Statistics of Flow through a Multichannel Supply System with Random Channel Capacitances

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***Abstract*—We consider a hypothetical flow of media transported by a system of N channels characterized by quenched random capacitances drawn independently from a given distribution. A question we address is how individual capacitances influence a resulting flow through the whole system. Under increasing flux of transported media channels successively clog and the system approaches its limit of functionality. This limiting state is characterized by the critical flux Φ_c and the smallest number $n_c < N$ of non-clogged channels. We assume that after each clog an amount of flux carried by the clogged channel is undertaken by neighbouring channels if they are accessible or else by the whole system. Using computer simulations we study distributions of Φ_c and n_c. We show that for capacitances distributed uniformly or according to the Weibull distribution Φ_c and n_c are skew-normally distributed. For varying shape parameter ρ of the Weibull distribution expectations $(1-\overline{\Phi}_c/N)$ and $(1-\overline{n}_c/N)$ scale as $\sim \rho^{-\beta} \cdot \exp(-\gamma \cdot \rho)$, where β, γ depend on N and are different for both quantities.**

***Index Terms*—clogging, failure, multicomponent system, supply network, statistics, transfer rule.**

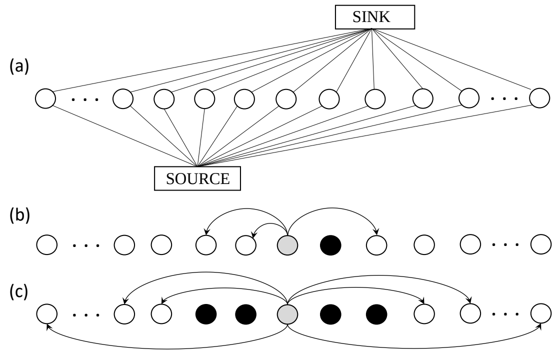

Fig. 1. Schematic view of a multichannel system with disks representing channels (a). White and black disks refer to open and clog channels, respectively. Shaded disk represents just clogged channel whereas arrows indicate transfer of blocked flux to (b) neighbouring channels if they are opened or (c) to all other channels if the neighbourhood is inaccessible.

I. Introduction

CONSIDER a supply network composed of a large number of functionally identical channels which preform a common task consisting on transmission of a specified medium. Such supply organisation is present in air filters, lubrication systems and piping networks to name a few. When the system operates some of its channels may become clogged reducing the system's efficiency. A question is how the system is resistant to subsequent clogs which appear among its channels. This question is important because clogs may accumulate in a way triggering blockade of the whole system [1], [2]. Once the set of channels is under an increasing flux it starts to clog when the internal flux exceeds the smallest-capacitance value. Then, the congestion develops, involves other over-saturated channels and migrates through the system as an avalanche of simultaneously clogged components.

To analyse avalanches of clogs we use so-called load transfer models. Specifically, we employ the Fibre Bundle Model (FBM) that offers an efficient framework to study evolving failures in technological processes and in Nature [3], [4], [5], [6], [7].

Our system is a set of channels located at nodes of a lattice presented in Fig. 1. Channels differ in fluxes they can transport. For example, when a high-flow air filter operates its channels capture contaminants and thus capacitances gradually decrease. Due to different sources of imperfections, capacitances are non-uniform and in simulations they are modelled by quenched random variables $\{c_i\}$. We consider two probability distributions of $\{c_i\}$: an uniform and the Weibull. Especially, the latter one is widely employed in analysis of failures in engineering systems [6], [8], [9], [10], [11], [12], [13], [14], [15], [16].

In our simulations, a set of N channels is subjected to a quasi-statically growing flux (Φ) of transported medium. Under the growing flux channels begin to congest and then clog. When a channel clogs, its flux is transferred to the other transporting channels which in turn increases the probability of subsequent clogs. A possible sequence of clogs among channels decreases the system performance and may eventually trigger a catastrophic avalanche of clogs. Such an avalanche develops when Φ reaches a specific, we call it critical, level Φ_c. This avalanche involves all still working channels and the whole system becomes blocked.

In this work, the set of channels is represented by a collection of nodes of a graph, see Fig. 1, and then analysed under a FBM framework. We assume there is no flow through clogged channels and thus we limit our study to the case where each channel can be in one of two states: conducted or clogged.

II. Simulation framework

The rule of flux transfer is a fundamental factor of the model. Among many different rules there are two extreme ones: global flux sharing (GLS) and local flux sharing (LLS) [17], [18], [19], [20].

