A Fishery Resource Optimal Harvesting Model with Reserved Area having Bird as Predator

Kulbhushan Agnihotri, Sheenu Nayyer

Abstract: **In this paper, a fishery model with a reserved and unreserved area for prey in the presence of bird predator has been proposed and analysed. Holling type II functional response is considered for this research work. The harvesting is applied on predator as well as on prey in an unreserved area due to some commercial value. Local stability of the system is discussed in this paper whereas threshold for the existence of biological and bionomical equilibrium points of the model have also determined. Further, optimal harvesting policy has been studied with the assistance of the Pontryagin's Maximum Principle. Finally, the theoretical results have obtained and verified with the help of numerical simulations through MATLAB.**

***Keywords*: Prey-predator; Stability; Optimal harvesting; Bionomical equilibrium.**

I. INTRODUCTION

In the past few decades, dynamics of associating natural species has been examined from various angles [1-6]. Many species have become threatened or endangered, and many others are on the verge of extinction due to various reasons like overexploitation, over-predation, environmental pollution, mismanagement of natural resources etc. To save these species, marine protected and marine reserved areas have been proposed as the most important tools to conserve the marine life and sustain ecosystems. Beaverton and Holt were the first in considering the idea of marine reserves. Clark [7] introduced the concept of economic and biological aspects of renewable resources of multispecies fisheries. Recently, it is proved by Dubey [8] that the reserved area has stabilizing effects on the predator-prey dynamics. It is demonstrated that regardless of whether the fishery is exploited constantly in the unreserved zone, fishery density can be kept up at an appropriate equilibrium level in the natural surroundings. Kar and Misra [9] have studied that the interior equilibrium level is never disturbed. It has been observed that, in the absence of predators, even under continuous harvesting in the unreserved zone, the fish population may be maintained at an appropriate equilibrium level. On the other hand, in the presence of predators, populations may be sustained at an appropriate equilibrium level if the population in the unreserved zone lies in a certain interval.

Dubey [10] proved that the role of the reserved zone is an essential coordinating idea in ecology and evolution. By creating reserved zones in the habitat, where the predatorhas no access or chance of settling, the prey species can grow without any external disturbances.

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Therefore, the prey species can be maintained at an appropriate level. As per observational information of Lake Kasumigaura in Japan, Kitabatake [11] built up a dynamic model for fishery assets with a prey–predator system in a two-patch environment. He examined the possibilities of the existence of bionomic equilibrium and an optimal harvesting policy is given by Pontryagin’s maximum principle. Srinivasu and Gayatri [12] observed that the reserve capacity has a vital part in guaranteeing eitherpresence of predators or their eradication. Kar and Matsuda [13] established the significance of marine protected areas (MPAs), from both economic and biological perspective. Rui Zhang [14] has demonstrated that in the absence or within the sight of predators, the fishing populations may be remaining at an suitable equilibrium level. Yunfei et al. [15] investigated that marine reserves ensured the sustainability of the system. Aquatic reserves secure both the species inside the reserve area as well as increase fish abundance in adjoining areas. An appropriate equilibrium level of prey population is always maintained irrespective of presence or absence of predators in the unreserved zone. Amit Sharma and Bhanu Gupta [16] studied the dynamics of fishery resource with reserve area in the presence of bird predator. In this work they have given criteria for finding the biological and bionomic equilibrium points of the system.Optimal harvesting policy has also established using Pontryagin’s maximum principle.

All the reserve and unreserved area models have been motivated by the marine national park in Kenya, and in the Iroise sea, a coastal sea west of Brittany (France). Where artificial boundary in the form of fencing of suitable mesh size, has been created to restrict the entry of predator in the reserved zone.The Latest researchersfound that MPAs are very effective tool for enhancing yield as well as assurance of stocks and sustainability of endangered species.

Keeping this in view,we have modified the model proposed by Amit Sharma *et al*. [16], within the sight of bird predator. Holling type-II predator functional response is considered, which seems to be more realistic then Holling type-I functional response.

II. THE MATHEMATICAL MODEL

Consider a habitat, in a biological community, with prey (fishes) dispersal in a two-patch environment, one is assumed to be a free fishing zone and other is a reserved zone, where fishing and other additional activities are restricted. Both zones are supposed to be homogeneous. There is also a bird predator in this system, which is feeding on fishes in both the reserved and unreserved zones. It is assumed that the bird predator population is also harvested in the unreserved zone. We assume that he fish species

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migrate between the two zones randomly. Keeping view the all above assumptions, a model is represented by the following ordinary differential equations.

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \sigma_1 x + \sigma_2 y - \frac{m_1 xz}{\alpha + x} - q_1 E_1 x$$

$$\frac{dy}{dt} = sy\left(1 - \frac{y}{L}\right) + \sigma_1 x - \sigma_2 y - \frac{m_2 yz}{\alpha + y}$$

$$\frac{dz}{dt} = -dz + \left(\frac{k_1 m_1 xz}{\alpha + x} + \frac{k_2 m_2 yz}{\alpha + y}\right) - q_2 E_2 z \qquad (1)$$

All the parameters of the system (1) are assumed to be positive and are defined in the following Table.

Table I
Variables and Parameters used in Proposed Model

Variable /Parameter	**Description**
$x(t)$	Biomass density of the prey species inside the unreserved area
$y(t)$	Biomass density of the prey species inside the reserved area
$z(t)$	Biomass density of the bird predator
r	Intrinsic growth rate of the prey species inside the unreserved area
s	Intrinsic growth rate of the prey species inside the reserved area
K	Carrying capacity of prey species inside the unreserved area
L	Carrying capacity of prey species inside the reserved area
σ_1, σ_2	Migration rate from the unreserved area to reserved area and reserved area to unreserved area respectively
E_1, E_2	Harvesting efforts applied on the prey (fishes) and the predator(bird) respectively
q_1, q_2	Catchability coefficient of prey in unreserved area and predator respectively
m_1, m_2	Capturing rates of prey by predator in unreserved and reserved area respectively
k_1, k_2	Conversion rates of prey to predator young ones in unreserved and reserved zone respectively
d	Death rate of a predator(bird)
α	Half saturation level coefficient

In above model, if there is no-relocation of fish population from the reserved zone to the unreserved zone *i.e* $(\sigma_2=0)$ and $r - \sigma_1 - q_1 E_1 < 0$, then $\frac{dx}{dt} < 0$, thus fish species will extinct from the unreserved area. Similarly if there is no migration of fish population from unreserved area to reserved area *i.e.* $(\sigma_1 = 0)$ and $s - \sigma_2 < 0$, then $\frac{dy}{dt} < 0$ holds consequently, fish species will extinct from the reserved area. To protect the fish species from extinction, migration of prey species from both the patches is mandatory. Therefore,we assume that

$$r - \sigma_1 - q_1 E_1 > 0 \text{ and } s - \sigma_2 > 0 \qquad (2)$$

To make the proposed model easier, the capturing rates and conversion rates are considered to be equal *i.e.*

