# Bäcklund Transformations and Exact Explicit Solutions for a High-order Classical Boussinesq-Burgers Equation 

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#### Abstract

In this work, a high-order classical BoussinesqBurgers equation is investigated. We establish a transforation which turns the high-order classical Boussinesq-Burgers equation into a single Sharma-Tasso-Olver equation, then we obtain Bäcklund transformation and abundant exact solutions, including multi-solitary wave solution, trigonometric function series solution, rational series solution and solution consisting of the three types of solutions.


Index Terms-high-order classical Boussinesq-Burgers (HCBB) equation, Bäcklund transformation, Soliton solution, Homogenous balance method.

## I. Introduction

IT is well known that nonlinear evolution equations play an important role in describing nonlinear scientific phenomena, such as marine engineering, fluid dynamics, plasma physics, chemistry, and physics. The research on qualitative and quantitative features of these equations has increased significantly in recent decades. It has become an extremely active area of study to solve these nonlinear evolution equations. With the aid of symbolic computation, a variety of powerful methods are presented, such as Hirota's bilinear method [1,2], Bäcklund transformation (BT) [3-5], Darboux transformation (DT) [6,7], Painlevé analysis [810], homogeneous balance method (HB) [11-13] and so on. In Refs. [14,15], Fan extended HB method to search for Bäcklund transformations and similarity reductions of nonlinear PDE. So more solutions can be obtained by the extended HB method.

In this work, we will discuss the following high-order classical Boussinesq-Burgers (HCBB) equation [16]

$$
\begin{align*}
u_{t}= & \frac{3}{2}(\beta-1)\left(u u_{x}\right)_{x}+\frac{3}{2}(u v)_{x}+3 u^{2} u_{x}+\frac{1}{4} u_{x x x} \\
v_{t}= & 3 \beta\left(1-\frac{1}{2} \beta\right)\left(2 u_{x} u_{x x}+u u_{x x x}\right)+\frac{3}{2}(1-\beta)  \tag{1}\\
& \left(u v_{x}\right)_{x}+\frac{3}{2} v v_{x}+3\left(u^{2} v\right)_{x}+\frac{1}{4} v_{x x x}
\end{align*}
$$

where $\beta$ is an arbitrary constant. When $\beta=1$, Eq. (1) becomes a high-order Boussinesq-Burgers (HBB) equation

$$
\begin{align*}
u_{t}= & \frac{3}{2}(u v)_{x}+3 u^{2} u_{x}+\frac{1}{4} u_{x x x} \\
v_{t}= & 3 u_{x} u_{x x}+\frac{3}{2} u u_{x x x}+\frac{3}{2} v v_{x}+3\left(u^{2} v\right)_{x}  \tag{2}\\
& +\frac{1}{4} v_{x x x}
\end{align*}
$$

When $\beta=0$, Eq. (1) becomes a high-order Boussinesq system

$$
\begin{align*}
& u_{t}=-\frac{3}{2}\left(u u_{x}\right)_{x}+\frac{3}{2}(u v)_{x}+3 u^{2} u_{x}+\frac{1}{4} u_{x x x}  \tag{3}\\
& v_{t}=\frac{3}{2}\left(u v_{x}\right)_{x}+\frac{3}{2} v v_{x}+3\left(u^{2} v\right)_{x}+\frac{1}{4} v_{x x x}
\end{align*}
$$

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In this work, we first establish a transforation which turns the HCBB equation into a single Sharma-Tasso-Olver (STO) equation. By the extended HB method [14,15], we reduce the STO equation to a linear PDE and obtain Bäcklund transformation of it. In addition, the self-transformation of solutions for the HCBB equation can be obtained. By the Bäcklund transformation and various series solutions of the linear PDE, we obtain abundant exact solutions of the HCBB equation, including multi-solitary wave solution, trigonometric function series solution, rational series solution and solution consisting of the three types of solutions. Furthermore, more exact solutions can be obtained by repeatedly using the self-transformation of solutions.