We transfer a flux from a clogged channel according to a rule symbolized by arrows in Fig. 1(b)-(c). To be concrete consider one of channels, say i-th channel and assume the local flux $\phi_i \leq c_i$. When Φ increases enough to exceed $\phi_i > c_i$ locally then the i-th channel become over-saturated and stops working. At this circumstance ϕ_i has to be undertaken by other channels. At first attempt we share

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ϕ_i equally among nearest-neighbouring channels if they are accessible. If no such neighbouring channels in the system the flux is uniformly distributed through entire set of undersaturated channels. The first attempt corresponds to the LLS transfer whereas the latter one is in the spirit of the GLS rule. Effectively such a flux-transfer process represents a mixed LLS-GLS transfer rule.

Because of mixed-range-flux transfer, internal fluxes are distributed non-uniformly and clusters of channels with flux accumulation appear across the whole system. Growing internal flux in the transporting channels induces other clogs, after which each surviving channel sustains increasing flux. If the flux transfer does not trigger further clogs, a stable fluxes' configuration emerges. This means that the given Φ is not high enough to block the entire system, and the applied flux of transported medium has to increase.

In the simulations we apply a quasi-static procedure: if the system is in a stable state the applied flux Φ increases by an amount sufficient to clog the channel with the smallest $(c_i - \phi_i)$. A sequence of increases in the value of external flux gives Φ_c, i.e. Φ_c corresponds to a marginally stable state of the system, whereas $\Phi_c + \delta\Phi$ induces an avalanche of clogs among all remaining channels. Application of quasi-static procedure allows us to identify a flux Φ necessary for clogging of all the channels and thus to get Φ_c and n_c that characterize the set of channels on the edge of its functionality.

To determine the initial state of the supply-system we assign capacitances of channels randomly. In our simulation capacitances $\{c_i\}$ are uniformly random or are assigned according to the Weibull distribution [6], [8]. The corresponding probability density function of the former distribution is given by

$$p_{\rho,\lambda}(c) = (\rho/\lambda)(c/\lambda)^{\rho-1} \exp[-(c/\lambda)^k] \qquad (1)$$

The distribution (1) involves two parameters: $\lambda > 0$ is the scale parameter and the shape parameter ($\rho > 0$) controls the amount of disorder in the system. The scale parameter my be used to tune distributed capacitances in order to relate them to capacitances governed by another distribution. Since in this work such another distribution is the uniform one we assume $\lambda = 1$ and thus the corresponding probability density reads

$$p_\rho(c) = \rho c^{\rho-1} \exp(-c^\rho) \qquad (2)$$

The main question is how quenched random capacitances, governed by (2) or capacitances distributed uniformly over $[c_{min}, c_{max}]$, determine the resulting critical flux Φ_c and limiting number of working channels n_c. Based on results of simulations, we have found that coefficient of skewness of distribution of Φ_c/n_c decreases with growing number of channels and becomes negative when $N \geq 900$. We have also observed that data are correctly fitted by a three-parameter skew-normal distribution [21], [22], [23] defined by the density function

$$\psi_{SN}(x) = \frac{\operatorname{erfc}\left(-\alpha \frac{x-\mu}{\sqrt{2}\sigma}\right)}{\sqrt{2\pi}\sigma} \exp\left[-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2\right] \qquad (3)$$

where μ, σ and α are respectively: location, scale and shape parameters.

III. Critical flux and minimal number of channels

Applying the mixed LLS-GLS rule depicted in Fig. (1), we simulated supply processes in systems with number of channels $400 \leq N \leq 14400$ employing uniform and Weibull distributions to quench channel's capacitances $\{c_i\}$. In the latter case we modelled non-uniformity of $\{c_i\}$ by changing values of the Weibull shape parameter in the range $2 \leq \rho \leq 9$. To achieve reliable estimates of Φ_c and n_c each simulation was repeated 10^4 times. We have collected large data sets containing detailed information about applied flux (Φ) and numbers of non-clogged channels (n). These Φ's and n's data sets have enabled us to determine appropriate statistics by merging Φ_c with corresponding number of transmitting channels n_c. Some empirical estimators as *e.g.* the mean values, the standard deviations or the skewness have been taken into account as well.

The simulation framework presented in Section II enabled us to collect data sets of critical fluxes Φ_c with corresponding minimal numbers of non-blocked channels n_c for various system sizes and distributions of capacitances. Then, based on these sets we have analysed resulting empirical probability density functions.