$k_1 = k_2 = k,\ m_1 = m_2 = m$ then $k_1 m_1 = k_2 m_2 = km = \alpha_1$

Thus model (1) becomes

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \sigma_1 x + \sigma_2 y - \frac{mxz}{\alpha + x} - q_1 E_1 x$$

$$\frac{dy}{dt} = sy\left(1 - \frac{y}{L}\right) + \sigma_1 x - \sigma_2 y - \frac{myz}{\alpha + y}$$

$$\frac{dz}{dt} = -dz + \alpha_1\left(\frac{xz}{\alpha + x} + \frac{yz}{\alpha + y}\right) - q_2 E_2 z \qquad (3)$$

III. STABILITY ANALYSIS

A.Boundedness of the solution

Theorem 1.*All the solutions of the model system (3) with the positive initial conditions* (x_0, y_0, z_0) *are uniformly bounded within* ψ*, and the set* $\psi \subseteq R_+^3$*, is positively invariant for the system (3),*

Where

$$\psi = \left\{ (x, y, z) \in R_+^3 : 0 \le x + y + \frac{m}{\alpha_1} z \le \frac{G}{d + q_2 E_2} \right\}$$

$$G = \frac{K}{4r}\left(r + d + q_2 E_2 - q_1 E_1\right)^2 + \frac{L}{4s}\left(s + d + q_2 E_2\right)^2$$

Proof**:** If $x(0) > 0,\ y(0) > 0$ and $z(0) > 0$

Let $\eta = x + y + \frac{m}{\alpha_1} z$ and $\eta > 0$ be a constant.

Then

$$\frac{d\eta}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{m}{\alpha_1}\frac{dz}{dt}$$

$$\frac{d\eta}{dt} = rx\left(1 - \frac{x}{K}\right) - q_1E_1x + sy\left(1 - \frac{y}{L}\right) - \frac{m}{\alpha_1}\left(d+q_2E_2\right)z$$

$$\frac{d\eta}{dt} + \eta\left(d+q_2E_2\right) \le -\frac{r}{K}x^2 + \left(r-q_1E_1+d+q_2E_2\right)x - \frac{s}{L}y^2 + \left(s+d+q_2E_2\right)y$$

Hence

$$\frac{d\eta}{dt} + \eta\left(d+q_2E_2\right) \le \frac{K}{4r}\left(r-q_1E_1+d+q_2E_2\right)^2 + \frac{L}{4s}\left(s+d+q_2E_2\right)^2 \underline{\underline{\Delta}}G$$

Applying the theory of differential inequality (Birkoff and Rota, 1982) , following inequality will be obtained

$$0 \le \eta(t) \le e^{-\left(d+q_2E_2\right)t}\left(\eta(0) - \frac{G}{d+q_2E_2}\right) + \frac{G}{d+q_2E_2} \qquad (4)$$

As $t \to \infty$ solution of (3) will be always within the set ψ . Thus, the system (3) is dissipative. The proof of the theorem is completed

B. The existence of non-negative equilibria

Following are the three possible equilibria of the system (3)

I. $P_0(0,\,0,\,0)$, which always exists; (extinction of all species)

II. $P_1(x_1, y_1, 0)$, (The predator free equilibrium point)

III. $P_2(x^*, y^*, z^*)$, (The interior equilibrium point)

The predator-free equilibrium point $P_1(x_1, y_1, 0)$

From the first two equations of (3)

$$\sigma_2 y = \frac{rx^2}{K} - \left(r-\sigma_1-q_1E_1\right)x, \quad \sigma_1 x = \left(\sigma_2 - s\right)y + \frac{sy^2}{L} \qquad (5)$$

After simplification, we get cubic equation in x as

$$ax^3 + bx^2 + cx + d = 0 \qquad (6)$$

Where

$$a = \frac{sr^2}{L\sigma_2^2K^2},\; b = \frac{-2rs\left(r-\sigma_1-q_1E_1\right)}{L\sigma_2^2K}$$

$$c = \frac{s\left(r-\sigma_1-q_1E_1\right)^2}{L\sigma_2^2} - \frac{\left(s-\sigma_2\right)r}{K\sigma_2},\; d = \frac{\left(s-\sigma_2\right)\left(r-\sigma_1-q_1E_1\right)}{\sigma_2} - \sigma_1$$

By Descartes rule of a sign, the equation (6) has a unique positive solution $x=x_1$ if the following inequalities hold.

$$\frac{s\left(r-\sigma_1-q_1E_1\right)^2}{L\sigma_2} < \frac{\left(s-\sigma_2\right)r}{K},$$

$$\left(s-\sigma_2\right)\left(r-\sigma_1-q_1E_1\right) < \sigma_1\sigma_2 \qquad (7)$$

Knowing the value of x_1 the value of y_1 can be computed from (5) as

$$y_1 = \frac{1}{\sigma_2}\left[\frac{rx_1^2}{K} - \left(r-\sigma_1-q_1E_1\right)x_1\right]$$, which exist provided

$$x_1 > \frac{K}{r}\left(r-\sigma_1-q_1E_1\right) \qquad (8)$$

Hence, $P_1(x_1, y_1, 0)$ exist, provided conditions (7) and (8) are Satisfied.

For the interior equilibrium point $P_2(x^*, y^*, z^*)$

On solving system (3) for non-zero point, we get

$$y = \alpha\left[\frac{\alpha\left(d+q_2E_2\right) - xE_6}{\left(E_6+\alpha_1\right)x + \alpha E_6}\right] \qquad (9)$$

$$z = \frac{\left(\alpha+x\right)}{mx}\left[\left(\frac{-rx}{K}+E_3\right)x + \frac{\alpha\sigma_2\{\alpha(d+q_2E_2)-xE_6\}}{\alpha E_6+(\alpha_1+E_6)x}\right] \qquad (10)$$

