $(MOFGR)$ becomes a matrix game model $(MOFGFP)$

$$(S^m, S^n, \tilde{A}^k, \tilde{V}_0^k, \lesssim^{min}, \tilde{W}_0^k, \lesssim_{max}, (k = 1, 2, \cdots, r)).$$

Here, it is assumed that Player I and Player II is optimizing with regard to the ordering $\lesssim^{min}$ and $\lesssim_{max}$, respectively.

Definition 12: (solution of $MOFGFP$). $(\bar{x}, \bar{y}) \in S^m \times S^n$ is called a solution of the multi-objective fuzzy matrix games $(MOFGFP)$ if

$$\tilde{V}_0^k \ \lesssim^{min} \ ((\bar{x})^T \tilde{A}^k y), \quad \forall y \in S^n,$$
$$(x^T \tilde{A}^k \bar{y}) \lesssim_{max} \ \tilde{W}_0^k, \quad \forall x \in S^m,$$

where $\bar{x}$ is called an optimal strategy for Player *I* and $\bar{y}$ is called an optimal strategy for Player *II* .

In this case, the multi-objective fuzzy optimization problems for Player I $(MOGFP1)$ and Player II $(MOGFD1)$, respectively, become

$(MOGFP2)$ Find $x \in S^m$ such that

$$\tilde{V}_0^k \ \lesssim^{min} \ (x^T \tilde{A}^k y), \quad \forall y \in S^n,$$

$(MOGFD2)$ Find $y \in S^n$ such that

$$(x^T \tilde{A}^k y) \lesssim_{max} \ \tilde{W}_0^k, \quad \forall x \in S^m, (k = 1, 2, \cdots, r).$$

Here, the membership functions for the fuzzy constraints are taken as

$$u_{\lesssim^{min}}(\tilde{V}_0^k \ , x^T \tilde{A}^k y) = \sup_{s, s' \in \mathbb{R}} \{min(u_{\leq}(s, s'), u_{x^T \tilde{A}^k y}(s), u_{\tilde{V}_0^k}(s'))\},$$

$$u_{\lesssim_{max}}(x^T \tilde{A}^k y \ , \tilde{W}_0^k) = \inf_{s, s'' \in \mathbb{R}} \{max(1 - u_{x^T \tilde{A}^k y}(s), 1 - u_{\tilde{W}_0^k}(s''), u_{\leq}(s'', s))\}.$$

Theorem 1: [3] Let $\tilde{B}, \tilde{D} \in \mathscr{F}(X)$ be normal and compact, $T = min$, $S = max$ and $\alpha \in (0, 1)$. Then

$$u_{\lesssim^T}(\tilde{B}, \tilde{D}) \geq \alpha \Leftrightarrow \inf[\tilde{B}]_\alpha \leq \sup[\tilde{D}]_\alpha,$$
$$u_{\lesssim_S}(\tilde{B}, \tilde{D}) \geq \alpha \Leftrightarrow \sup(\tilde{B})_{1-\alpha} \leq \inf(\tilde{D})_{1-\alpha}.$$

Theorem 2: Let $\tilde{A}^k$, $\tilde{V}_0^k$ and $\tilde{W}_0^k$ $(k = 1, 2, \cdots, r)$ be fuzzy quantities. Let $T = min$, $S = max$, $\alpha \in (0, 1)$ and $\leq$ be the usual binary relation "less than or equal to". Then, we obtain

$$u_{\lesssim^T}(\tilde{V}_0^k \ , x^T \tilde{A}^k y) \geq \alpha \Leftrightarrow (V_0^L(\alpha))^k \leq x^T (A^R(\alpha))^k y,$$
$$x \in S^m, y \in S^n (k = 1, 2, \cdots, r).$$

$$u_{\lesssim_S}(x^T \tilde{A}^k y \ , \tilde{W}_0^k) \geq \alpha \Leftrightarrow x^T (A^R(1 - \alpha))^k y \leq (W_0^L(1 - \alpha))^k,$$
$$x \in S^m, y \in S^n (k = 1, 2, \cdots, r).$$

Proof. Since $\tilde{A}^k$ $(k = 1, 2, \cdots, r)$ are fuzzy quantities, according to Definition 10 and Lemma 1, we obtain that $x^T \tilde{A}^k y$ $(k = 1, 2, \cdots, r)$ are fuzzy quantities. By Theorem 1, we have

$$u_{\lesssim^T}(\tilde{V}_0^k \ , x^T \tilde{A}^k y) \geq \alpha \Leftrightarrow \inf[\tilde{V}_0^k]_\alpha \leq \sup[x^T \tilde{A}^k y]_\alpha, \tag{11}$$