## II. BÄCKLUND TRANSFORMATION FOR THE HCBB EQUATION

For simplicity, we consider the function transformation $v=\lambda u_{x}+\mu$, which converts Eq. (1) into

$$
\begin{aligned}
& u_{t}=\frac{3}{2}(\beta-1)\left(u u_{x}\right)_{x}+\frac{3}{2}\left[u\left(\lambda u_{x}+\mu\right)\right]_{x}+3 u^{2} u_{x}+\frac{1}{4} u_{x x x}, \\
& \lambda u_{x t}=3 \beta\left(1-\frac{1}{2} \beta\right)\left(2 u_{x} u_{x x}+u u_{x x x}\right)+\frac{3}{2} \lambda(1-\beta)\left(u u_{x x}\right)_{x} \\
& +\frac{3}{2} \lambda\left(\lambda u_{x}+\mu\right) u_{x x}+3\left[u^{2}\left(\lambda u_{x}+\mu\right)\right]_{x}+\frac{1}{4} \lambda u_{x x x x}
\end{aligned}
$$

Especially, if we set $\lambda=2-\beta, \mu=0$, the above equations convert into a simple STO equation [17]

$$
\begin{equation*}
u_{t}=3 u^{2} u_{x}+\frac{3}{2} u_{x}^{2}+\frac{3}{2} u u_{x x}+\frac{1}{4} u_{x x x} . \tag{4}
\end{equation*}
$$

According to the extended HB method [14,15], we suppose that the solution of Eq. (4) has the following form

$$
\begin{equation*}
u(x, t)=f^{\prime}(\omega) \omega_{x}(x, t)+u_{0}(x, t) \tag{5}
\end{equation*}
$$

where $f, \omega$ are functions to be determined later, and $u_{0}(x, t)$ is a solution of Eq. (4). Then we have

$$
\begin{align*}
u_{t}= & f^{\prime \prime}(\omega) \omega_{x} \omega_{t}+f^{\prime}(\omega) \omega_{x t}+u_{0 t} \\
u_{x}= & f^{\prime \prime}(\omega) \omega_{x}^{2}+f^{\prime}(\omega) \omega_{x x}+u_{0 x} \\
u_{x x} & =f^{\prime \prime \prime}(\omega) \omega_{x}^{3}+3 f^{\prime \prime}(\omega) \omega_{x} \omega_{x x}+f^{\prime}(\omega) \omega_{x x x}  \tag{6}\\
& +u_{0 x x}, \\
u_{x x x} & =f^{(4)}(\omega) \omega_{x}^{4}+6 f^{\prime \prime \prime}(\omega) \omega_{x}^{2} \omega_{x x}+f^{\prime \prime}(\omega) \\
& \left(4 \omega_{x} \omega_{x x x}+3 \omega_{x x}^{2}\right)+f^{\prime}(\omega) \omega_{x x x x}+u_{0 x x x}
\end{align*}
$$

Substituting (6) into Eq. (4), we have

$$
\begin{align*}
& -\left(\frac{3}{2} f^{\prime} f^{\prime \prime \prime}+3 f^{\prime 2} f^{\prime \prime}+\frac{3}{2} f^{\prime \prime 2}+\frac{1}{4} f^{(4)}\right) w_{x}^{4} \\
& -\left(6 f^{\prime} f^{\prime \prime}+\frac{3}{2} f^{\prime \prime \prime}\right) u_{0} w_{x}^{3}-\left(\frac{15}{2} f^{\prime} f^{\prime \prime} w_{x x}+3 f^{\prime 3} w_{x x}\right. \\
& \left.+\frac{3}{2} f^{\prime \prime \prime} w_{x x}+3 f^{\prime \prime} u_{0 x}+3 f^{\prime 2} u_{0 x}+3 f^{\prime \prime} u_{0}^{2}\right) w_{x}^{2} \\
& +\left(f^{\prime \prime} w_{t}-6 f^{\prime 2} u_{0} w_{x x}-6 f^{\prime} u_{0} u_{0 x}-\frac{9}{2} f^{\prime \prime} u_{0} w_{x x}\right.  \tag{7}\\
& \left.-f^{\prime \prime} w_{x x x}-\frac{3}{2} f^{\prime 2} w_{x x x}-\frac{3}{2} f^{\prime} u_{0 x x}\right) w_{x} \\
& +u_{0 t}+f^{\prime} w_{x t}-3 f^{\prime} u_{0}^{2} w_{x x}-3 f^{\prime} u_{0 x} w_{x x} \\
& -\frac{3}{2} f^{\prime} u_{0} w_{x x x}-\frac{3}{2} u_{0 x}^{2}-3 u_{0}^{2} u_{0 x}-\frac{3}{2} f^{\prime 2} w_{x x}^{2} \\
& -\frac{3}{2} u_{0} u_{0 x x}-\frac{3}{4} f^{\prime \prime} w_{x x}^{2}-\frac{1}{4} f^{\prime} w_{x x x x}-\frac{1}{4} u_{0 x x x}=0 .
\end{align*}
$$

