Analysis of Composite Runge Kutta Methods and New One-Step Technique for Stiff Delay Differential Equations

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Abstract- **This paper presents fourth order Composite Runge Kutta (RK) methods based on Arithmetic Mean (AM), Harmonic Mean (HaM), Centroidal Mean (CeM), Contraharmonic Mean (CoM) and a new one-step technique which uses non-linear polynomial interpolating function to solve the stiff Delay differential equations (DDEs). The stability polynomials of these methods are derived. The Local Grid Search Algorithm is used to determine the stability regions. The efficiency of these methods is compared through three numerical examples of stiff Delay Differential Equations (DDEs).**

Index Terms- **Stiff Delay Differential Equations, Means, New One-Step technique, Lagrange Interpolation, Stability polynomial and region.**

I. INTRODUCTION

Delay differential equations have been rapidly developing in chemical kinetics [1], population dynamics [2], management systems [3] and in several areas of science and engineering. Recently there has been a growing interest in the numerical solutions of DDEs. Ismail and Read Ali [4] obtained the numerical solution of DDEs by RK method by using Hermite interpolation. Fatemeh [5] and Rostann [6] et al proposed homotopy perturbation and adomian decomposition methods. Radzi [7] et al investigated the two and three point one-step block method for solving DDEs. Toheeb et al. [8] found the exact solution by using the combination of Laplace and the variational iteration method. Ismail and Suleiman [9] studied the p-stability and q-stability of singly diagonally implicit Runge-Kutta method for DDEs.

Many researchers have tried to extend and modify the RK methods according to their needs. This is because of the nature of flexibility, efficiency and accuracy. In this way, RK methods based on variety of means have been developed by Murugesan et al. [10, 11] in ordinary differential equations (ODEs). Dingwen and Tingting [12] developed the fourth order singly diagonally implicit Runge-Kutta method for solving one-dimensional Burger's equations.

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Several one-step techniques using variety of interpolating polynomials and functions have been developed to solve ODEs. In 1976, Fatunla [13] gave a new algorithm which consists of the interpolating function of two complex parameters for solving ordinary differential equations. Kama and Ibijola [14] have developed the new one-step polynomial and exponential interpolating function technique for solving initial value problems in ordinary differential equations. In 2017, Abolarin et al. [15] derived the fourth stage inverse polynomial scheme to initial value problems. Fadugba and Falodun [16] have developed the new one-step power series polynomial scheme for initial value problems in ODEs. In 2000, Asgari found the numerical solution for solving a system of fractional integro-differential equations [17].

Most of the models in differential equations are 'stiff' in nature. For solving stiff equations, the step size is taken to be extremely small. Also, many problems may be stiff in some intervals and non-stiff in others. Several numerical methods have been used for solving stiff DDEs. In 1980, Roth [18] investigated the difference method for stiff delay differential equations. Bocharov [19] gave the numerical solution by LMMs of stiff delay differential systems modelling immune response. El-Safty [20] and Zhu [21] have found the Chebyschev method and parallel two-step ROW methods for stiff delay differential equations.

The purpose of this study is to present Composite RK methods based on different means such as AM, CeM, HaM, CoM and a new one-step technique which uses non-linear polynomial interpolating function while solving DDEs. Lagrange interpolation is used to approximate the delay argument. The stability polynomials are derived for these numerical methods. Their corresponding stability regions are also plotted.

II. FOURTH ORDER COMPOSITE RK METHODS FOR ODES

Consider the first order equation of the form

$$y' = f(x, y) \text{ with } y(x_0) = y_0$$

The classical fourth order RK method is given by

$$y_{n+1} = y_n + \frac{h}{6}[k_1 + 2k_2 + 2k_3 + k_4] \qquad (1)$$

which involve $k_i,\ 1 \le i \le 4$, given by

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$k_i = f\left(t_n + c_i h, y_n + h\sum_{j=1}^{i} a_{ij}k_j\right)$ where $c_i = \sum_{j=1}^{i} a_{ij}$

The formula (1) can be rewritten as

$$y_{n+1} = y_n + \frac{h}{3}\sum_{i=1}^{3}\left[\frac{k_i+k_{i+1}}{2}\right] = y_n + \frac{h}{3}\left[\sum_{i=1}^{3}(\text{AM})\right] \quad (2)$$

The above formula (2) can be called as the fourth order RKAM formula. The values of AM can be replaced by variety of means such as HaM, CeM and CoM which are given in Table I.
The formulae for HaM, CeM and CoM in terms of AM and GM and also the corresponding values of and in all these cases can be referred in Murugesan et al [11].

III. FOURTH ORDER COMPOSITE RK METHODS FOR DDES

In this section, the fourth order RK formulae have been extended to solve DDEs with constant delays. Consider the first order DDEs with constant delay τ,

$$y'(t) = f\left(t, y(t), y\left(t - \tau(t, y(t))\right)\right), \quad t > t_0$$
$$y(t) = \Phi(t), \quad t \le t_0$$

where $\Phi(t)$ is the initial function. The fourth order RK method using variety of means can be written as

$$y_{n+1} = y_n + \frac{h}{3}\left[\sum_{i=1}^{3} Means\right]$$

which involve k_i, $1 \le i \le 4$, given by

$$k_i = f\left(t_n + c_i h, y_n + h\sum_{j=1}^{i} a_{ij}k_j, y(t_n + c_i h - \tau)\right)$$

where $c_i = \sum_{j=1}^{i} a_{ij}$

Here Means include AM, HaM, CeM and CoM. The fourth order RK formulae based on various means for solving DDEs are same as given in Table I. These RK formulae can also be extended to solve DDEs with multiple delays. In this paper, Lagrange interpolation is used to approximate the delay argument.