Value of x is the root of following 6[th] degree equation

$$S_1x^6 + S_2x^5 + S_3x^4 + S_4x^3 + S_5x^2 + S_6x + S_7 = 0 \qquad (11)$$

Where

$$S_1 = \frac{L\alpha E_6 rE_5^2}{K}$$

$$S_2 = \left[\begin{array}{l} -E_6L\alpha\left(E_3E_5 - \frac{r\alpha E_6}{K}\right) + \frac{L\alpha^2 r}{K}\left(E_6^2+E_5E_6-E_5E_7\right) \\ +L\alpha\sigma_1E_5\left(E_6-E_5\right) \end{array}\right]E_5$$

$$S_3 = \left[\begin{array}{l} L\alpha^2\left(E_3E_5-\frac{r\alpha E_6}{K}\right)\left((-E_6+E_7)E_5-E_6^2\right)+ \\ \alpha^2E_6^2E_5\left(s\alpha-L\left(E_4+2\sigma_1-E_3+\sigma_2\right)\right) \end{array}\right] - s\alpha^3E_6^3$$

$$-\frac{L\alpha^3rE_5}{K}\left(-E_6^2+E_6E_7+E_5E_7\right) + L\alpha^2E_5^2\left(E_4E_6-E_7\sigma_1-3\sigma_1E_6\right)$$

$$S_4 = \left[\begin{array}{l} L\alpha^3\left(E_3E_5-\frac{r\alpha E_6}{K}\right)\left(E_7E_5-E_6^2+E_6E_7\right)+\alpha^3E_6\left(s\alpha-LE_4\right)\left(-2E_5E_7+E_6^2\right) \\ +L\alpha^3\left(E_6^3\sigma_1-E_4E_7E_5^2-3\sigma_1E_6^2E_5+2E_6^2E_4E_5-2E_6E_5E_7\sigma_1-E_6E_5E_7\sigma_2\right) \\ +\alpha^4E_7E_6\left(3sE_6-\frac{LrE_5}{K}\right)+L\alpha^3E_6\left(E_3-\sigma_2\right)\left(-E_5E_6+E_7E_5-E_6^2\right) \end{array}\right]$$

$$S_5 = L\alpha^4E_6E_7\left(\left(E_3-\sigma_2+E_3-2E_4\right)E_5 - \left(\frac{r\alpha}{K}+\sigma_2\right)E_6\right)$$

$$+L\alpha^4\left(-E_6+E_7\right)\left(E_6^2(E_3-\sigma_2)+E_5\sigma_2E_7\right) + L\alpha^4E_6^2\left(E_4E_6-E_7\sigma_1\right)$$

$$+\alpha^4\left(s\alpha-LE_4\right)\left(E_7E_5-2E_6^2\right)E_7 - L\sigma_1E_6^3\alpha^4 - 3\alpha^5sE_7^2E_6$$

$$S_6 = \left[\begin{array}{l} L\alpha^5E_6^2E_7\left(-E_4+E_3-\sigma_2\right) + L\alpha^5\sigma_2E_7 \\ \left(E_7E_5-E_6^2+E_7E_6\right) + \alpha^5E_7^2\left(-LE_4E_6+s\alpha\left(E_6+E_7\right)\right) \end{array}\right]$$

$$S_7 = L\alpha^6\sigma_2E_6E_7^2 > 0$$

$$E_3 = r-\sigma_1-q_1E_1 > 0 \;(by\ assumption)$$

$$E_4 = s-\sigma_2 > 0 \;(by\ assumption)$$

$E_5 = 2\alpha_1 - d - q_2E_2$, $E_6 = \alpha_1 - d - q_2E_2$, $E_7 = d + q_2E_2 > 0$

Let $F(x) = S_1x^6 + S_2x^5 + S_3x^4 + S_4x^3 + S_5x^2 + S_6x + S_7$

It is obvious that

$F(0) = S_7 > 0$

and if

$$F(\mathrm{K}) = S_1K^6 + S_2K^5 + S_3K^4 + S_4K^3 + S_5K^2 + S_6K + S_7 < 0 \quad (12)$$

Then there exists a positive value of x $\left(\text{say x*}\right)$ in the interval $[0, K]$

Now, the sufficient condition for $x*$ to be unique is

$$F'(x^*) = 6S_1\left(x*\right)^5 + 5S_2\left(x*\right)^4 + 4S_3\left(x*\right)^3 + 3S_4\left(x*\right)^2 + 2S_5\left(x*\right) + S_6 < 0 \quad (13)$$

Value of y^* and z^* will be obtained from (9) & (10), and will be positive, if following inequality holds

$$E_6 > 0 \text{ and } x < \text{Min}\left[\frac{\alpha\left(d+q_2E_2\right)}{E_6}, K\left(1-\frac{\left(\sigma_1+q_1E_1\right)}{r}\right)\right] \quad (14)$$

Hence, the equilibrium $P_2\left(x^*, y^*, z^*\right)$ exists provided conditions (12), (13), (14) are satisfied.

C. Stability analysis

Theorem 2. *If the equilibrium point $P_0\,(0,\,0,\,0)$, exist, then it will be always unstable.*

Proof: The characteristic equation of the system (3) is

$$\begin{vmatrix} r-\sigma_1-q_1E_1-\lambda & \sigma_2 & 0 \\ \sigma_1 & s-\sigma_2-\lambda & 0 \\ 0 & 0 & -d-q_2E_2-\lambda \end{vmatrix} = 0$$

On solving the characteristic equation of system (3) at P_0 (0, 0, 0)is given by

$$\left(\lambda+d+q_2E_2\right)\left[\lambda^2-\lambda(r-\sigma_1-q_1E_1+s-\sigma_2)+(r-\sigma_1-q_1E_1)(s-\sigma_2)-\sigma_1\sigma_2\right] = 0$$

One of the eigenvalue is $\lambda_1 = -\left(d + q_2E_2\right) < 0$

Other two eigenvalues are given by

$$\lambda^2 - \lambda\left(r-\sigma_1-q_1E_1+s-\sigma_2\right) + \left(r-\sigma_1-q_1E_1\right)\left(s-\sigma_2\right) - \sigma_1\sigma_2 = 0$$

As $\left(r\text{-}\sigma_1\text{-}q_1E_1\right) + \left(s\text{-}\sigma_2\right) > 0$ (by assumptions)

So, all the eigenvalues of the above characteristics equation have not negative real parts as there is at least one change of sign. Therefore, the equilibrium point $P_0\,(0,\,0,\,0)$ is always unstable.

Biological meaning: With the creation of marine reserves, it is seen that the equilibrium point $P_0\,(0,\,0,\,0)$ remains always unstable. So it may be concluded that even if the system is exploited constantly in the unreserved zone, the prey or the predator populations persist and doesn't extinct for sufficiently large time.