A. Uniformly distributed capacitances

In Fig. (2) we show exemplary distributions of critical flux Φ_c scaled by corresponding numbers of channels N, for systems with uniform random capacitances. It turns out that Φ_c/N are skew-normally distributed (3) with parameters related to N through power-law functions

$$\mu \sim N^{\delta_\mu}, \sigma \sim N^{\delta_\sigma}, \alpha \sim N^{\delta_\alpha} \qquad (4)$$

The exponents $\delta_\mu \approx -0.085, \delta_\sigma \approx -0.236$ are computed using best-fit lines presented in Fig. (3). Albeit not displayed

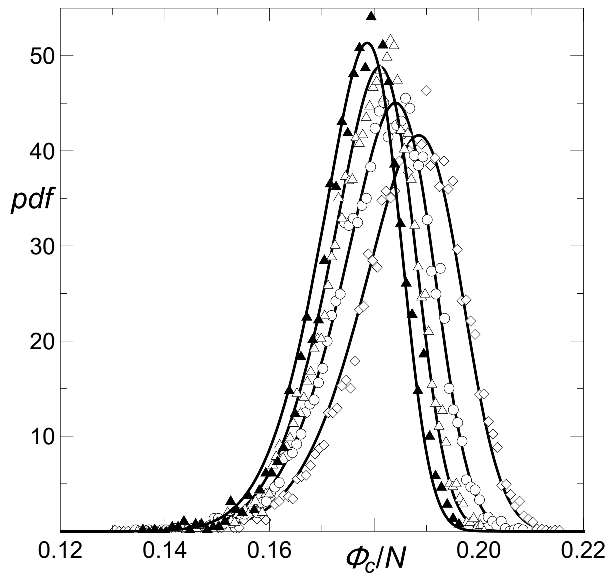

Fig. 2. Probability density of scaled critical flux Φ_c/N for systems with growing number of channels: $N = 3600$ (rhombus), 6400 (circles), 10000 (triangles) and 14400 (black triangles). Capacitances are distributed uniformly over $[0, 1]$. The solid lines represent skew normally distributed Φ_c/N with the parameters computed from the simulations. The results are obtained from 10^4 samples for each value of N

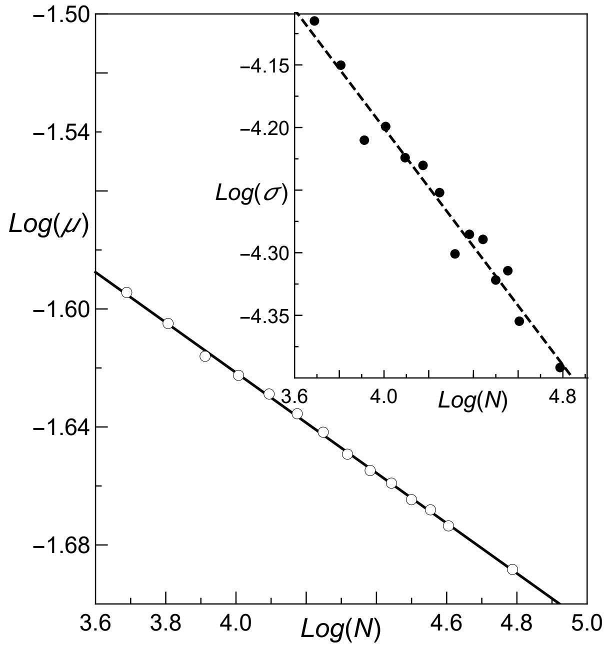

Fig. 3. Parameters μ and σ (inset) of skew-normally fitted empirical distributions of Φ_c/N, Eq. (3) for number of channels N ranging from 1600 to 14400. Capacitances are distributed uniformly over $[0, 1]$. Slopes of best-fit lines represent values of δ_μ and δ_σ in Eq. (4)

in a figure, the exponent δ_α is also computed using best-fit line on our scatter plot $\log(\alpha)$ vs $\log(N)$. We estimated its value as $\delta_\alpha \approx -0.7$.

Relations (4) enable us to estimate mean $\overline{\Phi}_c/N$. Using (3) the mean value reads

$$\overline{\Phi}_c/N = \mu + \frac{2\alpha\sigma}{\pi\sqrt{1+\alpha^2}}, \quad (5)$$

and thus, for $N \gg 1$

$$\overline{\Phi}_c \sim N^{1+\delta_\mu} \cdot (1 + \xi N^{\delta_\sigma+\delta_\alpha-\delta_\mu}) \sim N^{0.915}. \quad (6)$$

The scaling (6) holds because of the mixed LLS-GLS rule. When the GLS is applied $\overline{\Phi}_c \sim N$, whereas the LLS rule introduces a logarithmic correction to $\overline{\Phi}_c$, namely $\overline{\Phi}_c \sim N/\log(N)^{0.41}$ [24].