Setting the coefficient of $w_{x}^{4}$ to zero, we obtain the ODE about $f(w)$

$$
\begin{equation*}
\frac{3}{2} f^{\prime} f^{\prime \prime \prime}+3 f^{\prime 2} f^{\prime \prime}+\frac{3}{2} f^{\prime \prime 2}+\frac{1}{4} f^{(4)}=0 \tag{8}
\end{equation*}
$$

which admits the solution $f(w(x, t))=\frac{1}{2} \ln w(x, t)$ or $f(w(x, t))=\ln w(x, t)$. In order to obtain Bäcklund transformation of Eq. (4), we choose the solution

$$
\begin{equation*}
f(w(x, t))=\frac{1}{2} \ln w(x, t) \tag{9}
\end{equation*}
$$

From (9), it holds that

$$
f^{\prime 2}=-\frac{1}{2} f^{\prime \prime}, \quad f^{\prime} f^{\prime \prime}=-\frac{1}{4} f^{\prime \prime \prime}, \quad f^{\prime 3}=\frac{1}{8} f^{\prime \prime \prime}
$$

We can use these expressions to linearize the derivative terms of $f(w)$ in (7) and obtain sum of some terms of $f^{\prime}$ and $f^{\prime \prime}$. Setting their coefficients to zero, we can obtain the following reduction equations

$$
\begin{array}{r}
u_{0 t}-3 u_{0}^{2} u_{0 x}-\frac{3}{2} u_{0 x}^{2}-\frac{3}{2} u_{0} u_{0 x x}-\frac{1}{4} u_{0 x x x}=0 \\
w_{x t}-6 u_{0} u_{0 x} w_{x}-3 u_{0}^{2} w_{x x}-\frac{3}{2} u_{0 x x} w_{x} \\
-3 u_{0 x} w_{x x}-\frac{3}{2} u_{0} w_{x x x}-\frac{1}{4} w_{x x x x}=0,  \tag{10}\\
w_{t} w_{x}-3 u_{0}^{2} w_{x}^{2}-\frac{3}{2} u_{0 x} w_{x}^{2}-\frac{3}{2} u_{0} w_{x} w_{x x} \\
-\frac{1}{4} w_{x} w_{x x x}=0 .
\end{array}
$$

We find that above conditions can be satisfied, provided that

$$
\begin{align*}
& u_{0 t}-3 u_{0}^{2} u_{0 x}-\frac{3}{2} u_{0 x}^{2}-\frac{3}{2} u_{0} u_{0 x x}-\frac{1}{4} u_{0 x x x}=0 \\
& w_{t}-3 u_{0}^{2} w_{x}-\frac{3}{2} u_{0 x} w_{x}-\frac{3}{2} u_{0} w_{x x}-\frac{1}{4} w_{x x x}=0 . \tag{11}
\end{align*}
$$

Substituting the expression of (9) into (5), we obtain a Bäcklund transformation

$$
\begin{equation*}
u(x, t)=\frac{1}{2} \frac{\partial}{\partial x} \ln \omega(x, t)+u_{0}, \tag{12}
\end{equation*}
$$

where $\omega(x, t)$ and $u_{0}(x, t)$ satisfy (11).
It is interesting to note that if the original solution $u_{0}(x, t)=\omega(x, t)$, Eqs. (11) become the same form

$$
\begin{equation*}
\omega_{t}-3 \omega^{2} \omega_{x}-\frac{3}{2} \omega_{x}^{2}-\frac{3}{2} \omega \omega_{x x}-\frac{1}{4} \omega_{x x x}=0 \tag{13}
\end{equation*}
$$

which is exactly Eq. (4). So, from the Bäcklund transformation (12), we can see that if $u(x, t)$ is a solution of Eq. (4), then