Theorem 3. *If the Equilibrium point $P_1(x_1, y_1, 0)$ exists, then it will be always locally asymptotically stable, provided.*

$$2\alpha_1 < d + q_2E_2,\ \left(r - \frac{2rx}{K} - \sigma_1 - q_1E_1\right) + \left(s - \frac{2sy}{L} - \sigma_2\right) < 0$$

$$\left(r - \frac{2rx}{K} - \sigma_1 - q_1E_1\right)\left(s - \frac{2sy}{L} - \sigma_2\right) > \sigma_1\sigma_2$$

Proof: The Characteristic equation at $P_1(x_1, y_1, 0)$ is given by

$$\begin{vmatrix} r-\frac{2rx_1}{K}-\sigma_1-q_1E_1-\lambda & \sigma_2 & -\frac{mx_1}{\alpha+x_1} \\ \sigma_1 & s-\frac{2sy_1}{L}-\sigma_2-\lambda & -\frac{my_1}{\alpha+y_1} \\ 0 & 0 & -d-q_2E_2+\frac{\alpha_1x_1}{\alpha+x_1}+\frac{\alpha_1y_1}{\alpha+y_1}-\lambda \end{vmatrix} = 0$$

One of the eigenvalues is

$$\lambda_1 = -d - q_2E_2 + \frac{\alpha_1x_1}{\alpha + x_1} + \frac{\alpha_1y_1}{\alpha + y_1}$$

It will be negative, if

$$i.e\ \frac{\alpha_1x_1}{\alpha + x_1} + \frac{\alpha_1y_1}{\alpha + y_1} < d + q_2E_2$$

After simplification *i.e.*

$$2\alpha_1 < d + q_2E_2 \quad (15)$$

Other two eigen values are given by obviously

$$\lambda^2 - \left\{\left(r - \frac{2rx_1}{K} - \sigma_1 - q_1E_1\right) + \left(s - \frac{2sy_1}{L} - \sigma_2\right)\right\}\lambda + \left(r - \frac{2rx_1}{K} - \sigma_1 - q_1E_1\right)\left(s - \frac{2sy_1}{L} - \sigma_2\right) - \sigma_1\sigma_2 = 0$$

It's two eigens values will have negative real parts, provided

$$\left(r - \frac{2rx_1}{K} - \sigma_1 - q_1E_1\right) + \left(s - \frac{2sy_1}{L} - \sigma_2\right) < 0 \quad (16)$$

$$\left(r - \frac{2rx_1}{K} - \sigma_1 - q_1E_1\right)\left(s - \frac{2sy_1}{L} - \sigma_2\right) > \sigma_1\sigma_2 \quad (17)$$

Thus equilibrium point $P_1\left(x_1, y_1, 0\right)$ of the system (3) is locally asymptotically stable provided, (15), (16) and (17) hold.

Biological meaning: From above, it may be concluded that with the suitable harvesting of prey in the unreserved zone and of bird predator, it may be possible to extinct a bird predator from the system, if they are harmful to the system.

Theorem 4. *For the system (3), if the Interior equilibrium point $P_2\left(x^*, y^*, z^*\right)$ exists, then it will be always locally asymptotically stable provided*

$C_1 > 0, C_3 > 0$ and $C_1C_2 - C_3 > 0$, *where* C_1, C_2, C_3 *are given in the proof.*

***Proof*:** The characteristic equation of the variational matrix of the system (3) at P_2 is

$$\begin{vmatrix} -\frac{r}{K}x^* - \sigma_2\frac{y^*}{x^*} - \frac{mz^*\alpha}{(\alpha+x^*)^2} + \frac{mz^*}{\alpha+x^*} - \lambda & \sigma_2 & \frac{-mx^*}{\alpha+x^*} \\ \sigma_1 & -\frac{\sigma_1 x^*}{y^*} - \frac{sy^*}{L} - \frac{mz^*\alpha}{(\alpha+y^*)^2} + \frac{mz^*}{(\alpha+y^*)} - \lambda & \frac{-my^*}{\alpha+y^*} \\ \frac{\alpha\alpha_1 z^*}{(\alpha+x^*)^2} & \frac{\alpha\alpha_1 z^*}{(\alpha+y^*)^2} & 0-\lambda \end{vmatrix} = 0$$

On solving

$$i.e.\ \lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 = 0 \qquad (18)$$

Where

$$C_1 = \frac{sy^*}{L} + \frac{\sigma_1 x^*}{y^*} - \frac{mz^*}{\alpha+y^*} + \frac{mz^*\alpha}{(\alpha+y^*)^2} + \frac{rx^*}{K} + \frac{\sigma_2 y^*}{x^*} - \frac{mz^*}{\alpha+x^*} + \frac{mz^*\alpha}{(\alpha+x^*)^2}$$

$$C_2 = \frac{m\alpha\alpha_1 y^* z^*}{(\alpha+y^*)^3} - \sigma_2\sigma_1 + \frac{m\alpha\alpha_1 x^* z^*}{(\alpha+x^*)^3}$$
$$+\left(\frac{rx^*}{K} + \frac{\sigma_2 y^*}{x^*} - \frac{mz^*}{\alpha+x^*} + \frac{mz^*\alpha}{(\alpha+x^*)^2}\right)\left(\frac{sy^*}{L} + \frac{\sigma_1 x^*}{y^*} - \frac{mz^*}{\alpha+y^*} + \frac{mz^*\alpha}{(\alpha+y^*)^2}\right)$$

$$C_3 = \frac{\sigma_2 m\alpha\alpha_1 y^* z^*}{(\alpha+y^*)(\alpha+x^*)^2} + \frac{m\alpha\alpha_1\sigma_1 x^* z^*}{(\alpha+x^*)(\alpha+y^*)^2} +$$
$$\frac{m\alpha\alpha_1 x^* z^*}{(\alpha+x^*)^3}\left(\frac{sy^*}{L} + \frac{\sigma_1 x^*}{y^*} - \frac{mz^*}{\alpha+y^*} + \frac{mz^*\alpha}{(\alpha+y^*)^2}\right)$$
$$\frac{m\alpha\alpha_1 y^* z^*}{(\alpha+y^*)^3}\left(\frac{rx^*}{K} + \frac{\sigma_2 y^*}{x^*} + \frac{mz^*\alpha}{(\alpha+x^*)^2} - \frac{mz^*}{\alpha+x^*}\right)$$

Using the Routh-Hurwitz criteria, it will be easy to check that all roots of the equation (18) will have negative real parts if $C_1 > 0, C_3 > 0$ and $C_1C_2 - C_3 > 0$ holds. So, the interior equilibrium point $P_2(x^*, y^*, z^*)$ of the system (3) will be locally asymptotically stable, provided $C_1 > 0,\ C_3 > 0$ and $C_1C_2 - C_3 > 0$

IV. BIONOMIC EQUILIBRIUM

The economic rent (revenue at any time) is given as $\Pi = TR(E) - TC(E)$

Let c_1 and c_2 be the harvesting cost per unit effort for prey (fish) in the unreserved area and predator (bird) respectively. Further considering p_1 and p_2 be the price per unit biomass of the prey in the unreserved area and bird predator respectively. Then

$$\Pi_1 = (p_1q_1x - c_1)E_1, \Pi_2 = (p_2q_2z - c_2)E_2$$

Will be representing net revenue for the prey in unreserved area and predator respectively. Therefore, net economic revenue from harvesting of prey in unreserved area and predator at any time *t* is given by

$$\Pi = (p_1q_1x - c_1)E_1 + (p_2q_2z - c_2)E_2$$

The Bionomic equilibrium point $(x_\infty, y_\infty, z_\infty, E_{1\infty}, E_{2\infty})$ will be obtained by solving the following simultaneous equations.

$$rx\left(1 - \frac{x}{K}\right) - \sigma_1 x + \sigma_2 y - \frac{mxz}{\alpha + x} - q_1E_1x = 0 \qquad (19)$$

$$sy\left(1 - \frac{y}{L}\right) + \sigma_1 x - \sigma_2 y - \frac{myz}{\alpha + y} = 0 \qquad (20)$$

$$-dz + \alpha_1\left(\frac{xz}{\alpha + x} + \frac{yz}{\alpha + y}\right) - q_2E_2z = 0 \qquad (21)$$

$$\Pi = (p_1q_1x - c_1)E_1 + (p_2q_2z - c_2)E_2 = 0 \qquad (22)$$

By considering the following cases, we will determine the bionomic equilibrium points.