We have mentioned in Sec. (II) that the quasi-statically growing flux systematically reduces number of non-clogged channels and the system approaches critical state at which the maximal flux (Φ_c) is carried by the reduced number of channels (n_c). This critical state is marginally stable and n_c is minimal due to the fact that any subsequent clog induces an avalanche of clogs involving the whole system. This means that Φ_c/n_c is directly related to an average density of flux. In Fig. (5) we display empirical distributions of Φ_c/n_c for systems with $N_1 = 3600$ and $N_2 = 10000$ channels. It is clearly seen that bigger N corresponds to smaller value of $\overline{\Phi_c/n_c}$. Qualitatively, this behaviour of $\overline{\Phi_c/n_c}$ my be deduced from Figs. (2) and (4) which show that Φ_c/N decreases with growing N whereas n_c/N increases. Obviously a quantitative analyse requires data displayed in Fig. (5).

B. Weibull distribution of capacitances

The Weibull probability distribution is of a particular interest in reliability engineering, especially when systems involve so-called "weakest link." We apply this distribution because channels with the smallest capacitances initiate congestions and ultimate failure of the system.

For all values of N and ρ explored in our simulations the null hypothesis that the corresponding data is distributed according to (3) with parameters computed from the data is not rejected based on the Cramér-von Mises goodness of fit test [25].

Let us assume that the system operates under a near-maximal flux and consider two quantities of primary interest: (i) mean number of non-clogged channels $\overline{n}_c/N$ and (ii) expected critical flux $\overline{\Phi}_c/N$. It is worth mentioning that for $\rho \gg 1$ both these quantities $\overline{\Phi}_c \sim N$ and $\overline{n}_c \sim N$. These limiting relations result from (2) because for large values of ρ the Weibull distribution localises capacitances $\{c_i\}$ strongly around the mean value $\overline{c}$ and thus

$$\overline{c} = \Gamma(1 + 1/\rho) \to 1 \quad \text{as} \quad \rho \to \infty \quad (7)$$

The above limit indicates that almost all capacitances c_i are in a close vicinity of 1 and the only avalanche of clogs is the critical one. Thus, since the system fails in one step then $\Phi_c/N \sim \min(c_i) \simeq 1$.

Based on numerical data we fitted $\overline{\Phi}_c/N$ and $\overline{n}_c/N$ to the following symbolic non-linear models

$$\overline{n}_c/N = 1 - \frac{\alpha_N}{\rho^{\beta_N} \cdot \exp(\gamma_N \cdot \rho)} \quad (8)$$

$$\overline{\Phi}_c/N = 1 - \frac{a_N}{\rho^{b_N} \cdot \exp(d_N \cdot \rho)} \quad (9)$$

where the parameters $\alpha_N, \dots, d_N$ are slowly decreasing functions of N. As an example, numerical values of $\alpha_{3600}, \dots, d_{3600}$ and corresponding errors are presented in Table I. We would like to underline that (8) and (9) reflect not only the Weibull distribution of capacitances but also the

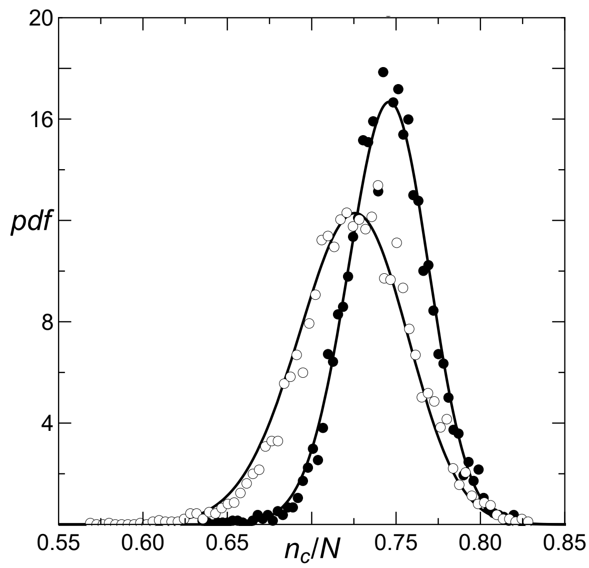

Fig. 4. Distributions of scaled minimal number of channels n_c/N for systems with $N = 3600$ (white disks) and $N = 10000$ (black disks) channels. Capacitances are distributed uniformly over $[0, 1]$. The solid lines represent skew normally distributed n_c/N with the parameters computed from the simulations. The results are obtained from 10^4 samples for each value of N