$$
U(x, t)=\frac{1}{2} \frac{u_{x}(x, t)}{u(x, t)}+u(x, t) .
$$

is still the solution of Eq. (4). Similarly, we can obtain that if $u(x, t)$ and $v(x, t)$ are a solution of the HCBB equation, then

$$
\begin{align*}
& U(x, t)=\frac{1}{2} \frac{u_{x}(x, t)}{u(x, t)}+u(x, t), \\
& V(x, t)=\frac{1}{2}\left[\frac{v(x, t)}{u(x, t)}\right]_{x}+v(x, t) . \tag{14}
\end{align*}
$$

is still the solution of the HCBB equation. It means (14) is a self-transformation of solutions for the HCBB equation.

## III. Abundant explicit and exact solutions to THE HCBB EQUATION

In order to obtain exact solutions, we choose the original solution $u_{0}(x, t)=b$ in Eqs. (11), where $b$ is an arbitrary constant. Eqs. (11) become a linear PDE

$$
\begin{equation*}
\omega_{t}-3 b^{2} \omega_{x}-\frac{3}{2} b \omega_{x x}-\frac{1}{4} \omega_{x x x}=0 \tag{15}
\end{equation*}
$$

Especially, choosing the original solution $u_{0}(x, t)=0$ in Eqs. (11), we can obtain a more simple PDE

$$
\begin{equation*}
\omega_{t}-\frac{1}{4} \omega_{x x x}=0 \tag{16}
\end{equation*}
$$

Case 1 (Multi-solitary solutions). We apply the traveling wave transformation $\omega(x, t)=\omega(\xi)=\omega(k(x+c t))$ to Eq. (15), which will yield

$$
\begin{equation*}
\left(c-3 b^{2}\right) \omega^{\prime}-\frac{3}{2} b k \omega^{\prime \prime}-\frac{1}{4} k^{2} \omega^{\prime \prime \prime}=0 . \tag{17}
\end{equation*}
$$

where ' denotes $d / d \xi$ and the constant $c$ is the velocity of traveling wave. Obviously, Eq. (17) admits the solution

$$
\begin{equation*}
\omega(\xi)=c_{0}+c_{1} e^{k\left(x+\left(3 b^{2}+\frac{3}{2} b k+\frac{1}{4} k^{2}\right) t\right)} \tag{18}
\end{equation*}
$$

where $c_{0}, c_{1}, k$ are arbitrary constants and the traveling wave velocity $c=3 b^{2}+\frac{3}{2} b k+\frac{1}{4} k^{2}$. Since (17) is linear, by the superposition principle,

$$
\begin{equation*}
\omega(\xi)=c_{0}+\sum_{i=1}^{n} c_{i} e^{k_{i}\left(x+\left(3 b^{2}+\frac{3}{2} b k_{i}+\frac{1}{4} k_{i}^{2}\right) t\right)} \tag{19}
\end{equation*}
$$

is still the solution of Eq. (17), where $c_{0}, c_{i}, k_{i}(i=$ $1,2, \ldots, n)$ are arbitrary constants.
From (19) and the Bäcklund transformation (12), multisolitary solution of Eq. (4) can be expressed by

$$
u(x, t)=\frac{1}{2} \frac{\sum_{i=1}^{n} c_{i} k_{i} e^{k_{i}\left(x+\left(3 b^{2}+\frac{3}{2} b k_{i}+\frac{1}{4} k_{i}^{2}\right) t\right)}}{c_{0}+\sum_{i=1}^{n} c_{i} e^{k_{i}\left(x+\left(3 b^{2}+\frac{3}{2} b k_{i}+\frac{1}{4} k_{i}^{2}\right) t\right)}}+b .
$$