Case 1: If $c_2 > p_2q_2z$ and $c_1 < p_1q_1x$

Here the cost of harvesting of predator (bird) is greater than the revenue received and cost of harvesting of prey (fish) is less than revenue. Subsequently, the harvesting of a predator (bird) will be ceased and the only prey (fish) harvesting (in an unreserved area) remains operational. Thus we have $E_{2\infty} = 0$

$$x_\infty = \frac{c_1}{p_1q_1},\ \ y_\infty = \frac{\alpha\left((d - \alpha_1)c_1 + d\alpha p_1q_1\right)}{\left(\alpha\alpha_1p_1q_1 + 2\alpha_1c_1 - dc_1 - d\alpha p_1q_1\right)}$$

Which will exist provided

$$2\alpha_1c_1 > d(c_1 + \alpha p_1q_1) > \alpha_1c_1 \qquad (23)$$

Now substituting the values x_∞ , y_∞ & $E_{2\infty}$ in equation (20)

$$z_\infty = \frac{(\alpha\alpha_1p_1q_1 + \alpha_1c_1)\begin{bmatrix} L(s - \sigma_2)\alpha\left[(d - \alpha_1)c_1 + d\alpha p_1q_1\right] \\ \left[\alpha\alpha_1p_1q_1 + 2\alpha_1c_1 - dc_1 - d\alpha p_1q_1\right] \\ + \frac{L\sigma_1c_1}{p_1q_1}\left[\alpha\alpha_1p_1q_1 + 2\alpha_1c_1 - dc_1 - d\alpha p_1q_1\right]^2 \\ -s\alpha^2\left[(d - \alpha_1)c_1 + d\alpha p_1q_1\right]^2 \end{bmatrix}}{Lm\left[(d - \alpha_1)c_1 + d\alpha p_1q_1\right]\left[\alpha\alpha_1p_1q_1 + 2\alpha_1c_1 - dc_1 - d\alpha p_1q_1\right]^2}$$

z_∞ will be positive provided $c_1\sigma_1L > s\alpha^2p_1q_1$ is satisfied (24)

$$E_{1\infty} = \frac{1}{q_1x_\infty}\left(rx_\infty\left(1 - \frac{x_\infty}{K}\right) - \sigma_1x_\infty + \sigma_2y_\infty - \frac{mx_\infty z_\infty}{\alpha + x_\infty}\right) \qquad (25)$$

$E_{1\infty}$ will be positive, provided

$$rx_\infty\left(1 - \frac{x_\infty}{K}\right) + \sigma_2y_\infty > \sigma_1x_\infty + \frac{mx_\infty z_\infty}{\alpha + x_\infty} \qquad (26)$$

Case 2: If $c_1 > p_1q_1x$ and $c_2 < p_1q_1z$

Here the harvesting cost of prey (fish) (in an unreserved area) is greater than the revenue received and harvesting cost of predator is less than revenue. Hence, the harvesting of a prey (fish) will be ceased and the only predator harvesting remains operational.

Thus, we have

$$z_\infty = \frac{c_2}{p_2 q_2},\ E_{1\infty} = 0$$

Substitute $z_\infty = \frac{c_2}{p_2 q_2}$ in (21)

We get

$$y_\infty = \frac{1}{\sigma_2}\left[\left(\sigma_1 - r + \frac{mc_2}{p_2 q_2(\alpha + x_\infty)}\right)x_\infty + \frac{rx_\infty^2}{K}\right] \tag{27}$$

Now $y_\infty > 0$ provided, $x_\infty < \left(\frac{mc_2}{p_2 q_2 (r - \sigma_1)} - \alpha\right)$ (28)

From (21), we get

$$\left(\frac{x}{\alpha + x} + \frac{y}{\alpha + y}\right) = \frac{d + q_2 E_2}{\alpha_1} \tag{29}$$

Adding (19) &(20) and using (29)

We get the following 6th degree equation in x

$$R_1 x^6 + R_2 x^5 + R_3 x^4 + R_4 x^3 + R_5 x^2 + R_6 x + R_7 = 0$$

Where

$$R_1 = \frac{sr^2}{L\sigma_2^2 K^2},\ R_2 = \left(\frac{2sr^2\alpha}{L\sigma_2^2 K^2} + \frac{2rsB}{KL\sigma_2^2 p_2 q_2}\right)$$

$$R_3 = \frac{r}{K}(\sigma_2 - s) + \frac{2rs}{KL\sigma_2^2 p_2 q_2}(A + B\alpha) + \frac{s}{L\sigma_2^2}\left(\frac{r^2\alpha^2}{K^2} + \frac{B^2}{p_2^2 q_2^2}\right)$$

$$R_4 = \left\{\frac{\alpha r}{K}\right\}\left\{2 - \frac{2s}{\sigma_2} - \frac{K}{\alpha}\right\} - \frac{sB}{p_2 q_2 \sigma_2} + \frac{2sA}{L\sigma_2^2 p_2 q_2}\left\{\frac{B}{p_2 q_2} + \frac{r\alpha}{K}\right\}$$

$$R_5 = \left(\frac{r}{K}\alpha^2\left(1 - \frac{s}{\sigma_2}\right) - \frac{s}{p_2 q_2 \sigma_2}(A + B\alpha) + \frac{A^2 s}{L\sigma_2^2 p_2^2 q_2^2} + \frac{mc_2(d + q_2 E_2)}{p_2 q_2 \alpha_1} - 2\alpha r\right)$$

$$R_6 = \left(\frac{2mc_2(d + q_2 E_2)\alpha}{p_2 q_2 \alpha_1} - \frac{A\alpha s}{p_2 q_2 \sigma_2} - r\alpha^2\right)$$

$$R_7 = \left(\frac{mc_2(d + q_2 E_2)\alpha^2}{p_2 q_2 \alpha_1}\right)$$

where $A = \sigma_1 p_2 q_2 \alpha - r p_2 q_2 \alpha + mc_2$, $B = p_2 q_2 (\sigma_1 - r)$

Let G(x) = $R_1 x^6 + R_2 x^5 + R_3 x^4 + R_4 x^3 + R_5 x^2 + R_6 x + R_7$

it is obvious that $G(0) = R_7 > 0$

And if

$$G(K) = R_1 K^6 + R_2 K^5 + R_3 K^4 + R_4 K^3 + R_5 K^2 + R_6 K + R_7 < 0 \tag{30}$$

Then there exists a positive value of x (say x_∞) in the interval[0,K].

