Almost Sure Exponential Stability of Nonlinear Stochastic Delayed Systems with Markovian Switching and Lévy Noises

Chao Wei

Abstract—**This paper is concerned with the almost sure exponential stability of nonlinear stochastic delayed systems with Markovian switching and Lévy noises. By the technique of Burkholder-Davis-Gundy inequality, Chebyshev inequality, Borel-Cantelli lemma and generalized Itô formula for Lévy stochastic integral, we propose the sufficient conditions to guarantee the almost sure exponential stability of the system. A numerical example is provided to show the usefulness of the proposed almost sure exponential stability criterion.**

Index Terms—**Nonlinear stochastic delayed system, Markovian switching, Lévy noises, almost sure exponential stability.**

I. Introduction

When the system has time lag, the performance of the system may change, sometimes the existence of time lag could damage the stability of the system. According to the time lag studied based on the relationship between the time lag and the mode, it can be divided into constant time delay, time-varying delay and time-varying delay associated with model, which makes it difficult to discuss the stability of time-delay systems. As far as we know, the sufficient conditions for stability of time-delay stochastic system given by scholars are mainly focused on it is whether related to time lag. The Lyapunov functional is always used to solve this problem and the results obtained are asymptotic stability in probability and second order mean square stability([1], [2]). However, Mao([8]) studied the almost sure stability of time-delay nonlinear stochastic system based on LaSalle theory. Then, Huang and Mao([5]), Yuang and Mao([24]) discussed the almost sure stability of time-delay nonlinear stochastic system with Markovian switching. When considering the almost sure convergence speed, due to the complexity of the mathematical tools to be used, there is little research in this area.

The problem of stability regarding jump diffusion systems([11], [12], [16]–[18], [21]) or systems with Lévy noise([4], [23]) has attracted scholars' attention in the past few decades. Exponential or asymptotic stability conditions have been presented for these stochastic systems. Applebaum([3]) proposed that Lévy noise can be decomposed into a continuous part and a jump part which respectively correspond to the diffusion and jump term in systems by Lévy-Itô decomposition. In the meantime, stability issues of stochastic systems with Markovian switching have become an increasing interest([6], [9], [10], [13], [20], [26]). A Markovian switching system is a hybrid system with state vector that has two components. The first one is regarded as the state while the second one as the mode. Governed by a Markov chain with a finite state space, the system switches from one mode to another in a random way([25]). This switching manner is more suitable for the description of random failures, abrupt changes or sudden disturbances arising in many real systems. Nowadays, stability analysis for jump diffusion systems with Markovian switching([7], [14], [15], [19], [22], [27]) or hybrid systems with jump([25]) tends to be a new research focus. On the other hand, time delays, which commonly appear in practical systems, are often the cause of instability. Hence, the stability of stochastic delay systems with Markovian switching and Lévy noises is always the hot area in many researches and there is little research in this area. In this paper, the almost sure exponential stability of nonlinear stochastic delayed systems with Markovian switching and Lévy noises is analyzed and the sufficient conditions are proposed by using the technique of Burkholder-Davis-Gundy inequality, Chebyshev inequality, Borel-Cantelli lemma and generalized Itô formula for Lévy stochastic integral. Moreover, the results obtained are extended to the generalization to semi-martingale noises and the proof is provided.

This paper is organized as follows. In Section 2, the n-dimensional nonlinear stochastic delayed systems with Markovian switching and Lévy noises is introduced and some important lemmas are given. In Section 3, some sufficient conditions are proposed to guarantee the almost sure exponential stability of the system. In Section 4, a numerical example is provided to show the usefulness of our results. The conclusion is given in Section 5.

II. Problem Formulation and Preliminaries

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a basic probability space equipped with a right continuous and increasing family of σ-algebras $(\{\mathscr{F}_t\}_{t\geq 0})$. Denote by $\mathcal{C}^{2,1}(\mathbb{R}^n \times [-\tau, \infty) \times \mathbb{S}; \mathbb{R}_+$ the family of positive real-valued functions defined on $\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}$ which are continuously twice differentiable in $x \in \mathbb{R}^n$ and once differentiable in $t \in \mathbb{R}_+$.

Let $r(t), t \geq 0$ be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, ..., N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & i \neq j \\ 1 + \gamma_{ii}\Delta + o(\Delta) & i = j \end{cases}$$

where $\Delta > 0$, $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\Sigma_{i\neq j}\gamma_{ij}$.

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Chao Wei is with the School of Mathematics and Statistics, Anyang Normal University, Anyang 455000, China.(Email: chaowei0806@aliyun.com..

Consider the n-dimensional nonlinear stochastic delayed systems with Markovian switching and Lévy noises

$$\begin{aligned} & dx(t) \qquad (1)\\ = \ & f(x(t), x(t-\tau(t,r(t))), t, r(t))dt \\ + \ & g(x(t), x(t-\tau(t,r(t))), t, r(t))dW(t) \\ + \ & \int_Y H(x(t^-), x((t-\tau(t,r(t)))^-), t, r(t), y) \\ & N(dt, dy), \end{aligned}$$

where $x(0) = x_0 \in \mathcal{C}^b_{\mathscr{F}_0}([-\tau, 0); \mathbb{R}^n)$, $r(0) = r_0 \in \mathbb{S}$, $x(t^- = \lim_{s \downarrow t} x(s)$, $\tau(t, r(t)) : \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}_+$ is is a Borel measurable function which stands for the time lag, $W(t)$ is an m-dimensional $\mathscr{F}_t$-adapted Brownian motion, $N(t, y)$ is an l-dimensional $\mathscr{F}_t$-adapted Poisson random measure on $[0, +\infty) \times \mathbb{R}^l$ with compensator $\widetilde{N}(t, y)$ which satisfies $\widetilde{N}(t, y) = N(dt, dy) - \nu(dy)dt$, $\nu(dy)$ is a Lévy measure, $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^{n\times m}$, $H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^{n\times l}$.