Similarly, we can obtain that multi-solitary solution of the HCBB equation can be expressed by

$$
\begin{align*}
u(x, t) & =\frac{1}{2} \frac{\sum_{i=1}^{n} c_{i} k_{i} e^{k_{i}\left(x+\left(3 b^{2}+\frac{3}{2} b k_{i}+\frac{1}{4} k_{i}^{2}\right) t\right)}}{c_{0}+\sum_{i=1}^{n} c_{i} e^{k_{i}\left(x+\left(3 b^{2}+\frac{3}{2} b k_{i}+\frac{1}{4} k_{i}^{2}\right) t\right)}}+b, \\
v(x, t) & =\frac{2-\beta}{2}\left\{\frac{\sum_{i=1}^{n} c_{i} k_{i}^{2} e^{k_{i}\left(x+\left(3 b^{2}+\frac{3}{2} b k_{i}+\frac{1}{4} k_{i}^{2}\right) t\right)}}{c_{0}+\sum_{i=1}^{n} c_{i} e^{k_{i}\left(x+\left(3 b^{2}+\frac{3}{2} b k_{i}+\frac{1}{4} k_{i}^{2}\right) t\right)}}\right.  \tag{20}\\
& \left.-\left[\frac{\sum_{i=1}^{n} c_{i} k_{i} e^{k_{i}\left(x+\left(3 b^{2}+\frac{3}{2} b k_{i}+\frac{1}{4} k_{i}^{2}\right) t\right)}}{c_{0}+\sum_{i=1}^{n} c_{i} e^{k_{i}\left(x+\left(3 b^{2}+\frac{3}{2} b k_{i}+\frac{1}{4} k_{i}^{2}\right) t\right)}}\right]^{2}\right\} .
\end{align*}
$$

Especially, choosing the original solution $u_{0}(x, t)=0$, we can obtain the solution of (16)

$$
\begin{equation*}
\omega(\xi)=c_{0}+\sum_{i=1}^{n} c_{i} e^{k_{i}\left(x+\frac{1}{4} k_{i}^{2} t\right)} \tag{21}
\end{equation*}
$$

and multi-solitary solution of Eq. (4) with compact form

$$
u(x, t)=\frac{1}{2} \frac{\sum_{i=1}^{n} c_{i} k_{i} e^{k_{i}\left(x+\frac{1}{4} k_{i}^{2} t\right)}}{c_{0}+\sum_{i=1}^{n} c_{i} e^{k_{i}\left(x+\frac{1}{4} k_{i}^{2} t\right)}} .
$$

Then, we can obtain that multi-solitary solution of the HCBB equation can be expressed by

$$
\begin{align*}
u(x, t) & =\frac{1}{2} \frac{\sum_{i=1}^{n} c_{i} k_{i} e^{k_{i}\left(x+\frac{1}{4} k_{i}^{2} t\right)}}{c_{0}+\sum_{i=1}^{n} c_{i} e^{k_{i}\left(x+\frac{1}{4} k_{i}^{2} t\right)}}, \\
v(x, t) & =\frac{2-\beta}{2}\left\{\frac{\sum_{i=1}^{n} c_{i} k_{i}^{2} e^{k_{i}\left(x+\frac{1}{4} k_{i}^{2} t\right)}}{c_{0}+\sum_{i=1}^{n} c_{i} e^{k_{i}\left(x+\frac{1}{4} k_{i}^{2} t\right)}}\right.  \tag{22}\\
& \left.-\left[\frac{\sum_{i=1}^{n} c_{i} k_{i} e^{k_{i}\left(x+\frac{1}{4} k_{i}^{2} t\right)}}{c_{0}+\sum_{i=1}^{n} c_{i} e^{k_{i}\left(x+\frac{1}{4} k_{i}^{2} t\right)}}\right]^{2}\right\} .
\end{align*}
$$

Taking $n=3, \beta=1, c_{0}=1, c_{i}=1, k_{i}=i(i=1,2,3)$ in (22), we can show the picture of multi-solitary wave solution in Fig. 1.


Fig. 1. Multi-solitary wave solutions of the HCBB equation.

Case 2 (Trigonometric function series solutions). We apply the traveling wave transformation $\omega(x, t)=\omega(\xi)=\omega(k(x+$ $c t)$ ) to Eq. (16), which will yield

$$
\begin{equation*}
c \omega^{\prime}-\frac{1}{4} k^{2} \omega^{\prime \prime \prime}=0 \tag{23}
\end{equation*}
$$