IV. ANALYSIS OF COMPOSITE RK METHODS

A. ORDER OF CONVERGENCE OF COMPOSITE RK METHODS

The interpolation order and hence the number of support points have to be adapted to the order of the method. For any given RK method, its adaptation to DDEs by means of interpolation procedure has an order of convergence equal to $\min\{p, q\}$where p denotes the order of consistency of the RK method and q is the number of support points of the interpolation procedure [22]. Here we use four support points of Lagrange interpolation so that the order of convergence of RK method is four for DDEs also.

B. STABILITY ANALYSIS OF COMPOSITE RK METHODS

The stability of numerical methods to DDEs depends on the test equation and the delay term involved. Here we consider a commonly used linear test equation with a constant delay $\tau = 1$ where N is a positive integer,

$$y'(t) = \lambda y(t) + \mu y(t - \tau), \quad t \ge 0$$
$$y(t) = \Phi(t), \quad t \in [-\tau, 0] \quad (3)$$

where $\lambda, \mu \in C$, $\tau > 0$, and Φ is continuous.

Assume that the numerical solution has been calculated up to point t_n with uniform step size h, satisfying $\tau = Nh$, where N is a positive integer. Lagrange interpolation is used to approximate the delay term using previously calculated value of y, giving

$$y(t_n + c_i h - Nh) = y(t_{n-N} + c_i h)$$
$$= \sum_{l=-r_1}^{s_1} L_l(c_i) y_{n-N+l} \quad (4)$$

with $L_l(c_i) = \prod_{j_1=-r_1}^{s_1} \frac{c_i - j_1}{l - j_1}$, $\quad j_1 \ne l \quad and\, r_1, s_1 > 0$

and y_{n-N+l} is calculated value of $y(t_{n-N+l})$. When RK method is applied to DDE (3) with delay $\tau = 1$, the following equations are obtained.

$$k_i = f\left(t_n + c_i h, y_n + h\sum_{=1}^{q} a_{ij}k_i, \sum_{l=-r_1}^{s_1} L_l(c_i) y_{n-N+l}\right) \quad (5)$$
$$y_{n+1} = y_n + h\sum_{i=1}^{q} a_{ij}k_i$$

Define $\mathbf{u} = (1, \ldots, 1)^{\mathrm{T}}$, $\quad \mathbf{k} = \left(k^{(1)}, k^{(2)}, \ldots, k^{(q)}\right)^{\mathrm{T}}$,

$\mathbf{b} = \left(b_1, b_2, \ldots, b_q\right)^{\mathrm{T}}$ and $\mathbf{L_l(c)} = \left(L_l(c_1), \ldots, L_l(c_q)\right)^{\mathrm{T}}$.

For $n \ge N$, considering f as in (3), (5) can be written as

$$\mathbf{k} = \lambda(y_n\mathbf{u} + hA\underline{k}) + \mu\left(\sum \mathbf{L_l}(\mathbf{c}) y_{n-N+l}\right) \quad (6)$$
$$y_{n+1} = y_n + h\mathbf{b}^{\mathbf{T}}\mathbf{k} \quad \textbf{(7)}$$

From equation (6),

$$\mathbf{k} = \lambda y_n \mathbf{u}[I - \lambda hA]^{-1} + \mu[I - \lambda hA]^{-1}\sum \mathbf{L_l}(\mathbf{c}) y_{n-N+l}$$
$$h\mathbf{k} = \alpha y_n \mathbf{u}\eta + \beta\eta \sum \mathbf{L_l}(\mathbf{c}) y_{n-N+l} \quad (8)$$

where $\alpha = \lambda h, \beta = \mu h, \eta = [I - \lambda hA]^{-1}$ and I is the identity matrix.

Substituting (8) in (7),

$$y_{n+1} = y_n + \alpha\mathbf{b}^{\mathbf{T}}\eta y_n \mathbf{u} + \beta\mathbf{b}^{\mathbf{T}}\eta \sum \mathbf{L_l}(\mathbf{c}) y_{n-N+l} \quad (9)$$

Taking $\mathbf{Y_n} = (y_n, h\mathbf{k})^T$ (8) and (9) can be written as the recurrence,

$$\mathbf{Y_{n+1}} = \mathbf{XY_n} + \mathbf{ZY}_{n-N+l} \text{ where}$$

$$\mathbf{X}=\begin{bmatrix}1+\alpha\mathbf{b}^{\mathrm{T}}\eta\mathbf{u} & 0,\dots,0\\ & 0\\ & .\\ \alpha\eta\mathbf{u} & .\\ & .\\ & 0\end{bmatrix} \text{ and } \mathbf{Z}=\begin{bmatrix}\beta\mathbf{b}^{\mathrm{T}}\eta\sum\mathbf{L}_{\mathrm{l}}(\mathbf{c}) & 0\\ & 0\\ & .\\ \beta\eta\sum\mathbf{L}_{\mathrm{l}}(\mathbf{c}) & .\\ & .\\ & 0\end{bmatrix}$$

By putting $n-N+l=0$, the stability polynomial will be in the standard form. The recurrence is stable if the zeros ζ_i of the stability polynomial

$$S(\alpha,\beta,\zeta)=\det\left[\zeta^{n+1}I-\zeta^{n}X-\sum_{l=r_1}^{s_1}\zeta^{1+l}Z_l\right] \text{and}$$

satisfies the root condition $|\zeta_i|\leq 1.$

To obtain the stability region of the method, we used four points interpolation to evaluate $y(t_n+c_ih-1)$. Then the stability polynomial for the method is,

$$S(\alpha,\beta:\zeta)=\zeta^{n+1}-(1+\alpha\mathbf{b}^T\eta\mathbf{u})\zeta^n-\beta\mathbf{b}^T\eta(L_{-1}(c)+L_0(c)\zeta+L_1(c)\zeta^2+L_2(c)\zeta^3)$$

To determine the stability polynomials for the RK method based on variety of means, we need the dense output $\mathbf{b}(\theta)$ where $\theta\in(0,1]$.