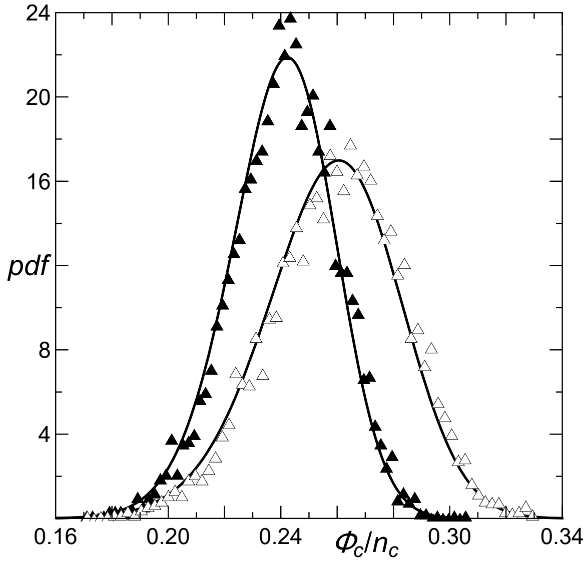

Fig. 5. Empirical probability density of Φ_c/n_c for systems with number of channels $N = 3600$ (white triangles) and 10000 (black triangles). Capacitances are distributed uniformly over $[0,1]$. The solid lines are drawn using (3) with parameters estimated from the simulations. Sample size is 10^4 for each value of N

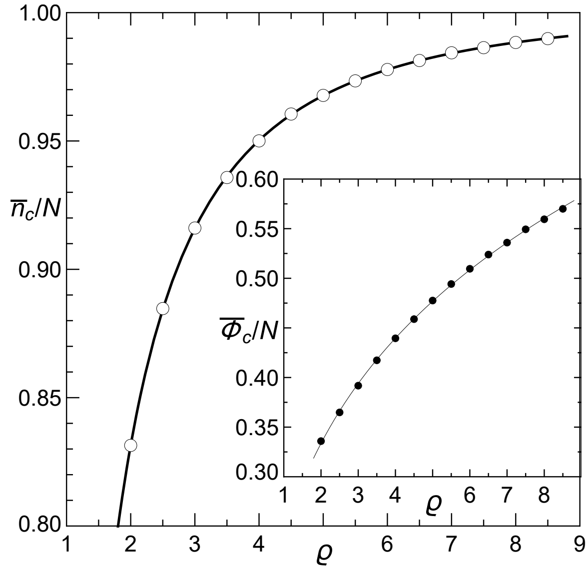

Fig. 6. Mean empirical $\overline{n}_c/N$ (white disks) and $\overline{\Phi}_c/N$ (black disks) for $N = 3600$ channels with capacitances distributed according to (2). The solid lines are drawn using (8) and (9) for the main plot and the inset, respectively. Estimated parameters are presented in Tab. I

TABLE I
PARAMETERS OF $\overline{n}_c/N$ (8) AND $\overline{\Phi}_c/N$ (9) FOR SYSTEMS WITH $N = 3600$ CHANNELS

Parameter	Estimate	Standard Error	Confidence Interval
α_{3600}	0.57059	0.00199	(0.56622, 0.57496)
β_{3600}	1.43106	0.01465	(1.39881, 1.46331)
γ_{3600}	0.11403	0.00433	(0.10450, 0.12356)
a_{3600}	0.79308	0.00340	(0.78559, 0.80057)
b_{3600}	0.15056	0.00918	(0.13035, 0.17077)
d_{3600}	0.03467	0.00208	(0.03010, 0.03924)

mixed LLS-GLS rule applied to transfer fluxes originated from clogged channels. If the same Weibull distribution is applied but with the LLS rule then the best-fitting (9) changes to

$$\overline{n}_c/N = 1 - \frac{\tilde{a}_N}{\rho^{7/4}} \tag{10}$$

as was observed in other over-loaded systems [24].

Presented in Fig. (7) distributions of Φ_c/n_c for variable N and constant ρ show that $\overline{\Phi_c/n_c}$ decreases with growing system size N. This is the same tendency, discussed already by the end of Sec. (III-A) and seen in Fig. (5).