Obviously, (23) admits two trigonometric function solutions

$$
\begin{align*}
& \omega_{1}(\xi)=c_{1}+c_{2} \sin k\left(x-\frac{1}{4} k^{2} t\right),  \tag{24}\\
& \omega_{2}(\xi)=d_{1}+d_{2} \cos l\left(x-\frac{1}{4} l^{2} t\right)
\end{align*}
$$

where $c_{1}, c_{2}, k, d_{1}, d_{2}, l$ are arbitrary constants and the traveling wave velocity $c=-\frac{1}{4} k^{2}$ (or $-\frac{1}{4} l^{2}$ ). Similarly, by the superposition principle, Eq. (16) admits the following trigonometric function series solution

$$
\begin{align*}
\omega(x, t) & =c_{0}+\sum_{i=1}^{n} c_{i} \sin k_{i}\left(x-\frac{1}{4} k_{i}^{2} t\right)  \tag{25}\\
& +\sum_{i=1}^{n} d_{i} \cos l_{i}\left(x-\frac{1}{4} l_{i}^{2} t\right)
\end{align*}
$$

where $c_{0}, c_{i}, d_{i}, k_{i}, l_{i}(i=1,2, \cdots, n)$ are arbitrary constants. From (25) and the Bäcklund transformation (12), the solution of Eq. (4) consisting of trigonometric functions
series can be expressed as

$$
u(x, t)=\frac{1}{2} \frac{\sum_{i=1}^{n} c_{i} k_{i} \cos k_{i}\left(x-\frac{1}{4} k_{i}^{2} t\right)-\sum_{i=1}^{n} d_{i} l_{i} \sin l_{i}\left(x-\frac{1}{4} l_{i}^{2} t\right)}{c_{0}+\sum_{i=1}^{n} c_{i} \sin k_{i}\left(x-\frac{1}{4} k_{i}^{2} t\right)+\sum_{i=1}^{n} d_{i} \cos l_{i}\left(x-\frac{1}{4} l_{i}^{2} t\right)} .
$$

Then, we can obtain that trigonometric functions series of the HCBB equation can be expressed by

$$
\begin{align*}
& u(x, t)= \\
& \frac{1}{2} \frac{\sum_{i=1}^{n} c_{i} k_{i} \cos k_{i}\left(x-\frac{1}{4} k_{i}^{2} t\right)-\sum_{i=1}^{n} d_{i} l_{i} \sin l_{i}\left(x-\frac{1}{4} l_{i}^{2} t\right)}{c_{0}+\sum_{i=1}^{n} c_{i} \sin k_{i}\left(x-\frac{1}{4} k_{i}^{2} t\right)+\sum_{i=1}^{n} d_{i} \cos l_{i}\left(x-\frac{1}{4} l_{i}^{2} t\right)} \\
& v(x, t)= \\
& \frac{2-\beta}{2}\left\{-\frac{\sum_{i=1}^{n} c_{i} k_{i}^{2} \sin k_{i}\left(x-\frac{1}{4} k_{i}^{2} t\right)+\sum_{i=1}^{n} d_{i} l_{i}^{2} \cos l_{i}\left(x-\frac{1}{4} l_{i}^{2} t\right)}{c_{0}+\sum_{i=1}^{n} c_{i} \sin k_{i}\left(x-\frac{1}{4} k_{i}^{2} t\right)+\sum_{i=1}^{n} d_{i} \cos l_{i}\left(x-\frac{1}{4} l_{i}^{2} t\right)}\right.  \tag{26}\\
& \left.+\left[\frac{\sum_{i=1}^{n} c_{i} k_{i} \cos k_{i}\left(x-\frac{1}{4} k_{i}^{2} t\right)-\sum_{i=1}^{n} d_{i} l_{i} \sin l_{i}\left(x-\frac{1}{4} l_{i}^{2} t\right)}{c_{0}+\sum_{i=1}^{n} c_{i} \sin k_{i}\left(x-\frac{1}{4} k_{i}^{2} t\right)+\sum_{i=1}^{n} d_{i} \cos l_{i}\left(x-\frac{1}{4} l_{i}^{2} t\right)}\right]^{2}\right\}
\end{align*}
$$

Taking $n=1, \beta=1, c_{0}=2, c_{1}=k_{1}=2, d_{1}=l_{1}=1$ in (26), we can show the picture of trigonometric function series solutions in Fig. 2.

(a) $u(x, t)$

(b) $v(x, t)$

Fig. 2. Trigonometric function series solutions of the HCBB equation.