Now, the sufficient condition for x_∞ to be unique is

$$G'(x_\infty) = 6R_1 x_\infty^5 + 5R_2 x_\infty^4 + 4R_3 x_\infty^3 + 3R_4 x_\infty^2 + 2R_5 x_\infty + R_6 < 0 \tag{31}$$

From(21)

$$E_{2\infty} = \frac{1}{q_2}\left[-d + \alpha_1\left(\frac{x_\infty}{\alpha + x_\infty} + \frac{y_\infty}{\alpha + y_\infty}\right)\right] \tag{32}$$

It will exist, provided

$$\frac{x_\infty}{\alpha + x_\infty} + \frac{y_\infty}{\alpha + y_\infty} > \frac{d}{\alpha_1} \tag{33}$$

Hence bionomic equilibrium $(x_\infty, y_\infty, z_\infty, 0, E_{2\infty})$ exists provided (28),(29), (30) and (33) are satisfied.

Case 3: If $c_1 > p_1 q_1 x$ and $c_2 > p_2 q_2 z$

In this case, the fishing cost exceeds the revenue for both the prey (fish) in unreserved area and predator (bird),then we will obtain negative economic rent. Thus no harvesting will be done.

Case 4: If $c_1 < p_1 q_1 x$ and $c_2 < p_2 q_2 z$

At that point, the income of both the species, prey(fish) in the unreserved area and predator(bird) are positive. Hence,the harvesting of prey in unreserved area and predator is possible.

$$x_\infty = \frac{c_1}{p_1 q_1},\ z_\infty = \frac{c_2}{p_2 q_2}$$

Substituting the value of x_∞ and z_∞ in eq. (20)

$$sy\left(1 - \frac{y}{L}\right) + \sigma_1 x_\infty - \sigma_2 y - \frac{myz_\infty}{\alpha + y} = 0$$

We get following cubic equation

$$D_{21} y^3 + D_{22} y^2 + D_{23} y + D_{24} = 0 \tag{34}$$

Where

$$D_{21} = s,\ D_{22} = s\alpha - (s - \sigma_2)L$$

$$D_{23} = L\left(\frac{mc_2}{p_2 q_2} - \frac{\sigma_1 c_1}{p_1 q_1}\right) - L\alpha(s - \sigma_2),\ D_{24} = -\frac{L\alpha\sigma_1 c_1}{p_1 q_1}$$

Thus, we get one unique positive root of (34) say y_∞ as there is one change of sign if

$$s\alpha < (s - \sigma_2)L \text{ and } \left(\frac{mc_2}{p_2 q_2} < \frac{\sigma_1 c_1}{p_1 q_1}\right) \tag{35}$$

Also

$$E_{2\infty} = \frac{1}{q_2}\left[-d + \alpha_1\left(\frac{x_\infty}{\alpha + x_\infty} + \frac{y_\infty}{\alpha + y_\infty}\right)\right] \tag{36}$$

$$E_{1\infty} = \frac{1}{q_1 x_\infty}\left(rx_\infty\left(1 - \frac{x_\infty}{K}\right) - \sigma_1 x_\infty + \sigma_2 y_\infty - \frac{mx_\infty z_\infty}{\alpha + x_\infty}\right) \tag{37}$$

$E_{1\infty} > 0$ and $E_{2\infty} > 0$ Provided following inequalities hold.

$$rx_\infty\left(1-\frac{x_\infty}{K}\right)+\sigma_2 y_\infty > \sigma_1 x_\infty + \frac{mx_\infty z_\infty}{\alpha + x_\infty}, \alpha_1\left(\frac{x_\infty}{\alpha + x_\infty}+\frac{y_\infty}{\alpha + y_\infty}\right) > d \quad (38)$$

Hence, the nontrivial bionomic equilibrium point $P_\infty\left(x_\infty, y_\infty, z_\infty, E_{1\infty}, E_{2\infty}\right)$ exists, provided conditions (35) and (38) are satisfied.

V. OPTIMAL HARVESTING POLICY

In this segment, the objective is to maximize the present value of "*J*"of a continuous time stream of revenues given by

$$J = \int_0^\infty e^{-\delta t}\left\{\left(p_1 q_1 x - c_1\right)E_1(t) + \left(p_2 q_2 z - c_2\right)E_2(t)\right\}dt \quad (39)$$

Here δ is supposed to be the instantaneous annual rate of discount. We intend to maximize (39)subject to the state equations (3)with the help of Pontryagin's maximal principle (Clark [1]),the control variable $E_i(t)$ (i = 1, 2) are subjected to the constraint

$0 \le E_i(t) \le (E_i)_{\max}$

The Hamiltonian function of the model (3) is given by

$$\begin{aligned} H = {} & e^{-\delta t}\left[\left(p_1 q_1 x - c_1\right)E_1(t) + \left(p_2 q_2 z - c_2\right)E_2(t)\right] \\ & + \lambda_1\left[rx\left(1-\frac{x}{K}\right) - \sigma_1 x + \sigma_2 y - \frac{mxz}{\alpha + x} - q_1 E_1 x\right] \\ & + \lambda_2\left[sy\left(1-\frac{y}{L}\right) + \sigma_1 x - \sigma_2 y - \frac{myz}{\alpha + y}\right] \\ & + \lambda_3\left[-dz + \alpha_1\left(\frac{xz}{\alpha + x}+\frac{yz}{\alpha + y}\right) - q_2 E_2 z\right] \end{aligned}$$

Where $\lambda_1, \lambda_2, \lambda_3$ are denoted as the adjoint variables. The control variables E_1 and E_2 appear linearly in the Hamiltonian Function *H*.

According to Pontryagin's maximum principle

$$\frac{\partial H}{\partial E_1} = 0,\ \frac{\partial H}{\partial E_2} = 0,\ \frac{\partial \lambda_1}{\partial t} = -\frac{\partial H}{\partial x},\ \frac{\partial \lambda_2}{\partial t} = -\frac{\partial H}{\partial y},\ \frac{\partial \lambda_3}{\partial t} = -\frac{\partial H}{\partial z} \quad (40)$$

Where

x, y, z, E_1 and E_2 are constants for finding the optimal Equilibrium solution of the model (3).