A complementary analyse concerns a question how Φ_c/n_c is affected by the strength of disorder imposed by the Weibull distribution alone. We consider systems with variable ρ and $N = const$. The Fig. (7) involves empirical distributions of Φ_c/n_c collected for $2 \le \rho \le 8$. It is clearly seen that these distributions are ordered, i.e.

$$\rho_1 \le \rho_2 \Rightarrow \overline{\Phi_c/n_c}(\rho_1) \le \overline{\Phi_c/n_c}(\rho_2) \tag{11}$$

We would underline that (11) can not be directly deduced from (8) and (9) because both Φ_c/N and n_c/N grow with growing ρ. A detailed information about Φ_c/n_c can be gained only by collating, sample by sample, the critical flux with the corresponding number of non-clogged channels.

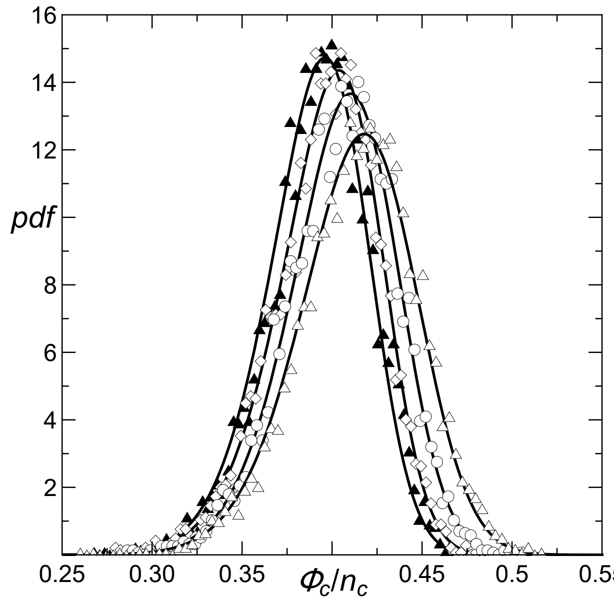

Fig. 7. Empirical distributions of Φ_c/n_c for systems with number of channels $N = 2500$ (white triangles), 3600 (disks), 4900 (rhombus) and 6400 (black triangles). Capacitances are assigned according to the Weibull *pdf* (2) with $\rho = 2$. The solid lines follow (3) with parameters estimated from the simulations.

C. Uniform versus Weibull distributions of capacitances

In this Subsection we contrast Φ_c and n_c related to systems with capacitances distributed according to the Weibull distribution to these ones characterising systems with uniformly random capacitances of channels. We chose the support of uniform distribution as $[0, 2\mu]$ so that both the distributions share the same mean value $\mu = \Gamma(1 + 1/\rho)$. Specifically, we apply $\rho = 2$ in order to generate the greatest amount of disorder in the system. In Fig. (9) we show distributions of Φ_c/n_c obtained for $N = 3600$. The mean $\overline{\Phi_c/n_c}$ for systems with uniformly random capacitances is $\sim 20\%$ higher than

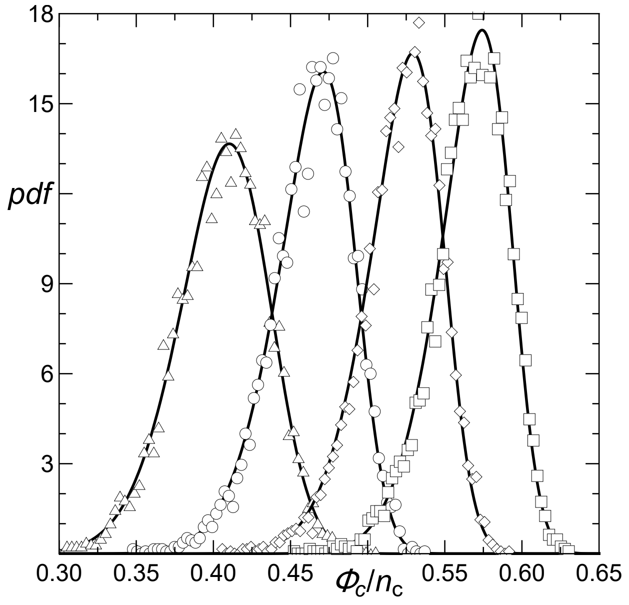

Fig. 8. Empirical distributions of Φ_c/n_c for systems with 3600 channels. Capacitances are distributed using (2) with $\rho = 2$ (triangles), 4 (disks), 6 (rhombus) and 8 (squares). The solid lines are plotted with (3) and parameters computed from the simulations.