From the order conditions for 4-stage fourth order method, we obtained the fourth order interpolant as given in Table II.

Generally, the stability polynomial for RK methods based on variety of means can be written as,

$$S(\alpha,\beta:\zeta)=\zeta^{n+1}-(1+\alpha\mathbf{b}^T(\theta)\eta\mathbf{u})\zeta^n -\beta\mathbf{b}^T(\theta)\eta(L_{-1}(c)+L_0(c)\zeta+L_1(c)\zeta^2+L_2(c)\zeta^3)$$

and they have been listed as follows.

For RKAM:

$$S(\alpha,\beta:\zeta)=\zeta^{n+1}-\zeta^n\left(1+\alpha+\frac{\alpha^2}{2}+\frac{\alpha^3}{6}+\frac{\alpha^4}{24}\right)-\zeta^3\left(\frac{-1}{24}\beta-\frac{1}{48}\alpha\beta-\frac{1}{192}\alpha^2\beta\right)-\zeta^2\left(\frac{13}{24}\beta+\frac{3}{16}\alpha\beta+\frac{3}{64}\alpha^2\beta\right)-\zeta\left(\frac{13}{24}\beta+\frac{17}{48}\alpha\beta+\frac{25}{192}\alpha^2\beta+\frac{1}{24}\alpha^3\beta\right)-\left(\frac{-1}{24}\beta-\frac{1}{48}\alpha\beta-\frac{1}{192}\alpha^2\beta\right)$$

For RKHaM:

$$S(\alpha,\beta:\zeta)=\zeta^{n+1}-\zeta^n\left(1+\alpha+\frac{\alpha^2}{2}+\frac{\alpha^3}{6}+\frac{\alpha^4}{24}\right)-\zeta^3\left(\frac{-19}{432}\beta-\frac{1}{48}\alpha\beta-\frac{1}{192}\alpha^2\beta\right)-\zeta^2\left(\frac{235}{432}\beta+\frac{3}{16}\alpha\beta+\frac{3}{64}\alpha^2\beta\right)-\zeta\left(\frac{235}{432}\beta+\frac{17}{48}\alpha\beta+\frac{25}{192}\alpha^2\beta+\frac{1}{24}\alpha^3\beta\right)-\left(\frac{-19}{432}\beta-\frac{1}{48}\alpha\beta-\frac{1}{192}\alpha^2\beta\right)$$

For RKCeM:

$$S(\alpha,\beta:\zeta)=\zeta^{n+1}-\zeta^n\left(1+\alpha+\frac{\alpha^2}{2}+\frac{\alpha^3}{6}+\frac{\alpha^4}{24}\right)-\zeta^3\left(\frac{-49}{1168}\beta-\frac{1}{48}\alpha\beta-\frac{1}{192}\alpha^2\beta\right)-\zeta^2\left(\frac{633}{1168}\beta+\frac{3}{16}\alpha\beta+\frac{3}{64}\alpha^2\beta\right)-\zeta\left(\frac{633}{1168}\beta+\frac{17}{48}\alpha\beta+\frac{25}{192}\alpha^2\beta+\frac{1}{24}\alpha^3\beta\right)-\left(\frac{-49}{1168}\beta-\frac{1}{48}\alpha\beta-\frac{1}{192}\alpha^2\beta\right)$$

For RKCoM:

$$S(\alpha,\beta:\zeta)=\zeta^{n+1}-\zeta^n\left(1+\alpha+\frac{\alpha^2}{2}+\frac{\alpha^3}{6}+\frac{\alpha^4}{24}\right)-\zeta^3\left(\frac{-19}{432}\beta-\frac{1}{48}\alpha\beta-\frac{1}{192}\alpha^2\beta\right)-\zeta^2\left(\frac{235}{432}\beta+\frac{3}{16}\alpha\beta+\frac{3}{64}\alpha^2\beta\right)-\zeta\left(\frac{235}{432}\beta+\frac{17}{48}\alpha\beta+\frac{25}{192}\alpha^2\beta+\frac{1}{24}\alpha^3\beta\right)-\left(\frac{-19}{432}\beta-\frac{1}{48}\alpha\beta-\frac{1}{192}\alpha^2\beta\right)$$

Their corresponding stability regions are given in Fig. 1-4.

V. A NEW ONE-STEP TECHNIQUE FOR SOLVING DDEs

Let us assume that the analytical solution $y(t)$ to the initial value problem (1) can be locally represented in the interval $[t_n,t_{n+1}]$, $n\geq 0$ by the non-linear polynomial interpolating function

$$F(t)=a_1e^{2t}+a_2t^4+a_3t^3+a_4t^2+a_5t+a_6 \quad (10)$$

where $a_1, a_2\ a_3, a_4, a_5$ and a_6 are undetermined coefficients. We shall assume y_n is a numerical approximation to the analytical solution $y(t)$ and using mesh points as follows:

$$t_n=a+nh, \qquad n=0,1,2,\dots \qquad (11)$$

Taking the following constraints on the interpolating function (3) in order to get the undetermined coefficients.