Remark 1: $\tau(t, r(t))$ is non-negative differential function and for all $t \geq 0$ and $i \in \mathbb{S}$, there exist non-negative constants $l_i, \tau_i, \delta_i, \overline{\delta_i}, \overline{\delta}, l, \tau$ satisfying

$$l_i \leq \tau(t, i) \leq \tau_i, \quad l \leq \tau(t, i) \leq \tau,$$

$$\tau_t(t, i) = \frac{\partial \tau(t, i)}{\partial t} \leq \delta_i.$$

$$\overline{\delta_i} = \delta_i + \gamma_{ii} l_i + \Sigma_{i \neq j} \gamma_{ij} \tau_j \leq \overline{\delta} < 1.$$

We further assume that $W(t)$, $N(t, y)$, $r(t)$ in system 1 are independent.

For the purpose of stability study in this paper we impose the following assumptions.

Assumption 1: The functions f, g and H satisfy $f(0, 0, t, i) = g(0, 0, t, i) = H(0, 0, t, i, y) \equiv 0$.

Assumption 2: $|f(x, \xi, t, i) - f(\overline{x}, \overline{\xi}, t, i)|^2 + |g(x, \xi, t, i) - g(\overline{x}, \overline{\xi}, t, i)|^2 + \int_Y |H(x, \xi, t, i, y) - H(\overline{x}, \overline{\xi}, t, i, y)|^2 \nu(dy) \leq L(|x - \overline{x}|^2 + |\xi - \overline{\xi}|^2)$.

Remark 2: According to Assumption 2, it is easy to check that when $t \geq -\tau$, for each $x_0 \in \mathcal{C}^b_{\mathscr{F}_0}([-\tau, 0); \mathbb{R}^n)$, system 1 has unique solution.

Definition 1: The solution of system 1 is said to be almost sure exponential stability if there exists $\lambda > 0$ satisfying

$$\lim_{t\to\infty} \sup \frac{1}{t} \log(|x(t; x_0, r_0)|) \leq -\lambda,$$

for any $\xi \in \mathcal{C}^b_{\mathscr{F}_0}([-\tau, 0); \mathbb{R}^n)$ and $r_0 \in \mathbb{S}$.

Given $V \in \mathcal{C}^{2,1}(\mathbb{R}^n \times [-\tau, \infty) \times \mathbb{S}; \mathbb{R}_+)$, we define the operator $\mathcal{L}V$ by

$$\begin{aligned} & \mathcal{L}V(x, \xi, t, i) \\ = \ & V_t(x, t, i) + V_x(x, t, i) f(x, \xi, t, i) \\ + \ & \frac{1}{2} trace[g^T(x, \xi, t, i) V_{xx}(x, t, i) g(x, \xi, t, i)] \\ + \ & \int_Y \sum_{k=1}^{l} [V(x + H^k(x, \xi, t, i, y_k), t, i) \\ & -V(x, t, i)] \nu_k(dy_k) + \sum_{j=1}^{N} \gamma_{ij} V(x, t, j). \end{aligned}$$

Then the generalized Itô formula can be given as follows:

$$\begin{aligned} & V(x, t, r(t)) \qquad (2)\\ = \ & V(x_0, 0, r_0) \\ + \ & \int_0^t \mathcal{L}V(x(s), x(s - \tau(s, r(s))), s, r(s))ds \\ + \ & \int_0^t V_x(x(s), s, r(s)) g(x(s), x(s - \tau(s, r(s))), s, \\ & r(s))dW(s) \\ + \ & \sum_{k=1}^{l} \int_0^t \int_Y [V(x(s^-) + H^k(x(s^-), \\ & x((s - \tau(s, r(s)))^-), s, r(s), y_k), s, r(s)) \\ & -V(x(s^-), s, r(s))] \widetilde{N}(ds, dy_k) \\ + \ & \int_0^t \int_Y [V(x(s^-), s, r_0 + c(r(s), u)) \\ & -V(x(s^-), s, r(s))] \mu(ds, du). \end{aligned}$$

The details of the function c and the martingale measure $\mu(ds, du)$ can be seen in [15]. Obviously (2) holds if we replace 0 and t with bounded stopping time τ_1 and τ_2 respectively. Thus the following lemma is derived.

Lemma 1: Let τ_1, τ_2 be bounded stopping times such that $0 \leq \tau_1 \leq \tau_2$ a.s. If $V(x(t), t, r(t))$ and $\mathcal{L}V(x(t), x(t - \tau(t, r(t)), t, r(t))$ are bounded on $t \in [\tau_1, \tau_2]$ with probability 1, then

$$\begin{aligned} & \mathbb{E}V(x(\tau_2), \tau_2, r(\tau_2)) \\ = \ & \mathbb{E}V(x(\tau_1), \tau_1, r(\tau_1)) \\ + \ & \mathbb{E} \int_{\tau_1}^{\tau_2} \mathcal{L}V(x(s), x(s - \tau(s, r(s))), s, r(s))ds. \end{aligned}$$

Proof: Replace 0 and t in (2) with τ_1 and τ_2, by taking expectation on both side of (2), it is easy to check the results. ■

We also need some lemmas such as Burkholder-Davis-Gundy inequality, Chebyshev inequality and Borel-Cantelli lemma as follows.