Case 3 (Rational series solution). In order to obtain the rational series solution of the HCBB equation, we still consider the linear PDE (16). Suppose that the solution of (16) can be expressed as the rational series form

$$
\begin{equation*}
\omega(x, t)=\sum_{i=0}^{n} k_{i}(x) t^{i} . \tag{27}
\end{equation*}
$$

Substituting (27) into (16) and setting the coefficient of $t^{i}$ to zero, we obtain the recursive ODEs for $k_{i}(x)$ as follows

$$
\begin{align*}
& k_{1}-\frac{1}{4} \frac{d^{3} k_{0}}{d x^{3}}=0, \\
& 2 k_{2}-\frac{1}{4} \frac{d^{3} k_{1}}{d x^{3}}=0, \\
& 3 k_{3}-\frac{1}{4} \frac{d^{3} k_{2}}{d x^{3}}=0, \\
& \vdots  \tag{28}\\
& n k_{n}-\frac{1}{4} \frac{d^{3} k_{n-1}}{d x^{3}}=0, \\
& -\frac{1}{4} \frac{d^{3} k_{n}}{d x^{3}}=0 .
\end{align*}
$$

Solving the recursive ODEs, we obtain

$$
\begin{align*}
k_{i}(x)=\frac{1}{4^{i} i!} & \sum_{j=0}^{3(n-i)+2} \frac{c_{j}(3 n+2-j)!}{(3 n+2-3 i-j)!} x^{3 n+2-3 i-j}  \tag{29}\\
& (i=0,1, \cdots, n) .
\end{align*}
$$

From (29), the rational series solution of (16) can be expressed as

$$
\begin{equation*}
\omega(x, t)=\sum_{i=0}^{n}\left(\sum_{j=0}^{N-3 i} \frac{c_{j}(N-j)!}{4^{i} i!(N-3 i-j)!} x^{N-3 i-j}\right) t^{i} . \tag{30}
\end{equation*}
$$

where $N=3 n+2$. From (30) and the Bäcklund transformation (12), rational series solution of Eq. (4) can be expressed as

$$
u(x, t)=\frac{1}{2} \frac{\sum_{i=0}^{n}\left(\sum_{j=0}^{N-3 i-1} \frac{c_{j}(N-j)!}{4^{i} i!(N-3 i-j-1)!} x^{N-3 i-j-1}\right) t^{i}}{\sum_{i=0}^{n}\left(\sum_{j=0}^{N-3 i} \frac{c_{j}(N-j)!}{4^{i} i!(N-3 i-j)!} x^{N-3 i-j}\right) t^{i}} .
$$

Then, we can obtain that rational series solutions of the HCBB equation can be expressed by

$$
\begin{align*}
& u(x, t)= \\
& \frac{1}{2} \frac{\sum_{i=0}^{n}\left(\sum_{j=0}^{N-3 i-1} \frac{c_{j}(N-j)!}{4^{i} i!(N-3 i-j-1)!} x^{N-3 i-j-1}\right) t^{i}}{\sum_{i=0}^{n}\left(\sum_{j=0}^{N-3 i} \frac{c_{j}(N-j)!}{4^{i} i!(N-3 i-j)!} x^{N-3 i-j}\right) t^{i}}, \\
& v(x, t)= \\
& \frac{2-\beta}{2}\left\{\frac{\sum_{i=0}^{n}\left(\sum_{j=0}^{N-3 i-2} \frac{c_{j}(N-j)!}{4^{i} i!(N-3 i-j-2)!} x^{N-3 i-j-2}\right) t^{i}}{\sum_{i=0}^{n}\left(\sum_{j=0}^{N-3 i} \frac{c_{j}(N-j)!}{4^{i} i!(N-3 i-j)!} x^{N-3 i-j}\right) t^{i}}\right.  \tag{31}\\
& \left.-\left[\frac{\sum_{i=0}^{n}\left(\sum_{j=0}^{N-3 i-1} \frac{c_{j}(N-j)!}{4^{2} i!(N-3 i-j-1)!} x^{N-3 i-j-1}\right) t^{i}}{\sum_{i=0}^{n}\left(\sum_{j=0}^{N-3 i} \frac{c_{j}(N-j)!}{4^{2} i!(N-3 i-j)!} x^{N-3 i-j}\right) t^{i}}\right]^{2}\right\}
\end{align*}
$$

where $c_{i}(i=1,2, \cdots, N)$ are arbitrary constants.
Taking $n=2, \beta=1, c_{i}=1(i=0,1,2, \cdots, 8)$ in (31), we can show the picture of rational series solutions in Fig. 3.