Firstly, the interpolating function must coincide with the analytical solution at

$$t=t_n \text{ and } t=t_{n+1}.$$

Hence we required that

$$F(t_n)=a_1e^{2t_n}+a_2{t_n}^4+a_3{t_n}^3++a_4{t_n}^2+a_5t_n+a_6 \qquad (12)$$

and

$$F(t_{n+1})=a_1e^{2t_{n+1}}+a_2{t_{n+1}}^4+a_3{t_{n+1}}^3+a_4{t_{n+1}}^2+a_5t_{n+1}+a_6 \qquad (13)$$

Secondly, the derivatives of the interpolating function are required to coincide with the differential equation as its first, second, and third derivatives with respect to t at $t=t_n$. We denote the i[th] total derivatives of $f(t,y(t),y(t-\tau(t,y(t))))$ with respect to t with $f^{(i)}$ such that

$$F^1(t_n)=f_n^{(1)} \qquad (14)$$

$$F^2(t_n) = f_n^{(2)} \tag{15}$$

$$F^3(t_n) = f_n^{(3)} \tag{16}$$

$$F^4(t_n) = f_n^{(4)} \tag{17}$$

$$F^5(t_n) = f_n^{(5)} \tag{18}$$

$$f_n^{(1)} = 2a_1 e^{2t_n} + 4a_2 t_n^3 + 3a_3 t_n^2 + 2a_4 t_n + a_5 \tag{19}$$

$$f_n^{(2)} = 4a_1 e^{2t_n} + 12a_2 t_n^2 + 6a_3 t_n + 2a_4 \tag{20}$$

$$f_n^{(3)} = 8a_1 e^{2t_n} + 24a_2 t_n + 6a_3 \tag{21}$$

$$f_n^{(4)} = 16a_1 e^{2t_n} + 24a_2 \tag{22}$$

$$f_n^{(5)} = 32a_1 e^{2t_n} \tag{23}$$

solving for a_1 from eqn. (23), we have

$$a_1 = \frac{f_n^{(5)}}{32e^{2t_n}} \tag{24}$$

Substituting (24) in (22), we have

$$a_2 = \frac{1}{24}\left[f_n^{(4)} - \frac{f_n^{(5)}}{2}\right] \tag{25}$$

Substituting (24) and (25) into (21), we have

$$a_3 = \frac{1}{6}\left[\left(f_n^{(3)} - \frac{f_n^{(5)}}{4}\right) - \left(f_n^{(4)} - \frac{f_n^{(5)}}{2}\right) t_n\right] \tag{26}$$

Substituting (24), (25) and (26) into (20), we have

$$a_4 = \frac{1}{2}\begin{bmatrix} \left(f_n^{(2)} - \frac{f_n^{(5)}}{8}\right) - \left(f_n^{(3)} - \frac{f_n^{(5)}}{4}\right) t_n \\ -\left(\frac{f_n^{(5)}}{4} - \frac{f_n^{(4)}}{2}\right) {t_n}^2 \end{bmatrix} \tag{27}$$

Substituting (24), (25), (26) and (27) into (19), we have

$$a_5 = \begin{bmatrix} \left(f_n^{(1)} - \frac{f_n^{(5)}}{16}\right) - \left(f_n^{(2)} - \frac{f_n^{(5)}}{8}\right) t_n \\ -\left(\frac{f_n^{(5)}}{8} - \frac{f_n^{(3)}}{2}\right) {t_n}^2 - \left(\frac{f_n^{(4)}}{6} - \frac{f_n^{(5)}}{12}\right) {t_n}^3 \end{bmatrix} \tag{28}$$

Since $F(t_{n+1}) = y(t_{n+1})$ and $F(t_n) = y(t_n)$ implies that $y(t_{n+1}) = y_{n+1}$ and $y(t_n) = y_n$

$$F(t_{n+1}) - F(t_n) = y_{n+1} - y_n \tag{29}$$

Then we shall have

$$\begin{aligned} y_{n+1} - y_n &= a_1 e^{2t_{n+1}} + a_2 {t_{n+1}}^4 + a_3 {t_{n+1}}^3 + a_4 {t_{n+1}}^2 \\ &\quad + a_5 t_{n+1} + a_6 - (a_1 e^{2t_n} + a_2 {t_n}^4 + a_3 {t_n}^3 \\ &\quad + a_4 {t_n}^2 + a_5 t_n + a_6) \\ &= a_1(e^{2t_{n+1}} - e^{2t_n}) + a_2({t_{n+1}}^4 - {t_n}^4) \\ &\quad + a_3({t_{n+1}}^3 - {t_n}^3) + a_4({t_{n+1}}^2 - {t_n}^2) \\ &\quad + a_5(t_{n+1} - t_n) \end{aligned} \tag{30}$$

Setting a = 0 in (11), we get $t_n = nh$ and $t_{n+1} = (n+1)h$ (31)

Then,

$$\begin{aligned} y_{n+1} &= y_n + a_1(e^{2t_{n+1}} - e^{2t_n}) \\ &\quad + a_2 h^4(1 + 4n + 6n^2 + 4n^3) \\ &\quad + a_3 h^3(1 + 3n + 3n^2) + a_4 h^2(1 + 2n) + a_5 h \end{aligned} \tag{32}$$

Eqn. (32) is the new one-step numerical technique.