Lemma 2: (Burkholder-Davis-Gundy inequality) For $t \geq 0$, let $x(t) = \int_0^t g(s)dB(s)$ $A(t) = \int_0^t |g(s)|^2 ds$. Then, for any $p > 0$, there exist positive constants c_p and C_p satisfying

$$c_p \mathbb{E}|A(t)|^{\frac{p}{2}} \leq \mathbb{E}(\sup_{0\leq s\leq t} |x(s)|^p) \leq C_p \mathbb{E}|A(t)|^{\frac{p}{2}},$$

where

$$\begin{cases} c_p = (\frac{p}{2})^p, \quad C_p = (\frac{32}{p})^{\frac{p}{2}}, \quad 0 < p < 2; \\ c_p = 1, \quad C_p = 4, \quad p = 2; \\ c_p = (2p)^{-\frac{p}{2}}, \quad C_p = (\frac{p^{p+1}}{2(p-1)^{p-1}}))^{\frac{p}{2}}, \quad p > 2. \end{cases}$$

Lemma 3: (Borel-Cantelli lemma) For the complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathrm{P})$,

(1) if $\{\mathcal{A}_k\} \subset \mathscr{F}$ and $\sum_{k=1}^{\infty} \mathrm{P}(\mathcal{A}_k) < \infty$, then

$$\mathrm{P}(\limsup_{k\to\infty} \mathcal{A}_k) = 0.$$

Namely, there exist a positive constant k_0 and set Ω_0, where $\Omega_0 \in \mathscr{F}$ and satisfying $\mathrm{P}(\Omega_0) = 1$, for any $\omega \in \Omega_0$, it follows that

$$\omega \notin \mathcal{A}_k \quad k \geq k_0.$$

(2) If $\{\mathcal{A}_k\} \subset \mathscr{F}$ are independent and $\sum_{k=1}^{\infty} \mathrm{P}(\mathcal{A}_k) = \infty$, then

$$\mathrm{P}(\limsup_{k\to\infty} \mathcal{A}_k) = 1.$$

Namely, there exist a set Ω_1 satisfying $\mathrm{P}(\Omega_1) = 1$, and $\mathcal{A}_{k_i}$, for any $\omega \in \Omega_1$, it follows that

$$\omega \in \mathcal{A}_{k_i} \quad \forall i \in \mathcal{I}_+.$$

Lemma 4: (Chebyshev inequality) If $c > 0$,$p > 0$,$X \in \mathrm{L}^p$, then

$$\mathbb{P}\{\omega | X(\omega) \geq c|\} \leq c^{-p}\mathbb{E}|X|^p.$$

III. Main Result and Proofs

In the following theorem, some sufficient conditions are proposed to guarantee the almost sure exponential stability of the system (17).

Theorem 1: Under Assumptions 1 and 2, if there exist a function $V(x,t,i) \in \mathcal{C}^{2,1}(\mathbb{R}^n \times [-\tau,\infty) \times \mathbb{S}; \mathbb{R}_+$ and positive constants $c_1, c_2, \lambda_1, \lambda_2$ such that

$$c_1|x|^2 \leq V(x,t,i) \leq c_2|x|^2,$$
$$\mathcal{L}V(x,\xi,t,i) \leq -\lambda_1|x|^2 + \lambda_2|\xi|^2, t > 0$$

for any $(x,t,i) \in (\mathbb{R}^n \times [-\tau,\infty) \times \mathbb{S})$, then the system 1 is almost sure exponential stability.

Proof: For any $i \in \mathbb{S}$, when $t > 0$, according to Lemma 1, it follows that

$$\begin{aligned} & \mathbb{E}[e^{\lambda t}V(x(t),t,i)] \\ = & \ \mathbb{E}[V(x_0,0,i)] + \mathbb{E}\int_0^t e^{\lambda s}[\lambda V(x(s),s,i) \\ + & \ \mathcal{L}V(x(s),x(s-\tau(s,i)),s,i)]ds. \end{aligned}$$

Since

$$\begin{aligned} & d\tau(s,i) \\ = & \ [\tau_s(s,i) + \sum_{j=1}^N \gamma_{ij}\tau(s,j)]ds \\ \leq & \ [\delta_i + \sum_{j=1}^N \gamma_{ij}\tau_j]ds \leq \bar{\delta}ds. \end{aligned}$$

Then we obtain that

$$\begin{aligned} & \mathbb{E}\int_0^t e^{\lambda s}|x(s-\tau(s,i))|^2 ds \\ \leq & \ \frac{e^{\gamma\tau}}{1-\bar{\delta}}\mathbb{E}[\int_{-\tau}^0 |x(s)|^2 ds + \int_0^t e^{\lambda s}|x(s)|^2 ds] \\ \leq & \ \frac{\tau|x_0|^2 e^{\gamma\tau}}{1-\bar{\delta}} + \frac{e^{\gamma\tau}}{1-\bar{\delta}}\mathbb{E}\int_0^t e^{\lambda s}|x(s)|^2 ds. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \mathbb{E}[e^{\lambda t}V(x(t),t,i)] \\ \leq & \ c_2|x_0|^2 + \mathbb{E}\int_0^t e^{\lambda s}[\lambda c_2|x_s|^2 - \lambda_1|x_s|^2 \\ & +\lambda_2|x(s-\tau(s,i))|^2]ds \\ \leq & \ c_2|x_0|^2 + \frac{\lambda_2\tau e^{\gamma\tau}}{1-\bar{\delta}}|x_0|^2 \\ + & \ \mathbb{E}\int_0^t e^{\lambda s}[\lambda c_2|x_s|^2 - \lambda_1|x_s|^2 + \frac{\lambda_2 e^{\gamma\tau}}{1-\bar{\delta}}|x_s|^2]ds \\ \leq & \ \frac{c_2 + \lambda_2\tau e^{\gamma\tau}}{1-\bar{\delta}}|x_0|^2. \end{aligned}$$