$$x \in S^m, y \in S^n (k = 1, 2, \cdots, r). \tag{12}$$

$$u_{\lesssim_S}(x^T \tilde{A}^k y \ , \tilde{W}_0^k) \geq \alpha \Leftrightarrow \sup(x^T \tilde{A}^k y)_{1-\alpha} \leq \inf(\tilde{W}_0^k)_{1-\alpha}, \tag{13}$$

$$x \in S^m, y \in S^n (k = 1, 2, \cdots, r). \tag{14}$$

Now combining (9) and (10), it follows that

$$(V_0^L(\alpha))^k = \inf[\tilde{V}_0^k]_\alpha, \tag{15}$$

$$x^T (A^R(\alpha))^k y = ((x^T \tilde{A}^k y)^R(\alpha))^k = \sup[x^T \tilde{A}^k y]_\alpha, \tag{16}$$

$$\begin{aligned} x^T (A^R(1 - \alpha))^k y &= ((x^T \tilde{A}^k y)^R(1 - \alpha))^k \\ &= \sup[x^T \tilde{A}^k y]_{1-\alpha} = \sup(x^T \tilde{A}^k y)_{1-\alpha}, \end{aligned} \tag{17}$$

$$(W_0^L(1 - \alpha))^k = \inf[\tilde{W}_0^k]_{1-\alpha} = \inf(\tilde{W}_0^k)_{1-\alpha}. \tag{18}$$

From (11)-(18), we conclude

$$u_{\lesssim^T}(\tilde{V}_0^k \ , x^T \tilde{A}^k y) \geq \alpha \Leftrightarrow (V_0^L(\alpha))^k \leq x^T (A^R(\alpha))^k y,$$
$$x \in S^m, y \in S^n (k = 1, 2, \cdots, r),$$
$$u_{\lesssim_S}(x^T \tilde{A}^k y \ , \tilde{W}_0^k) \geq \alpha \Leftrightarrow x^T (A^R(1 - \alpha))^k y \leq (W_0^L(1 - \alpha))^k,$$
$$x \in S^m, y \in S^n (k = 1, 2, \cdots, r).$$

Furthermore, by Theorem 2, the multi-objective fuzzy optimization problems of $(MOGFP2)$ and Player II $(MOGFD2)$ are equivalent to the following crisp linear programming problems $(MOGCLP1)$ and $(MOGCLD1)$, respectively,

$$\max \ ((V_0^L(\lambda))^1, (V_0^L(\lambda))^2, \cdots, (V_0^L(\lambda))^r)$$
$$\begin{aligned} such \ that \quad & (V_0^L(\lambda))^k \leq x^T (A^R(\lambda))^k y, \\ & 0 \leq \lambda \leq 1, \\ & \forall x \in S^m, \forall y \in S^n, \ (k = 1, 2, \cdots, r). \end{aligned}$$

$$\min \ ((W_0^L(1 - \eta))^1, (W_0^L(1 - \eta))^2, \cdots, (W_0^L(1 - \eta))^r)$$
$$\begin{aligned} such \ that \quad & x^T (A^R(1 - \eta))^k y \leq (W_0^L(1 - \eta))^k, \\ & 0 \leq \eta \leq 1, \\ & \forall x \in S^m, \forall y \in S^n, \ (k = 1, 2, \cdots, r). \end{aligned}$$

That is equivalent to

$$(MOGCLP2) \ \max \ ((V_0^L(\lambda))^1, (V_0^L(\lambda))^2, \cdots, (V_0^L(\lambda))^r)$$
$$\begin{aligned} such \ that \quad & (V_0^L(\lambda))^k \leq x^T (A^R(\lambda))^k y, \\ & 0 \leq \lambda \leq 1, \\ & \forall x \in S^m, \forall y \in S^n, \ (k = 1, 2, \cdots, r). \end{aligned}$$

$$(MOGCLD2) \ \min \ ((W_0^L(\lambda))^1, (W_0^L(\lambda))^2, \cdots, (W_0^L(\lambda))^r)$$
$$\begin{aligned} such \ that \ & x^T (A^R(\lambda))^k y \leq (W_0^L(\lambda))^k, \\ & 0 \leq \lambda \leq 1, \\ & \forall x \in S^m, \forall y \in S^n, \ (k = 1, 2, \cdots, r). \end{aligned}$$