Considering the interior equilibrium $F(x^*, y^*, z^*)$, the first equation of (40) is given by

$$\frac{\partial H}{\partial E_1} = 0 \Rightarrow \lambda_1 = e^{-\delta t}\left(p_1 - \frac{c_1}{q_1 x}\right) \quad (41)$$

Similarly the 2nd equation of (40) can be written as

$$\frac{\partial H}{\partial E_2} = 0 \Rightarrow \lambda_3 = e^{-\delta t}\left(p_2 - \frac{c_2}{q_2 z}\right) \quad (42)$$

By using 3rd Equation of (40)

$$\frac{\partial \lambda_1}{\partial t} - \lambda_1 B_1 = -e^{-\delta t}B_2$$

$$\lambda_1 = \frac{B_2}{B_1 + \delta}e^{-\delta t} \quad (43)$$

Where

$$B_1 = -r + \frac{2rx^*}{K} + \sigma_1 + \frac{\alpha m z^*}{(\alpha + x^*)^2} + q_1 E_1$$

$$B_2 = p_1 q_1 E_1(t) + \frac{\alpha\alpha_1 z^*}{(\alpha + x^*)^2}\left(p_2 - \frac{c_2}{q_2 z^*}\right) + \frac{N_2\sigma_1}{N_1 + \delta}$$

By using 4th equation of (40)

$$\frac{\partial \lambda_2}{\partial t} - N_1\lambda_2 = -N_2 e^{-\delta t}$$

$$\lambda_2 = \frac{N_2}{N_1 + \delta}e^{-\delta t} \quad (44)$$

Where

$$N_1 = -s + \frac{2sy^*}{L} + \sigma_2 + \frac{mz^*\alpha}{\left(\alpha + y^*\right)^2}$$

$$N_2 = \sigma_2\left(p_1 - \frac{c_1}{q_1 x^*}\right) + \frac{\alpha\alpha_1 z^*}{\left(\alpha + y^*\right)^2}\left(p_2 - \frac{c_2}{q_2 z^*}\right)$$

It is easy to verify that N_1 ,N_2 , B_1 , B_2 can be written as the function of x^*only because y^*, z^* are in the form of x^*

Where

$$z^* = \frac{\left(\alpha + x^*\right)}{mx^*}\left[\left(\frac{-rx^{*2}}{K} + x^* E_3\right) + \frac{\alpha\sigma_2\left(\alpha\left(d + q_2 E_2\right) - x^* E_6\right)}{x^* E_6 + \alpha_1 x^* + \alpha E_6}\right] \quad (45)$$

$$y^* = \alpha\left[\frac{\alpha\left(d + q_2 E_2\right) - x^* E_6}{x^* E_6 + \alpha_1 x^* + \alpha E_6}\right] \quad (46)$$

From (41) and (43) we get desired equation of the singular path as

$$\left(p_1 - \frac{c_1}{q_1 x^*}\right) = \frac{B_2}{B_1 + \delta} \quad (47)$$

$$\Rightarrow p_1 q_1 x^* - c_1 = \frac{B_2 q_1 x^*}{B_1 + \delta}$$

Where equation (47) can be written as

$p_1 q_1 x^* - c_1 = h\left(x^*\right)$

Here

$$h\left(x^*\right) == \frac{\left[p_1 q_1 E_1(t) + \frac{\alpha\alpha_1 z^*}{(\alpha + x^*)^2}\left(p_2 - \frac{c_2}{q_2 z^*}\right) + \frac{N_2\sigma_1}{N_1 + \delta}\right]q_1 x^*}{B_1 + \delta}$$

Let $F\left(x^*\right) = h\left(x^*\right) - \left(p_1 q_1 x^* - c_1\right)$

Then positive root of $F(x^*) = 0$ gives the optimal level of fish population in unreserved area at $x^* = x_\delta$ It may be noted that there exists a unique $x^* = x_\delta$ in the interval $0 < x_\delta < K$ if the following inequalities hold:

$$F(0) < 0,\ \ F(K) > 0,\ \ F'(x^*) > 0,\ \ \text{for } x^* > 0 \tag{48}$$

Knowing the value of $x^* = x_\delta$, the optimal level of prey population in the reserved zone ($y^* = y_\delta$), predator ($z^*=z_\delta$) will be obtained from (11), (12)and optimal level of efforts are given by respectively.

$$E_{1\delta} = \frac{1}{q_1 x_\delta}\left(rx_\delta\left(1-\frac{x_\delta}{K}\right) - \sigma_1 x_\delta + \sigma_2 y_\delta - \frac{m x_\delta z_\delta}{\alpha + x_\delta}\right),$$
$$E_{2\delta} = \frac{1}{q_2}\left[-d + \alpha_1\left(\frac{x_\delta}{\alpha + x_\delta} + \frac{y_\delta}{\alpha + y_\delta}\right)\right] \tag{49}$$

$$\text{if } rx_\delta\left(1-\frac{x_\delta}{K}\right) + \sigma_2 y_\delta > \sigma_1 x_\delta + \frac{m x_\delta z_\delta}{\alpha + x_\delta}$$
$$\text{and } \left(\frac{x_\delta}{\alpha + x_\delta} + \frac{y_\delta}{\alpha + y_\delta}\right) > \frac{d}{\alpha_1} \tag{50}$$

Consequently, once the optimal equilibrium $(x_\delta, y_\delta, z_\delta)$ is obtained then optimal harvesting efforts $E_{1\delta}$ and $E_{2\delta}$ will be determined from (49) provided(50)is satisfied.

VI. NUMERICAL SIMULATIONS

In order to investigate the dynamics of the system (3) with help of numerical simulation, we choose different set of parameters.

Let

$$r = 6,\ s = 2,\ \sigma_1 = 0.8,\ \sigma_2 = 0.4,\ m = 0.3,\ q_1 = 4,\ q_2 = 3,\ E_1 = 1.5,$$
$$E_2 = 0.08,\ d = 0.9,\ K = 56,\ L = 100,\ \alpha = 0.0702,\ \alpha_1 = 0.1 \tag{51}$$

For this set of parameters, equilibrium point $P_1(14.6458,\ 86.7534,\ 0)$ exists and it is also locally asymptotically stable as existence conditions (7), (8) and stability conditions of Theorem 3 are satisfied. As P_2 does not exist and P_0 is always unstable for this set of data, so $P_1(14.6458,\ 86.7534,\ 0)$ will always be globally stable in the absence of limit cycles whenever it is locally stable (**Fig. 1**).