This one-step technique can also be extended to solve DDEs with multiple delays. In this paper, Lagrange interpolation is used to approximate the delay argument.

VI. ANALYSIS OF NEW ONE-STEP TECHNIQUE

A. ORDER OF CONVERGENCE OF NEW ONE-STEP TECHNIQUE

A slight rearrangement of (32) and comparing with Taylor's series,

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2} y_n'' + \frac{h^3}{6} y_n''' + \frac{h^4}{24} y_n^{(iv)}$$

We found that the new one-step technique is of order four.

B. STABILITY ANALYSIS OF NEW ONE-STEP TECHNIQUE

A slight arrangement of (32), we obtain that

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2} y_n'' + \frac{h^3}{6} y_n''' + \frac{h^4}{24} y_n^{(iv)} \tag{33}$$

This implies,

$$\begin{aligned} y_{n+1} &= y_n + h\big(\lambda y_n + \mu y(t_n - \tau)\big) \\ &\quad + \frac{h^2}{2}\big(\lambda y'_n + \mu y'(t_n - \tau)\big) \\ &\quad + \frac{h^3}{6}\big(\lambda y''_n + \mu y''(t_n - \tau)\big) \\ &\quad + \frac{h^4}{24}\big(\lambda y'''_n + \mu y'''(t_n - \tau)\big) \end{aligned} \tag{34}$$

Here Lagrange interpolation is used to approximate the delay term.

$$y(t_n - mh) = y(t_{n-m}) = \sum_{l=-r_1}^{s_1} L_l(c_i) y_{n-m+l}$$

with

$$L_l(c_i) = \prod_{j_1=-r_1}^{s_1} \frac{c_i - j_1}{l - j_1}, \qquad j_1 \neq l \quad and\ r_1, s_1 > 0$$

Now $y(t_n - \tau) = \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l}$,

$$\begin{aligned} y'(t_n - \tau) &= \lambda \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l} \\ &\quad + \mu \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l} \end{aligned}$$

$$y''(t_n - \tau) = \lambda\begin{pmatrix} \lambda \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l} \\ +\mu \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l} \end{pmatrix} + \mu\begin{pmatrix} \lambda \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l} \\ +\mu \sum_{l=-r_1}^{s_1} L_l(c) y_{n-3m+l} \end{pmatrix}$$

and

$$y'''(t_n-\tau)=\lambda\left(\lambda\begin{pmatrix}\lambda\sum_{l=-r_1}^{s_1}L_l(c)y_{n-m+l}\\+\mu\sum_{l=-r_1}^{s_1}L_l(c)y_{n-2m+l}\end{pmatrix}+\mu\begin{pmatrix}\lambda\sum_{l=-r_1}^{s_1}L_l(c)y_{n-2m+l}\\+\mu\sum_{l=-r_1}^{s_1}L_l(c)y_{n-3m+l}\end{pmatrix}\right)+\mu\left(\lambda\begin{pmatrix}\lambda\sum_{l=-r_1}^{s_1}L_l(c)y_{n-2m+l}\\+\mu\sum_{l=-r_1}^{s_1}L_l(c)y_{n-3m+l}\end{pmatrix}+\mu\begin{pmatrix}\lambda\sum_{l=-r_1}^{s_1}L_l(c)y_{n-3m+l}\\+\mu\sum_{l=-r_1}^{s_1}L_l(c)y_{n-4m+l}\end{pmatrix}\right)\tag{35}$$

Substituting (35) in (34), we get

$$\begin{aligned}y_{n+1}&=y_n+h\left(\lambda y_n+\mu\sum_{l=-r_1}^{s_1}L_l(c)y_{n-m+l}\right)\\&+\frac{h^2}{2}\begin{pmatrix}\lambda^2y_n+2\lambda\mu\sum_{l=-r_1}^{s_1}L_l(c)y_{n-m+l}\\+\mu^2\sum_{l=-r_1}^{s_1}L_l(c)y_{n-2m+l}\end{pmatrix}\\&+\frac{h^3}{6}\begin{pmatrix}\lambda^3y_n+3\lambda^2\mu\sum_{l=-r_1}^{s_1}L_l(c)y_{n-m+l}\\+3\lambda\mu^2\sum_{l=-r_1}^{s_1}L_l(c)y_{n-2m+l}\\+\mu^3\sum_{l=-r_1}^{s_1}L_l(c)y_{n-3m+l}\end{pmatrix}\\&+\frac{h^4}{24}\begin{pmatrix}\lambda^4y_n+4\lambda^3\mu\sum_{l=-r_1}^{s_1}L_l(c)y_{n-m+l}\\+6\lambda^2\mu^2\sum_{l=-r_1}^{s_1}L_l(c)y_{n-2m+l}\\+4\lambda\mu^3\sum_{l=-r_1}^{s_1}L_l(c)y_{n-3m+l}\\+\mu^4\sum_{l=-r_1}^{s_1}L_l(c)y_{n-4m+l}\end{pmatrix}\end{aligned}$$

$$\begin{aligned}y_{n+1}&=y_n+\lambda hy_n+\frac{\lambda^2h^2}{2}y_n+\frac{\lambda^3h^3}{6}y_n+\frac{\lambda^4h^4}{24}y_n\\&+\sum_{l=-r_1}^{s_1}L_l(c)y_{n-m+l}\left(\mu h+\mu\lambda h^2+\frac{\lambda^2\mu h^3}{2}+\frac{\lambda^3\mu h^4}{6}\right)\\&+\sum_{l=-r_1}^{s_1}L_l(c)y_{n-2m+l}\left(\frac{\mu^2h^2}{2}+\frac{\lambda\mu^2h^3}{2}+\frac{\lambda^2\mu^2h^4}{4}\right)\\&+\sum_{l=-r_1}^{s_1}L_l(c)y_{n-3m+l}\left(\frac{\mu^3h^3}{6}+\frac{\lambda\mu^3h^4}{6}\right)\\&+\sum_{l=-r_1}^{s_1}L_l(c)y_{n-4m+l}\left(\frac{\mu^4h^4}{24}\right)\end{aligned}$$