Let $M = \frac{c_2+\lambda_2\tau e^{\gamma\tau}}{c_1(1-\bar{\delta})}$, it can be check that

$$\mathbb{E}|x(t)|^2 \leq Me^{-\lambda t}, t \geq 0. \tag{3}$$

Then, for any $\varepsilon \in (0, \frac{\lambda}{2})$,

$$\mathbb{E}|x(t)|^2 \leq Me^{-(\lambda-\varepsilon)t}, t \geq 0. \tag{4}$$

For any $\delta > 0$, there exists $k_0(\delta)$ satisfying $(k_0-1)\delta \geq \tau$. Let $k = k_0, k_0+1, ...$, we get

$$\begin{aligned} & \mathbb{E}[\sup_{(k-1)\delta\leq t\leq k\delta} |x(t)|^2] \\ \leq & \ 4\mathbb{E}|x((k-1)\delta)|^2 \\ + & \ 4\mathbb{E}[\int_{(k-1)\delta}^{k\delta} |f(x(s),x(s-\tau(s)),s,r(s))|ds]^2 \\ + & \ 4\mathbb{E}[\sup_{(k-1)\delta\leq s\leq k\delta}\int_{(k-1)\delta}^t |g(x(s),x(s-\tau(s)), \\ & s,r(s))dW(s)|^2] \\ + & \ 4\mathbb{E}[\sup_{(k-1)\delta\leq s\leq k\delta}|\int_{(k-1)\delta}^t \\ & \int_Y H(x(s^-),x((s-\tau(s)^-),s,r(s),y)N(ds,dy)|^2]. \end{aligned}$$

By using Burkholder-Davis-Gundy inequality and Lévy stochastic integral, it follows that

$$\begin{aligned} & \mathbb{E}[\sup_{(k-1)\delta\leq s\leq k\delta}|\int_{(k-1)\delta}^t \\ & \int_Y H(x(s^-),x((s-\tau(s)^-),s,r(s),y)N(ds,dy)|^2] \\ \leq & \ \mathbb{E}[\sup_{(k-1)\delta\leq s\leq k\delta}\int_{(k-1)\delta}^t \\ & \int_Y |H(x(s^-),x((s-\tau(s)^-),s,r(s),y)|^2\nu(dy)ds] \\ \leq & \ L\delta\mathbb{E}[\sup_{(k-1)\delta\leq s\leq k\delta}(|x(s)|^2 + |x((s-\tau(s))|^2)], \end{aligned}$$

$$\begin{aligned} & \mathbb{E}[\int_{(k-1)\delta}^{k\delta} |f(x(s),x(s-\tau(s)),s,r(s))|ds]^2 \\ \leq & \ \mathbb{E}[\delta\sup_{(k-1)\delta\leq s\leq k\delta}|f(x(s),x(s-\tau(s)),s,r(s))|]^2 \\ \leq & \ \delta^2 L\mathbb{E}[\sup_{(k-1)\delta\leq s\leq k\delta}(|x(s)|^2 + |x((s-\tau(s))|^2)], \end{aligned}$$

and

$$\begin{aligned} & \mathbb{E}[\sup_{(k-1)\delta\leq s\leq k\delta}|\int_{(k-1)\delta}^t g(x(s),x(s-\tau(s)),s, \\ & r(s))dW(s)|^2] \\ \leq & \ C_p\mathbb{E}[\int_{(k-1)\delta}^{k\delta} g^2(x(s),x(s-\tau(s)),s,r(s))ds] \\ \leq & \ C_p\delta L\mathbb{E}[\sup_{(k-1)\delta\leq s\leq k\delta}(|x(s)|^2 + |x((s-\tau(s))|^2)], \end{aligned}$$

where C_p is a constant.

Assume that $L\delta(1+\delta+C_p) < \frac{1}{8}$.

Hence, it can be checked that

$$\begin{aligned}
& \mathbb{E}[\sup_{(k-1)\delta\le t\le k\delta}|x(t)|^2] \\
\le\ & 4Me^{-(\lambda-\varepsilon)(k-1)\delta} \\
+\ & 4L\delta(1+\delta+C_p) \\
& \mathbb{E}[\sup_{(k-1)\delta\le s\le k\delta}(|x(s)|^2+|x((s-\tau(s))|^2)] \\
\le\ & 4Me^{-(\lambda-\varepsilon)(k-1)\delta} \\
+\ & \frac{1}{2}\mathbb{E}[\sup_{(k-1)\delta\le s\le k\delta}(|x(s)|^2+|x((s-\tau(s))|^2)].
\end{aligned}$$

Since

$$\mathbb{E}[\sup_{(k-1)\delta\le s\le k\delta}|x(s-\tau(s))|^2]\le Me^{-(\lambda-\varepsilon)(k-1)\delta-\tau}, \quad (5)$$

we obtain that

$$\mathbb{E}[\sup_{(k-1)\delta\le t\le k\delta}|x(t)|^2]\le 5Me^{-(\lambda-\varepsilon)(k-1)\delta-\tau}. \quad (6)$$

From Chebyshev inequality, it is easy to check that

$$\begin{aligned}
& \mathbb{P}\{\omega:\sup_{(k-1)\delta\le t\le k\delta}|x(t)|>e^{-(\lambda-2\varepsilon)((k-1)\delta-\tau)/2}\} \\
\le\ & \frac{\mathbb{E}[\sup_{(k-1)\delta\le t\le k\delta}|x(t)|^2]}{e^{-(\lambda-2\varepsilon)(k-1)\delta-\tau}} \\
\le\ & 5Me^{-\varepsilon((k-1)\delta-\tau)}.
\end{aligned}$$

According to Borel-Cantelli lemma, for all $\omega\in\Omega$ and except some k, it follows that

$$\sup_{(k-1)\delta\le t\le k\delta}|x(t)|\le e^{-(\lambda-2\varepsilon)((k-1)\delta-\tau)/2}. \quad (7)$$

Therefore, for almost every $\omega\in\Omega$, if $(k-1)\delta\le t\le k\delta$ and $k\ge\max\{k_0,k_1\}$, we obtain

$$\lim_{t\to\infty}\sup\frac{1}{t}\log(|x(t)|)\le-\frac{\lambda-2\varepsilon}{2}. \quad (8)$$

Let $\varepsilon\to 0$, the system 1 is almost sure exponential stability.