Since S^m and S^n are convex polytopes. And the problems $(MOGCLP2)$ and $(MOGCLD2)$ are crisp multi-objective linear programming problems, it is sufficient to consider only the extreme points of S^m and S^n. Thus solving the above problems $(MOGCLP2)$ and $(MOGCLD2)$ is equivalent to solving the following problems $(MOGCLP3)$ and $(MOGCLD3)$

$$(MOGCLP3) \ \max \ ((V_0^L(\lambda))^1, (V_0^L(\lambda))^2, \cdots, (V_0^L(\lambda))^r)$$
$$\begin{aligned} & such \ that \\ & \sum_{i=1}^{m} (a_{ij}^R(\lambda))^k x_i \geq (V_0^L(\lambda))^k, \\ & 0 \leq \lambda \leq 1, \\ & \forall x \in S^m, \forall y \in S^n, \ (k = 1, 2, \cdots, r, j = 1, 2, \cdots, n). \end{aligned}$$

$$(MOGCLD3)\ \min\ \ ((W_0^L(\lambda))^1,(W_0^L(\lambda))^2,\cdots,(W_0^L(\lambda))^r)$$

$$\begin{aligned}&such\ that\\&\sum_{j=1}^{n}((a_{ij}^R(\lambda))^k y\le (W_0^L(\lambda))^k,\\&0\le\lambda\le 1,\\&\forall x\in S^m,\forall y\in S^n,\ (k=1,2,\cdots,r,i=1,2,\cdots,m).\end{aligned}$$

Equivalently, for solving the multi-objective matrix game with fuzzy goals and fuzzy payoffs $(MOFGR)$, we have to solve $(MOGCLP3)$ and $(MOGCLD3)$ for Player I and Player II, respectively. Moreover, if $(\bar{x}(\lambda),((\bar{V}_0^L(\lambda))^1,(\bar{V}_0^L(\lambda))^2,\cdots,(\bar{V}_0^L(\lambda))^r))$ is an optimal solution of $(MOGCLP3)$ then $\bar{x}(\lambda)$ is an optimal strategy of Player I with $((\bar{V}_0^L(\lambda))^1,(\bar{V}_0^L(\lambda))^2,\cdots,(\bar{V}_0^L(\lambda))^r)$ as a lower bound of the greatest aspiration of at the level λ. Similarly, $(\bar{y}(\lambda),((\bar{W}_0^L(\lambda))^1,(\bar{W}_0^L(\lambda))^2,\cdots,(\bar{W}_0^L(\lambda))^r))$ is an optimal solution of $(MOGCLD3)$ then $\bar{y}(\lambda)$ is an optimal strategy of Player II with $((\bar{W}_0^L(\lambda))^1,(\bar{W}_0^L(\lambda))^2,\cdots,(\bar{W}_0^L(\lambda))^r)$ as a upper bound of the greatest aspiration of at the level λ. we shall call $((\bar{V}_0^L(\lambda))^1,(\bar{V}_0^L(\lambda))^2,\cdots,(\bar{V}_0^L(\lambda))^r)$ as the λ-acceptable value for Player I and $((\bar{W}_0^L(\lambda))^1,\bar{W}_0^L(\lambda))^2,\cdots,(\bar{W}_0^L(\lambda))^r)$ as the $(1-\lambda)$-acceptable value for Player II.

The above discussion then leads to the following theorem:

Theorem 3: Let $\lambda \in (0,1]$ be fixed. Suppose $(\bar{x}(\lambda),((\bar{V}_0^L(\lambda))^1,(\bar{V}_0^L(\lambda))^2,\cdots,(\bar{V}_0^L(\lambda))^r))$ and $(\bar{y}(\lambda),((\bar{W}_0^L(\lambda))^1,(\bar{W}_0^L(\lambda))^2,\cdots,(\bar{W}_0^L(\lambda))^r))$ be the optimal solutions of the pair multi-objective linear programming problems $(MOGCLP3)$ and $(MOGCLD3)$ for Player I and Player II, respectively. Then $\bar{x}(\lambda)$ (respectively, $\bar{y}(\lambda)$) is an optimal strategy for Player I (respectively, Player II), and $((\bar{V}_0^L(\lambda))^1,(\bar{V}_0^L(\lambda))^2,\cdots,(\bar{V}_0^L(\lambda))^r)$ (respectively, $((\bar{W}_0^L(\lambda))^1,(\bar{W}_0^L(\lambda))^2,\cdots,(\bar{W}_0^L(\lambda))^r)$) is an λ-acceptable value (respectively, $(1-\lambda)$-acceptable value) for Player I (respectively, Player II).

V. CONCLUSION

In this paper, a generalized model for a multi-objective matrix game with fuzzy goals and fuzzy payoffs via fuzzy relation approach is studied. The inspiration of the model is from [1], and it is shown that solving such a game is equivalent to solving a pair of multi-objective non-linear optimization problems. And then the pair of multi-objective non-linear optimization problems can be reduced to the two mutually dual multi-objective linear programming problems. we have also concluded that the model with entropy is becoming more and more significant and it is related to practical problems of our real life in such a competitive system [6], [14], [16], [17], [20], [24], [26]. Furthermore, we will put the certain results of the model into multi-objective fuzzy matrix games under (fuzzy) entropy environment. It would be interesting and challenging for us to explore this approach for multi-objective fuzzy matrix games in our future research.

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