Let $\alpha=\lambda h$ and $\beta=\mu h$. Then the above equation becomes,

$$\begin{aligned}y_{n+1}&=y_n\left(1+\alpha+\frac{\alpha^2}{2!}+\frac{\alpha^3}{3!}+\frac{\alpha^4}{4!}\right)\\&+\left(\beta\left(1+\alpha+\frac{\alpha^2}{2}+\frac{\alpha^3}{6}\right)\right)\sum_{l=-r_1}^{s_1}L_l(c)y_{n-m+l}\\&+\left(\beta^2\left(\frac{1}{2}+\frac{\alpha}{2}+\frac{\alpha^2}{4}\right)\right)\sum_{l=-r_1}^{s_1}L_l(c)y_{n-2m+l}\\&+\left(\beta^3\left(\frac{1}{6}+\frac{\alpha}{6}\right)\right)\sum_{l=-r_1}^{s_1}L_l(c)y_{n-3m+l}\\&+\left(\frac{\beta^4}{24}\right)\sum_{l=-r_1}^{s_1}L_l(c)y_{n-4m+l}\end{aligned}$$

To obtain the stability region of the method, the delay term is approximated using four points Lagrange interpolation. By putting $n-m+l=0$, $n-2m+l=0$, $n-3m+l=0$, $n-4m+l=0$ and by taking $l=-1,0,1,2$, the stability polynomial will be in the standard form. The recurrence is stable if the zeros ζ_i of the stability polynomial

$$\begin{aligned}S(\alpha,\beta:\zeta)&=\zeta^{n+1}-\left(1+1+\alpha+\frac{\alpha^2}{2!}+\frac{\alpha^3}{3!}+\frac{\alpha^4}{4!}\right)\zeta^n\\&-\left(\beta\left(1+\alpha+\frac{\alpha^2}{2}+\frac{\alpha^3}{6}\right)\right)(L_{-1}(c)+L_0(c)\zeta+L_1(c)\zeta^2+L_2(c)\zeta^3)\\&-\left(\beta^2\left(\frac{1}{2}+\frac{\alpha}{2}+\frac{\alpha^2}{4}\right)\right)(L_{-1}(c)+L_0(c)\zeta+L_1(c)\zeta^2+L_2(c)\zeta^3)\\&-\left(\beta^3\left(\frac{1}{6}+\frac{\alpha}{6}\right)\right)(L_{-1}(c)+L_0(c)\zeta+L_1(c)\zeta^2+L_2(c)\zeta^3)\\&-\left(\frac{\beta^4}{24}\right)(L_{-1}(c)+L_0(c)\zeta+L_1(c)\zeta^2+L_2(c)\zeta^3)\end{aligned}$$

satisfies the root condition $|\zeta_i|\le 1$.

Then the stability polynomial with delay $\tau=1$ for this method is

$$\begin{aligned}S(\alpha,\beta:\zeta)&=\zeta^{n+1}-\left(1+1+\alpha+\frac{\alpha^2}{2!}+\frac{\alpha^3}{3!}+\frac{\alpha^4}{4!}\right)\zeta^n\\&-\left(\beta+\frac{\beta^2}{2}+\frac{\beta^3}{6}+\frac{\beta^4}{24}+\frac{\alpha\beta^2}{2}+\frac{\alpha^2\beta}{2}+\frac{\alpha\beta^3}{6}+\frac{\alpha^3\beta}{6}+\frac{\alpha^2\beta^2}{4}+\alpha\beta\right)\zeta\end{aligned}$$

The corresponding stability region is given in Fig. 5.

VII. NUMERICAL EXAMPLES

Example 1:

Consider the stiff linear DDE

$y'(t)=-24y(t)-e^{-25}y(t-1),\quad t\ge 0$

with the initial function

$y(t)=\mathrm{e}^{-25t},\quad t\le 0$

which has the exact solution

$y(t)=\mathrm{e}^{-25t},\quad t\ge 0$

By taking the step size h=0.01 in all the RK formulae and one-step technique, the absolute errors are given in Table III.

Example 2:

Consider the system of stiff DDEs

$y_1'(t)=-\frac{1}{2}y_1(t)-\frac{1}{2}y_2(t-1)+f_1(t),$

$y_2'(t)=-y_2(t)-\frac{1}{2}y_1\left(t-\frac{1}{2}\right)+f_2(t),\quad 0\le t\le 1$

with initial conditions

$y_1(t)=e^{-t/2},\qquad \frac{-1}{2}\le t\le 0,$

$y_2(t) = e^{-t}, \qquad -1 \leq t \leq 0$

and $f_1(t) = \frac{1}{2}e^{-(t-1)}, \quad f_2(t) = \frac{1}{2}e^{-(t-1/2)/2}$

The exact solution is

$y_1(t) = e^{-t/2}, \qquad y_2(t) = e^{-t}$

By taking the step size h=0.01 in all the RK formulae and the new one-step technique, the absolute errors are given in Table IV-V.