The proof is complete. ∎

Remark 3: When system 1 is changed to another expression as follows:

$$\begin{aligned}
& dx(t) && (9)\\
=\ & f(x(t),x(t-\tau(t,r(t))),t,r(t))dt \\
+\ & g(x(t),x(t-\tau(t,r(t))),t,r(t))dL(t),
\end{aligned}$$

where

$$L_t=B_t+\int_0^t\int_{|z|>1}zN(ds,dz)+\int_0^t\int_{|z|\le1}z\widetilde{N}(ds,dz),$$

$(B_t,t\ge 0)$ is a standard Brownian motion, $N(ds,dz)$ is a Poisson random measure independent of $(B_t,t\ge 0)$ with characteristic measure $dt\nu(dz)$, and $\widetilde{N}(ds,dz)=N(ds,dz)-\nu(dz)$ is a martingale measure.

Therefore, the Assumptions 1-2 are changed to

Assumption 3: The functions f, g satisfy $f(0,0,t,i)=g(0,0,t,i)\equiv 0$.

Assumption 4: $|f(x,\xi,t,i)-f(\overline{x},\overline{\xi},t,i)|^2+|g(x,\xi,t,i)-g(\overline{x},\overline{\xi},t,i)|^2\le L(|x-\overline{x}|^2+|\xi-\overline{\xi}|^2)$.

Then, under the Assumptions 3-4, by using the same methods in Theorem 1, it is easy to check that

$$\lim_{t\to\infty}\sup\frac{1}{t}\log(|x(t;x_0,r_0)|)\le 0.$$

Therefore, the system 9 is almost sure exponential stability.

Remark 4: When system 1 is changed to another expression as follows:

$$\begin{aligned}
& dx(t) && (10)\\
=\ & f(x(t),x(t-\tau(t,r(t))),t,r(t))dt \\
+\ & g(x(t),x(t-\tau(t,r(t))),t,r(t))dQ(t),
\end{aligned}$$

where

$$Q_t=Q_0+M_t+A_t$$

be a semi-martingale, M_t is a local martingale and A_t is a finite variation process.

Therefore, the Assumptions 1-2 are changed to

Assumption 5: The functions f, g satisfy $f(0,0,t,i)=g(0,0,t,i)\equiv 0$.

Assumption 6: $|f(x,\xi,t,i)-f(\overline{x},\overline{\xi},t,i)|^2+|g(x,\xi,t,i)-g(\overline{x},\overline{\xi},t,i)|^2\le L(|x-\overline{x}|^2+|\xi-\overline{\xi}|^2)$.

For any $i\in\mathbb{S}$, when $t>0$, according to Lemma 1, it follows that

$$\begin{aligned}
& \mathbb{E}[e^{\lambda t}V(x(t),t,i)] \\
=\ & \mathbb{E}[V(x_0,0,i)]+\mathbb{E}\int_0^t e^{\lambda s}[\lambda V(x(s),s,i) \\
+\ & \mathcal{L}V(x(s),x(s-\tau(s,i)),s,i)]ds.
\end{aligned}$$

Since

$$\begin{aligned}
& d\tau(s,i) \\
=\ & [\tau_s(s,i)+\sum_{j=1}^N\gamma_{ij}\tau(s,j)]ds \\
\le\ & [\delta_i+\sum_{j=1}^N\gamma_{ij}\tau_j]ds\le\overline{\delta}ds.
\end{aligned}$$

Then we obtain that

$$\begin{aligned}
& \mathbb{E}\int_0^t e^{\lambda s}|x(s-\tau(s,i))|^2ds \\
\le\ & \frac{e^{\gamma\tau}}{1-\overline{\delta}}\mathbb{E}[\int_{-\tau}^0|x(s)|^2ds+\int_0^t e^{\lambda s}|x(s)|^2ds] \\
\le\ & \frac{\tau|x_0|^2e^{\gamma\tau}}{1-\overline{\delta}}+\frac{e^{\gamma\tau}}{1-\overline{\delta}}\mathbb{E}\int_0^t e^{\lambda s}|x(s)|^2ds.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& \mathbb{E}[e^{\lambda t}V(x(t),t,i)] \\
\le\ & c_2|x_0|^2+\mathbb{E}\int_0^t e^{\lambda s}[\lambda c_2|x_s|^2-\lambda_1|x_s|^2 \\
& +\lambda_2|x(s-\tau(s,i))|^2]ds \\
\le\ & c_2|x_0|^2+\frac{\lambda_2\tau e^{\gamma\tau}}{1-\overline{\delta}}|x_0|^2 \\
+\ & \mathbb{E}\int_0^t e^{\lambda s}[\lambda c_2|x_s|^2-\lambda_1|x_s|^2+\frac{\lambda_2e^{\gamma\tau}}{1-\overline{\delta}}|x_s|^2]ds \\
\le\ & \frac{c_2+\lambda_2\tau e^{\gamma\tau}}{1-\overline{\delta}}|x_0|^2.
\end{aligned}$$

Let $M=\frac{c_2+\lambda_2\tau e^{\gamma\tau}}{c_1(1-\bar{\delta})}$, it can be check that

$$\mathbb{E}|x(t)|^2 \le Me^{-\lambda t}, t\ge 0. \tag{11}$$

Then, for any $\varepsilon \in (0,\frac{\lambda}{2})$,

$$\mathbb{E}|x(t)|^2 \le Me^{-(\lambda-\varepsilon)t}, t\ge 0. \tag{12}$$

For any $\delta>0$, there exists $k_0(\delta)$ satisfying $(k_0-1)\delta \ge \tau$. Let $k=k_0, k_0+1, ...$, we get

$$\begin{aligned}
&\mathbb{E}[\sup_{(k-1)\delta\le t\le k\delta}|x(t)|^2]\\
\le\;& 4\mathbb{E}|x((k-1)\delta)|^2\\
+\;& 4\mathbb{E}[\int_{(k-1)\delta}^{k\delta}|f(x(s),x(s-\tau(s)),s,r(s))|ds]^2\\
+\;& 4\mathbb{E}[\sup_{(k-1)\delta\le s\le k\delta}\int_{(k-1)\delta}^{t}|g(x(s),x(s-\tau(s)),s,r(s))\\
& dQ(s)|^2].
\end{aligned}$$