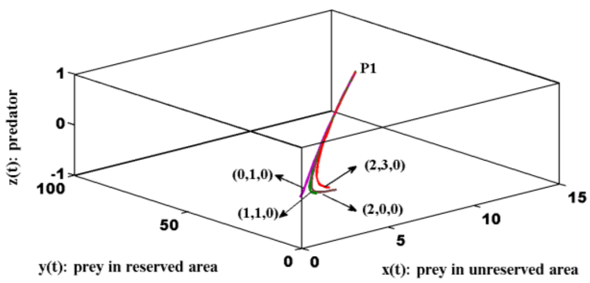

Fig 1: The Phase diagram showing the global stability of $P_1$for data set (51)

Now for another set of parameters

$$r = 7.8, s = 3.7, K = 56, L = 100, \sigma_1 = 0.6, \sigma_2 = 0.4, \alpha = 0.3,$$
$$q_1 = 4, q_2 = 3, E_1 = 1.5, E_2 = 0.08,\ m = 2,\ d = 1.5, \alpha_1 = 1 \tag{52}$$

Equilibrium points $P_0(0,\ 0,\ 0)$, $P_1(21.199,\ 92.8903,\ 0)$ and $P_2(0.8719,\ 74.5977,\ 20.685)$ existbut P_2 is the only locally stable equilibrium point by Theorem 4. As other equilibrium points are unstable, therefore P_2 is also globally stable in the absence of limit cycles (**Fig. 2**).

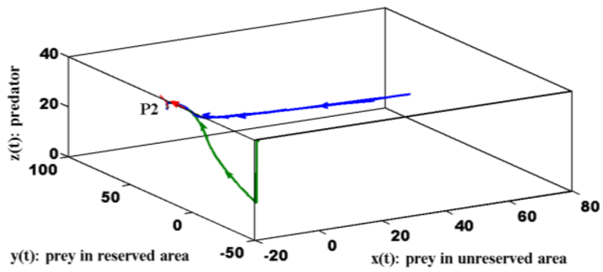

Fig 2: The Phase diagram showing the global stability of equilibrium point $P_2$for data set (52)

We also notice here that when $1.43 \le \alpha_1 < 2,$ limit cycles are observed in this range. For e.g. on taking α_1=1.45, keeping all other parameters same as in (52), limit cycles are seen(**Fig. 3a, Fig. 3b**).

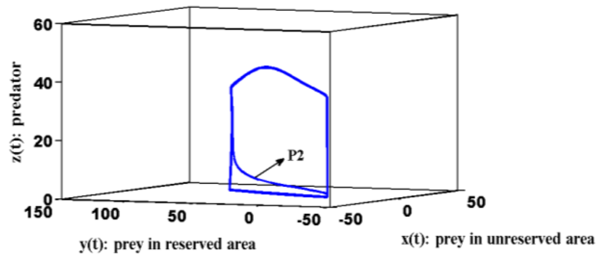

Fig 3a: The Phase diagram showing the limit cycle of P_2on taking α_1=1.45 along with data set (52)

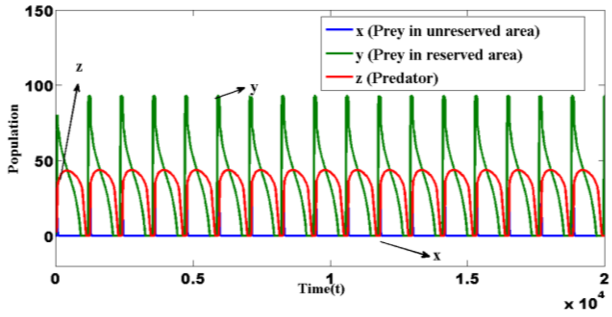

Fig. 3b: Time series plot of $x(t)$, $y(t)$ and $z(t)$ for same data set (52)

Now on considering $0.6 \le \sigma_1 \le 2.1,\ 0.4 \le \sigma_2 \le 1.1$ keeping all other parameters same as in (52), the non-zero point P_2islocally stable. We also observe here that when $\sigma_1 \ge 2.2,\ \sigma_2 \ge 1.2$ limit cycles are observed in this range. As on taking $\sigma_1 = 2.2,\ \sigma_2 = 1.2$ keeping all other parameters same as in (52), limit cycles occur (**Fig.4a, Fig.4b**).

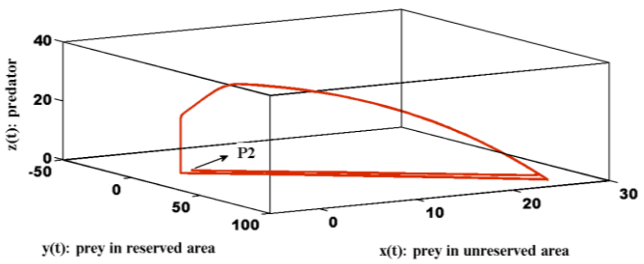

Fig 4a: The Phase diagram showing the limit cycle of P_2 for same data set (52)taking $\sigma_1 = 2.2,\ \sigma_2 = 1.2$

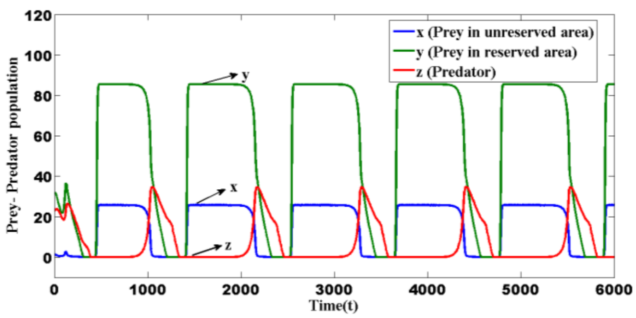

Fig4b: Time series plot of $x(t)$, $y(t)$ and $z(t)$ for same data set (52)in which $\sigma_1 = 2.2,\ \sigma_2 = 1.2$

VII. CONCLUSIONS

In this paper, a mathematical model has been presented and examined to understand the dynamics of a prey-predator system with bird as a predator, which moves in both reserve (where harvesting of prey is restricted) and unreserved area of prey. Thresholds for existence and local stability of the system at various equilibrium points have been examined. The global stability of equilibrium points and existence of Hopf bifurcation of the system have been shown with the help of numerical simulation. Dubey concluded that reserved zone has a stabilizing effect on prey-predator dynamics. But this is not true for Holling type II predator functional response. As ranges for Hopf bifurcation have been found in our analysis of the model. It has been observed that if the system is exploited persistently in the unreserved zone, even then prey species will not extinct from a system and prey species in the unreserved zone will always remain at appropriate equilibrium level due to migration between two zones and restriction of harvesting in reserved zone. The volume of these equilibrium points will primarily depend upon intrinsic growth rates, the coefficient of migration of the prey population and the carrying capacities of the prey in unreserved and reserved zones. Further, it is examined that with the appropriate harvesting of prey in the unreserved zone and bird predator, we may be able to extinct a bird predator from the system if they are destructive to the system. It is also concluded that with the creation of reserved zone, prey in both the areas never extinct.

In the next section, Pontryagin's maximum principle has been used to discuss the optimal harvesting policy. It has been concluded that with the increase in discount rate, the economic rent decreases and as discount rate tends to infinity, then the economic rent even may tend to zero.

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