Example 3:

Consider the stiff linear DDE

$$y'(t) = \left(\frac{-1}{0.03}\right)y(t) + \left(\frac{0.8}{0.03}\right)y(t-1), \qquad 0 \leq t \leq 1$$

with the initial function

$y(t) = cos(t), \quad t \leq 1$

which has the exact solution

$$y(t) = 0.41167612\cos(t) + 0.68552722\sin(t) + 0.58832388\exp(-33.333333t)$$

By taking the step size h=0.001 in all the RK formulae and the new one-step technique, the absolute errors are given in Table VI.

VIII. CONCLUSION

This paper presents Composite RK methods based on different means such as AM, CeM, HaM, CoM and a new one-step technique which uses non-linear polynomial interpolating function while solving DDEs. The stability polynomials are derived for both numerical methods. Their corresponding stability regions are also plotted.

The efficiency of these methods is demonstrated through examples of system of stiff DDEs with single and multiple delays. To interpolate the delay term, Lagrange interpolation is used.

The numerical results by Composite RK methods based on AM, CeM, CoM and HaM are well comparable with the newly proposed one-step technique. It is evident that these numerical methods are suitable for solving stiff DDEs.

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TABLE I
FOURTH ORDER COMPOSITE RK FORMULAE

Means	y_{n+1} =
AM	$y_n + \frac{h}{6}(k_1 + 2(k_2 + k_3) + k_4)$
HaM	$y_n + \frac{2h}{3}\left(\frac{k_1 k_2}{k_1 + k_2} + \frac{k_2 k_3}{k_2 + k_3} + \frac{k_3 k_4}{k_3 + k_4}\right)$
CeM	$y_n + \frac{2h}{9}\left(\frac{{k_1}^2 + k_1 k_2 + {k_2}^2}{k_{1+}k_2} + \frac{{k_2}^2 + k_2 k_3 + {k_3}^2}{k_{2+}k_3} + \frac{{k_3}^2 + k_3 k_4 + {k_4}^2}{k_{3+}k_4}\right)$
CoM	$y_n + \frac{h}{3}\left(\frac{{k_1}^2 + {k_2}^2}{k_{1+}k_2} + \frac{{k_2}^2 + {k_3}^2}{k_{2+}k_3} + \frac{{k_3}^2 + {k_4}^2}{k_{3+}k_4}\right)$

TABLE II
INTERPOLANT FOR RK METHOD

$\mathbf{b}^{\mathrm{T}}(\boldsymbol{\theta})$ / Means	$b_1(\theta)$	$b_2(\theta)$	$b_3(\theta)$	$b_4(\theta)$
RKAM	$\theta - \theta^2 + \frac{1}{6}\theta^4$	$\frac{2}{3}\theta^3 - \frac{1}{3}\theta^4$	$\frac{2}{3}\theta^3 - \frac{1}{3}\theta^4$	$\frac{1}{6}\theta^4$
RKHaM	$\theta - \theta^2 + \frac{4}{27}\theta^4$	$\theta^2 - \frac{8}{15}\theta^3$	$\frac{8}{15}\theta^3 - \frac{8}{27}\theta^4$	$\frac{4}{27}\theta^4$
RKCeM	$\theta - \theta^2 + \frac{12}{73}\theta^4$	$\theta^2 - \frac{8}{11}\theta^3$	$\frac{8}{11}\theta^3 - \frac{24}{73}\theta^4$	$\frac{12}{73}\theta^4$
RKCoM	$\theta - \theta^2 + \frac{4}{27}\theta^4$	$\theta^2 - \frac{8}{9}\theta^3$	$\frac{8}{9}\theta^3 - \frac{8}{27}\theta^4$	$\frac{4}{27}\theta^4$

TABLE III
ABSOLUTE ERROR IN EXAMPLE 1

Time	Composite RK methods based on				One-step Technique
	AM	CeM	HaM	CoM	
0.2	1.29e-06	1.64e-06	1.02e-05	8.33e-06	1.49e-06
0.4	1.93e-08	2.45e-08	1.52e-07	1.25e-07	2.23e-08
0.6	2.18e-10	2.76e-10	1.71e-09	1.40e-09	2.51e-10
0.8	2.19e-12	2.77e-12	1.72e-11	1.41e-11	2.52e-12
1.0	2.07e-14	2.62e-14	1.63e-13	1.33e-13	2.38e-14

TABLE IV
ABSOLUTE ERRORS OF y_1 IN EXAMPLE 2

Time	Composite RK methods based on				One-step Technique
	AM	CeM	HaM	CoM	
0.2	5.22e-14	3.55e-13	1.39e-12	1.36e-12	4.74e-13
0.4	1.30e-13	6.05e-13	2.55e-12	2.42e-12	8.57e-13
0.6	2.23e-13	7.76e-13	3.50e-12	3.24e-12	1.16e-12
0.8	3.21e-13	8.84e-13	4.28e-12	3.86e-12	1.40e-12
1.0	2.24e-11	2.38e-11	1.79e-11	2.72e-11	1.59e-12