By using Lévy stochastic integral, it follows that

$$\begin{aligned}
&\mathbb{E}[\sup_{(k-1)\delta\le s\le k\delta}\int_{(k-1)\delta}^{t}|g(x(s),x(s-\tau(s)),s,r(s))\\
& dQ(s)|^2]\\
\le\;& \mathbb{E}[\sup_{(k-1)\delta\le s\le k\delta}\int_{(k-1)\delta}^{t}\\
& |g(x(s),x(s-\tau(s)),s,r(s))|^2\nu(dy)ds]\\
\le\;& L\delta\mathbb{E}[\sup_{(k-1)\delta\le s\le k\delta}(|x(s)|^2+|x((s-\tau(s))|^2)],
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}[\int_{(k-1)\delta}^{k\delta}|f(x(s),x(s-\tau(s)),s,r(s))|ds]^2\\
\le\;& \mathbb{E}[\delta\sup_{(k-1)\delta\le s\le k\delta}|f(x(s),x(s-\tau(s)),s,r(s))|]^2\\
\le\;& \delta^2 L\mathbb{E}[\sup_{(k-1)\delta\le s\le k\delta}(|x(s)|^2+|x((s-\tau(s))|^2)].
\end{aligned}$$

Assume that $L\delta(1+\delta)<\frac{1}{8}$.
Hence, it can be checked that

$$\begin{aligned}
&\mathbb{E}[\sup_{(k-1)\delta\le t\le k\delta}|x(t)|^2]\\
\le\;& 4Me^{-(\lambda-\varepsilon)(k-1)\delta}\\
+\;& 4L\delta(1+\delta)\\
& \mathbb{E}[\sup_{(k-1)\delta\le s\le k\delta}(|x(s)|^2+|x((s-\tau(s))|^2)]\\
\le\;& 4Me^{-(\lambda-\varepsilon)(k-1)\delta}\\
+\;& \frac{1}{2}\mathbb{E}[\sup_{(k-1)\delta\le s\le k\delta}(|x(s)|^2+|x((s-\tau(s))|^2)].
\end{aligned}$$

Since

$$\mathbb{E}[\sup_{(k-1)\delta\le s\le k\delta}|x(s-\tau(s))|^2]\le Me^{-(\lambda-\varepsilon)(k-1)\delta-\tau}, \tag{13}$$

we obtain that

$$\mathbb{E}[\sup_{(k-1)\delta\le t\le k\delta}|x(t)|^2]\le 5Me^{-(\lambda-\varepsilon)(k-1)\delta-\tau}. \tag{14}$$

From Chebyshev inequality, it is easy to check that

$$\begin{aligned}
&\mathbb{P}\{\omega:\sup_{(k-1)\delta\le t\le k\delta}|x(t)|>e^{-(\lambda-2\varepsilon)((k-1)\delta-\tau)/2}\}\\
\le\;& \frac{\mathbb{E}[\sup_{(k-1)\delta\le t\le k\delta}|x(t)|^2]}{e^{-(\lambda-2\varepsilon)(k-1)\delta-\tau}}\\
\le\;& 5Me^{-\varepsilon((k-1)\delta-\tau)}.
\end{aligned}$$

According to Borel-Cantelli lemma, for all $\omega\in\Omega$ and except some k, it follows that

$$\sup_{(k-1)\delta\le t\le k\delta}|x(t)|\le e^{-(\lambda-2\varepsilon)((k-1)\delta-\tau)/2}. \tag{15}$$

Therefore, for almost every $\omega\in\Omega$, if $(k-1)\delta\le t\le k\delta$ and $k\ge\max\{k_0,k_1\}$, we can obtain

$$\lim_{t\to\infty}\sup\frac{1}{t}\log(|x(t)|)\le-\frac{\lambda-2\varepsilon}{2}. \tag{16}$$

Let $\varepsilon\to 0$, the system 10 is almost sure exponential stability.

IV. Numerical simulation

Let $W(t)$ and $N(t,y)$ are all one-dimensional, The character measure υ of Poisson jump satisfies $\upsilon(dy)=\zeta\phi(dy)$, where $\zeta=1.5$ is the intensity of Poisson distribution and ϕ is the probability intensity of the standard normal distributed variable y, $r(t)\in\mathbb{S}=\{1,2\}$ and $\Gamma=(\gamma_{ij})_{2\times 2}=$

$$\begin{pmatrix}-0.5 & 0.5\\ 0.3 & -0.3\end{pmatrix}$$

Consider the following scalar nonlinear stochastic delayed systems with Markovian switching and Lévy noises:

$$\begin{aligned}
& dx(t) \qquad\qquad (17)\\
=\;& f(x(t),x(t-\tau(t,r(t))),t,r(t))dt\\
+\;& g(x(t),x(t-\tau(t,r(t))),t,r(t))dW(t)\\
+\;& \int_Y H(x(t^-),x((t-\tau(t,r(t)))^-),t,r(t),y)N(dt,dy),
\end{aligned}$$

where

$$\begin{aligned}
& f(x(t),x(t-\tau(t,1)),t,1)\\
=\;& -3x(t)+\frac{1}{2}sin(x(t))+\frac{1}{2}x(t-\tau(t,1)),
\end{aligned}$$