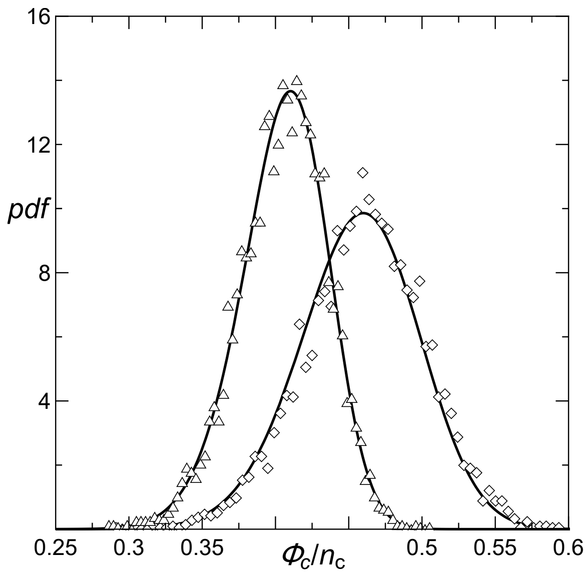

Fig. 9. Empirical probability density functions of Φ_c/n_c for systems with 3600 channels. Capacitances $\{c_i\}$ are distributed uniformly over $[0, 2\mu]$ (open rhombus), and according to the Weibull distribution with $\rho = 2$ (triangles). Both distributions of $\{c_i\}$ have equal mean $\mu = \Gamma(3/2)$. The results are obtained from 10^4 samples for each distribution.

that for the Weibull distribution. This difference indicates that systems with uniformly distributed capacitances operate under higher local fluxes and smaller number of non-blocked channels than systems whose c_i are governed by the corresponding Weibull distribution.

IV. Final remarks

We have examined statistics of flows in a multi-channel supply system subjected to a quasi-statically increasing flux of transported media. We have characterised channels by quench random capacitances distributed uniformly or according to the Weibull probability distribution and introduced the mixed LLS-GLS rule to transfer fluxes from clogged channels to non-clogged ones. Based on results of simulations we have shown that the experimental distributions of the critical flux Φ_c, minimal number of non-blocked channels n_c as well as the local-flux intensity Φ_c/n_c follow the skew-normal distribution for both considered distributions of capacitances. By fitting discrete distributions we have found that:

(i) for uniformly random capacitances expected maximal flux transported by the system depends mainly on the rule of flux transfer, i.e. $\Phi_c \sim N^{-0.915}$ for the mixed LLS-GLS rule examined in this work, $\Phi_c \sim N$ for the GLS rule and $\Phi_c \sim N/(\log N)^{0.41}$ when the LLS rule is applied

(ii) for capacitances governed by the Weibull distribution with the shape parameter ρ the mixed LLS-GLS rule yields the scaling $(1 - \overline{n}_c/N) \sim \rho^{-b_N} \cdot \exp(-d_N \cdot \rho)$ whereas under the LLS rule $(1 - \overline{n}_c/N) \sim \rho^{-7/4}$

(iii) minimal numbers of non-clogged channels n_c are skew-normally distributed when the mixed LLS-GLS rule is used in contrast to normally distributed n_c when the LLS rule is applied.

We are aware that our choice of flux transfer rule is arbitrary. The mixed LLS-GLS rule introduced in this work represents a reasonable alternative to other rules, however. This rule attempts to distribute locally accumulated over-flux among accessible neighbouring channels. The entire system is engaged only when such a neighbourhood does no longer exist.

References

[1] S. Redner and S. Datta, “Clogging Time of a Filter”, *Phys. Rev. Lett.*, vol. 84, pp. 6018-6021, June 2000.

[2] J.H.D. Hampton, S.B. Savage and R.A.L.Drew, “Computer modelling of filter pressing and clogging in a random tube network”, *Chem. Eng. Sci.*, vol. 48, no. 9, pp. 1601-1611, 1993.

[3] A. Hansen, P.C. Hemmer, and S. Pradhan, “The Fiber Bundle Model: Modeling Failure in Materials,” Weinheim, Wiley-VCH, 2015.

[4] F. Kun, F. Raischel, R.C. Hidalgo, and H.J. Herrmann, “Extensions of fibre bundle models,” in *Modelling Critical and Catastrophic Phenomena in Geoscience, Lecture Notes in Physics*, P. Bhattacharyya and B.K. Chakrabarti, Eds, vol. 705, Berlin: Springer 2006, pp. 57-92.