TABLE V
ABSOLUTE ERRORS OF y_2 IN EXAMPLE 2

Time	Composite RK methods based on				One-step Technique
	AM	CeM	HaM	CoM	
0.2	9.45e-07	9.45e-07	9.45e-07	9.45e-07	8.95e-07
0.4	1.63e-06	1.63e-06	1.63e-06	1.63e-06	1.54e-06
0.6	2.11e-06	2.11e-06	2.11e-06	2.11e-06	1.99e-06
0.8	2.42e-06	2.42e-06	2.42e-06	2.42e-06	2.29e-06
1.0	2.62e-06	2.62e-06	2.62e-06	2.62e-06	1.85e-06

TABLE V1
ABSOLUTE ERROR IN EXAMPLE 3

Time	Composite RK methods based on				One-step Technique
	AM	CeM	HaM	CoM	
0.2	1.16e-07	2.46e-07	5.06e-07	2.03e-07	1.16e-07
0.4	1.39e-07	1.47e-07	1.14e-07	1.64e-07	1.39e-07
0.6	1.57e-07	1.67e-07	1.29e-07	1.86e-07	1.57e-07
0.8	1.69e-07	1.79e-07	1.40e-07	1.98e-07	1.69e-07
1.0	4.19e-05	2.34e-05	1.00e-05	1.24e-04	1.74e-07

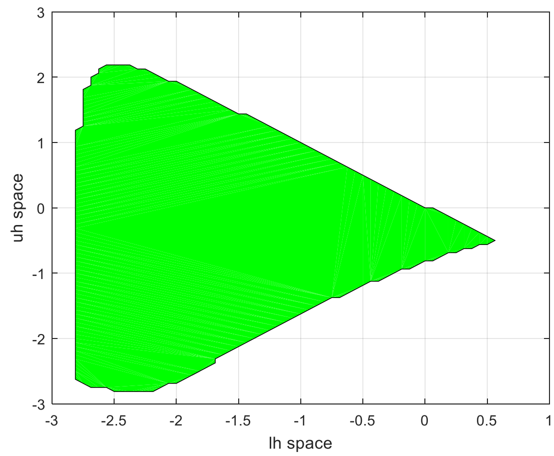

Fig. 1 (RKAM)

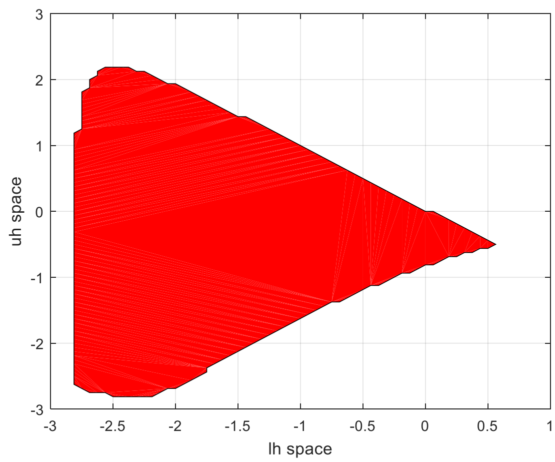

Fig. 2 (RKCeM)

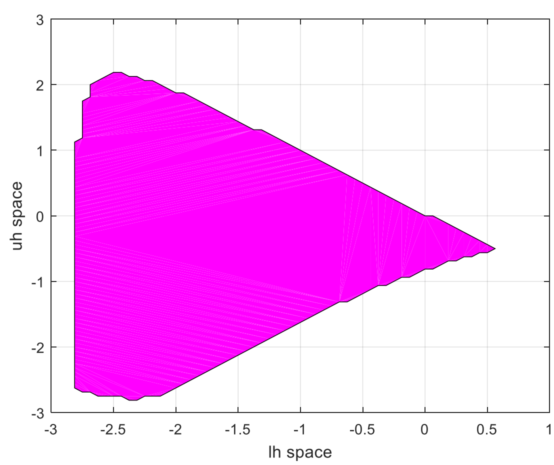

Fig. 3 (RKCoM)

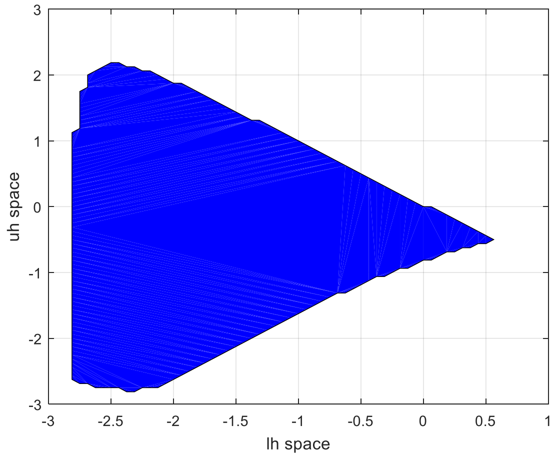

Fig. 4 (RKHaM)

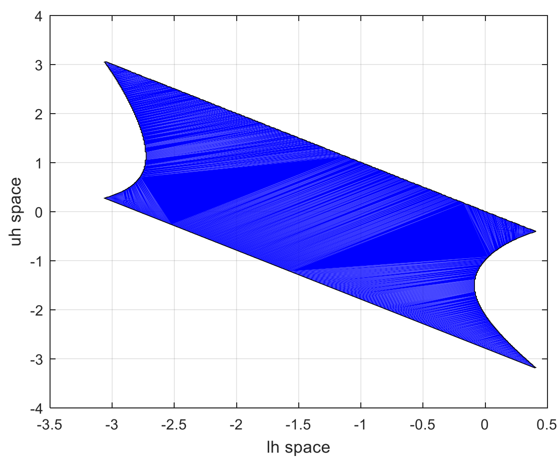

Fig. 5 Stability region of the new one-step technique