$$g(x(t),x(t-\tau(t,1)),t,1)=\frac{1}{3}x(t),$$

$$\begin{aligned}
& f(x(t),x(t-\tau(t,1)),t,2)\\
=\;& -\frac{5}{2}x(t)+\frac{1}{5}sin(x(t))+\frac{1}{3}x(t-\tau(t,2)),
\end{aligned}$$

$$g(x(t),x(t-\tau(t,1)),t,2)=\frac{1}{2}x(t),$$

$$H(x(t),x(t-\tau(t,r(t))),t,1,y)=\frac{x(t-\tau(t,1))y}{2}$$

$$H(x(t),x(t-\tau(t,r(t))),t,2,y)=-x(t)+x(t-\tau(t,1))y.$$

Let $V(x,i)=x^2,\quad i=1,2$, then we obtain that

$$\mathcal{L}V(x,\xi,t,1)\le -2x^2+\frac{1}{4}\xi^2,$$

$$\mathcal{L}V(x,\xi,t,2)\le 1.5x^2+\xi^2,$$

namely, $c_1=c_2=1$, $\lambda_1=1.5$, $\lambda_2=1$.

Let $\tau(t,1) = 1 + 0.3\sin(t)$, $\tau(t,2) = 1 + 0.2\cos(t)$, then $\tau = 1.3$, $\bar{\delta} = 0.45$. Therefore, it follows from Theorem 1 that the system is almost sure exponential stability.

Let $x(0) = 2$, $r(0) = 1$, step size $\Delta = 0.0001$, Figure 1 shows the state trajectory and the usefulness of the proposed almost sure exponential stability criterion. Figure 2 shows the Poisson trajectory and Figure 3 shows the Markov chain.

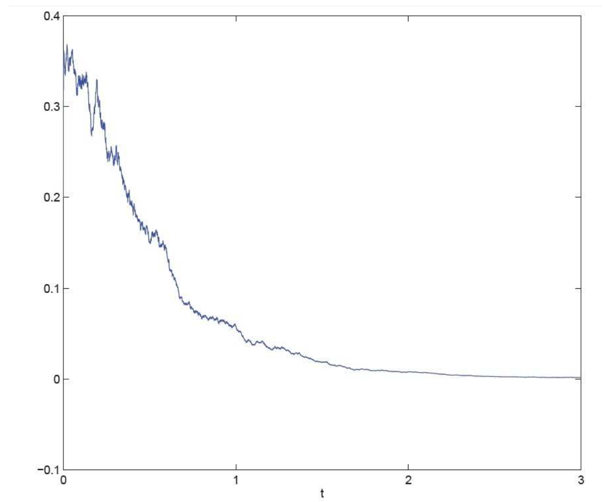

Fig. 1. State trajectory

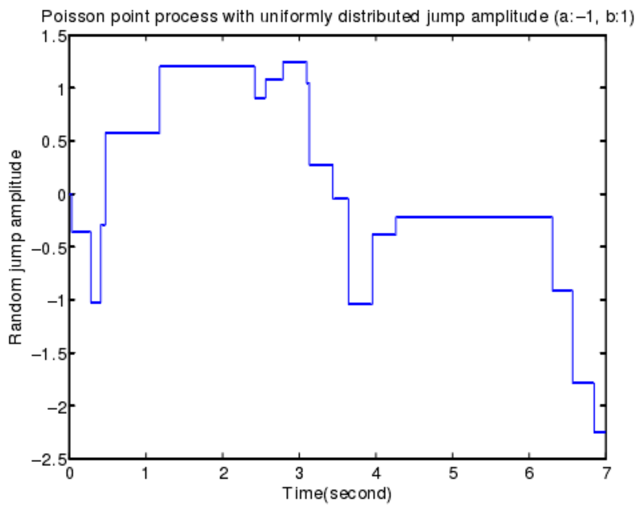

Fig. 2. Poisson trajectory

Next, let $r(t) \in \mathbb{S} = \{1,2,3\}$ and $\Gamma = (\gamma_{ijk})_{3\times 3} =$

$$\begin{pmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 2 & -5 \end{pmatrix}$$

$$\begin{aligned} & dx(t) \\ = \; & f(x(t), x(t-\tau(t,r(t))), t, r(t))dt \\ + \; & g(x(t), x(t-\tau(t,r(t))), t, r(t))dW(t) \\ + \; & \int_Y H(x(t^-), x((t-\tau(t,r(t)))^-), t, r(t), y) \\ & N(dt, dy), \end{aligned}$$

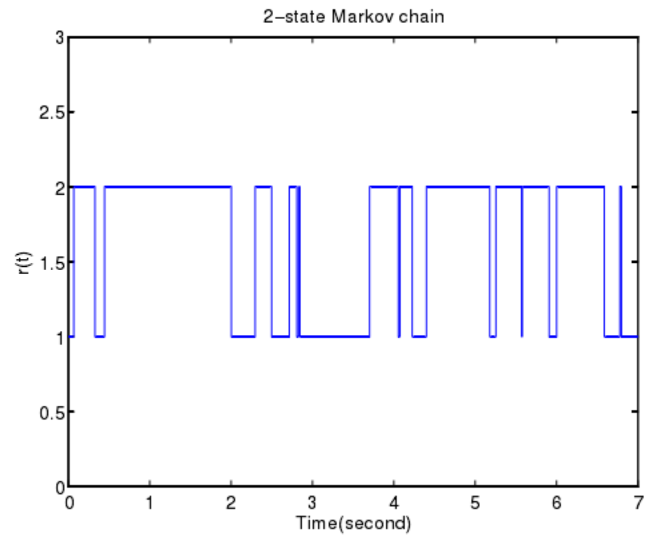

Fig. 3. Markov chain

where

$$\begin{aligned} & f(x(t), x(t-\tau(t,1)), t, 1) \\ = \; & -3x(t) + \frac{1}{2}sin(x(t)) + \frac{1}{2}x(t-\tau(t,1)), \end{aligned}$$