[5] M.J. Alava, P.K.V.V. Nukala, and S. Zapperi, “Statistical models of fracture,” *Adv. In Physics*, vol. 55, pp. 349-476, April 2006.

[6] W. Weibull, “A statistical distribution function of wide applicability,” *J. Appl. Mech.*, vol. 18, pp. 293-297, Sept. 1951.

[7] D. Cohen, P. Lehmann and D. Or,“Fiber bundle model for multiscale modeling of hydromechanical triggering of shallow landslides,” *Water Resources Res.*, vol. 45, W10436, Oct. 2009.

[8] N.M. Pugno and R.S. Ruoff, “Nanoscale Weibull statistics,” *J. Appl. Phys.*, vol. 99, id. 024301, Jan. 2006.

[9] Z. Bertalan, A. Shekhawat, J.P. Sethna, and S. Zapperi, “Fracture strength: Stress concentration, extreme value statistics and the fate of the Weibull distribution,” *Phys. Rev. Applied*, vol. 2, id. 034008, Sept. 2014.

[10] C. Manzato, A. Shekhawat, P. K. V. V. Nukala, M. J. Alava, J. P. Sethna, and S. Zapperi, “Fracture Strength of Disorder Media: Universality, Interactions, and Tail Asymptotics,” *Phys. Rev. Lett.*, vol. 108, id. 065504, Feb. 2012.

[11] F. Raischel, F. Kun, and H.J. Herrmann, “A simple beam model for the shear failure of interfaces,” *Phys. Rev. E*, vol. 72, id. 046126, Oct. 2005.

[12] J. Knudsen and A.R. Massih, “Breakdown of disordered media by surface loads,” *Phys. Rev. E*, vol. 72, id. 036129, Sept. 2005.

[13] S. Zapperi, P. Ray, H.E. Stanley, and A. Vespignani, “Analysis of damage clusters in fracture processes,” *Phys. A*, vol. 270, pp. 57-62, Aug. 1999.

[14] Z. Domański and T. Derda, “Distribution of critical load in arrays of nanopillars,” *Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2017*, 5-7 July, 2017, London, UK, pp. 797-801.

[15] Z. Domański, "Simulation Study of Failures in Progressively Loaded Multicomponent Systems," *Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017*, 25-27 October, 2017, San Francisco, USA, pp. 742-746.

[16] Z. Domański and T. Derda, "Critical Load of Cyclically Preloaded Arrays of Pillars," *Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2018*, 23-25 October, 2018, San Francisco, USA, pp. 606-610.

[17] R.C. Hidalgo, Y. Moreno, F. Kun, and H.J. Herrmann, "Fracture model with variable range of interaction," *Phys. Rev. E*, vol. 65, id. 046148, April 2002.

[18] R.C. Hidalgo, S. Zapperi, and H.J. Herrmann, "Discrete fracture model with anisotropic load sharing," *J. Stat. Mech.*, id. P01004, Jan. 2008.

[19] T. Derda, "Analysis of damage processes in nanopillar arrays with hierarchical load transfer," *J. Appl. Math. Comput. Mech.*, vol. 15(3), pp. 27-36, 2016.

[20] "Statistics of critical avalanches in vertical nanopillar arrays," in *Transactions on Engineering Technologies. Lecture Notes in Electrical Engineering*, G.C. Yang, S.I. Ao, X. Huang, O. Castillo, Eds., vol. 275, Dordrecht: Springer, 2014, pp. 1-11.

[21] A. OHagan, and T. Leonard, "Bayes estimation subject to uncertainty about parameter constraints," *Biometrika*, vol. 63, 201-202, April 1976.

[22] A. Azzalini and A.R. Massih, "A class of distributions which includes the normal ones," *Scand. J. Statist.*, vol. 12, pp. 171-178, June 1985.

[23] A. Azzalini, "The Skew-Normal and Related Families," Cambridge: Cambridge University Press, 2013.

[24] Z. Domanski, "Damage Statistics in Progressively Compressed Arrays of Nano-pillars", *Eng. Letters*, vol. 27, no. 1, pp. 18-23, February 2019.

[25] T.B. Arnold and J.W. Emerson, "Nonparametric Goodness-of-Fit Tests for Discrete Null Distributions," *The R Journal*, Vol. 3/2, pp. 34-39, December 2011.