$$g(x(t), x(t-\tau(t,1)), t, 1) = \frac{1}{3}x(t),$$

$$\begin{aligned} & f(x(t), x(t-\tau(t,1)), t, 2) \\ = \; & -\frac{5}{2}x(t) + \frac{1}{5}sin(x(t)) + \frac{1}{3}x(t-\tau(t,2)), \end{aligned}$$

$$g(x(t), x(t-\tau(t,1)), t, 2) = \frac{1}{2}x(t),$$

$$H(x(t), x(t-\tau(t,r(t))), t, i, y) = ix(t-\tau(t,i))y^2$$

Let $V(x,i) = x^2$, $\quad i = 1,2$, then we obtain that

$$\mathcal{L}V(x,\xi,t,1) \leq -2x^2 + \frac{1}{4}\xi^2,$$

$$\mathcal{L}V(x,\xi,t,2) \leq 2x^2 + \xi^2,$$

namely, $c_1 = c_2 = 1$, $\lambda_1 = 2$, $\lambda_2 = 1$.

Let $\tau(t,1) = 2 + 0.5\cos(t)$, $\tau(t,2) = 1 + 0.3\sin(t)$, then $\tau = 1.5$, $\bar{\delta} = 0.6$. Therefore, it follows from Theorem 1 that the system is almost sure exponential stability.

Let $x(0) = 1$, $r(0) = 0.5$, step size $\Delta = 0.0005$, Figure 4 shows the state trajectory and the usefulness of the proposed almost sure exponential stability criterion. Figure 5 shows the Poisson trajectory and Figure 6 shows the Markov chain.

V. Conclusion

The aim of this paper is to study the almost sure exponential stability of nonlinear stochastic delayed systems with Markovian switching and Lévy noises. By using Burkholder-Davis-Gundy inequality, Chebyshev inequality, Borel-Cantelli lemma and generalized Itô formula for Lévy stochastic integral, the sufficient conditions to guarantee the almost sure exponential stability of the system has been proposed. A numerical example has been provided to show

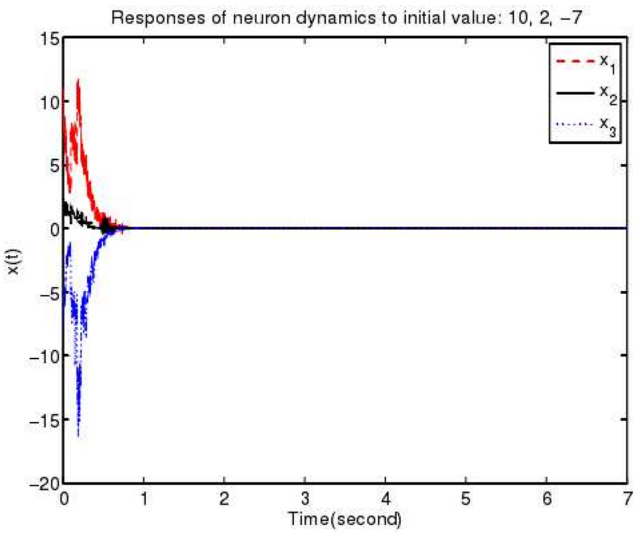

Fig. 4. State trajectory

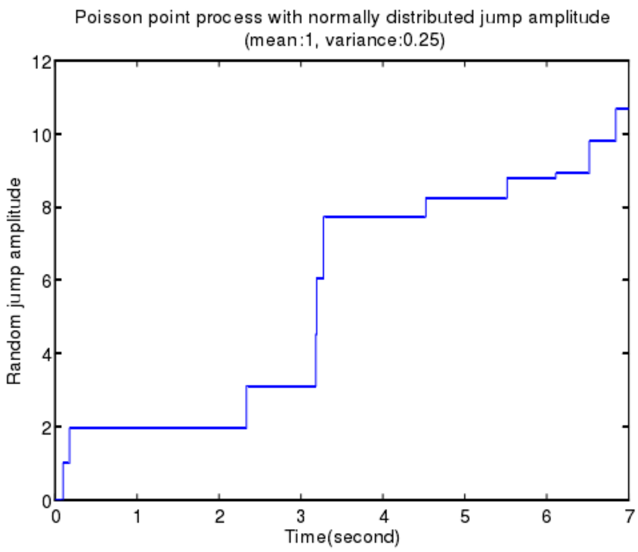

Fig. 5. Poisson trajectory

the usefulness of the proposed almost sure exponential stability criterion. Further research topics will include almost sure stability for stochastic hybrid systems with Markovian switching and Lévy noises.

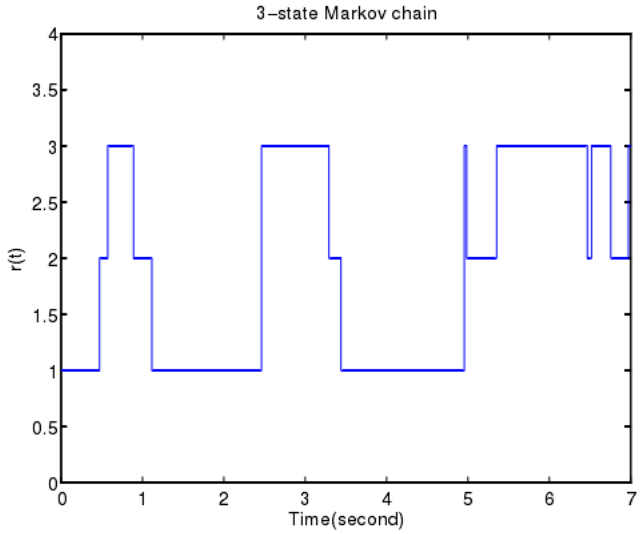

Fig. 6. Markov chain

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