Case 4 (Mixed solutions). We note that linear combinations of (21), (25) and (30) will yield the solutions of (16) by the superposition principle. Then, by Bäcklund

(a) $u(x, t)$

(b) $v(x, t)$

Fig. 3. Rational series solutions of the HCBB equation.
transformation (12), the mixed exact solutions of the HCBB equation can be expressed in the following form

$$
\begin{align*}
& u(x, t)=\frac{1}{2} \frac{\partial}{\partial x}\left(l_{1} \widehat{U_{1}}+l_{2} \widehat{U_{2}}+l_{3} \widehat{U_{3}}\right), \\
& v(x, t)=\frac{2-\beta}{2}\left\{\frac{\partial^{2}\left(l_{1} \widehat{U_{1}}+l_{2} \widehat{U_{2}}+l_{3} \widehat{U_{3}}\right)}{\partial l_{0}+l_{1} \widehat{U_{1}}+l_{2} \widehat{U_{2}}+l_{3} \widehat{U_{3}}}\right.  \tag{32}\\
& \left.-\left[\frac{\frac{\partial}{\partial x}\left(l_{1} \widehat{U_{1}}+l_{2} \widehat{U_{2}}+l_{3} \widehat{U_{3}}\right)}{l_{0}+l_{1} \widehat{U_{1}}+l_{2} \widehat{U_{2}}+l_{3} \widehat{U_{3}}}\right]^{2}\right\} .
\end{align*}
$$

where $l_{i}(i=0,1,2,3)$ are arbitrary constants and $\widehat{U_{i}}(i=$ $1,2,3$ ) present the solutions (24), (29) and (35), respectively.

## IV. Discussion

In this work, we first established a suitable transformation which converts the HCBB equation into a simple STO equation. By the Bäcklund transformation and various series solutions of the linear PDE, we obtain abundant exact solutions of the HCBB equation, including multi-solitary wave solution, trigonometric function series solution, rational series solution and solution consisting of the three types of
solutions. Furthermore, more exact solutions can be obtained by repeatedly using the self-transformation of solutions. These results are important and may have significant impact on future research. It is also worth noting that this method can be applied to other nonlinear evolution equations, especially those with high-order nonlinear terms.

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## References

[1] R. Hirota, "Exact N -soliton solutions of the wave equation of long waves in shallow-water and in nonlinear lattices," J. Math. Phys., vol. 14, pp. 810-814, 1973.
[2] A.M. Wazwaz, "Multiple kink solutions for two coupled integrable (2+1)-dimensional systems," Appl. Math. Lett., vol. 58, pp. 1-6, 2016.
[3] Y. Shang, "Bäcklund transformation, Lax pairs and explicit exact solutions for the shallow water waves equation," Appl. Math. Comput., vol. 187, pp. 1286-1697, 2007.
[4] I.E. Inan, Y. Ugurlu, and H. Bulut,"Auto-Bäcklund transformation for some nonlinear partial differential equation," Optik, pp. 10780-10785, 2016.
[5] A.M. Wazwaz, "Multiple-soliton solutions for extended (3+1)dimensional Jimbo-Miwa equations," Appl. Math. Lett., vol. 64, pp. 21-26, 2017.
[6] E.G. Fan, "Darboux transformation and soliton-like solutions for the Gerdjikov-Ivanov equation," J. Phys. A: Math. Gen., vol. 33, pp. 69256933, 2000.
[7] Y. Li, W.X. Ma and J.E. Zhang, "Darboux transformations of classical Boussinesq system and its new solutions," Phys. Lett. A, vol. 275, pp. 60-66, 2000.
[8] J. Weiss, M. Tabor and G. Carnevale, "The Painlevé property for partial differential equations," J. Math. Phys., vol. 24, pp. 522-526, 1983.
[9] S. Zhang and J.H. Li, "On Nonisospectral AKNS System with Infinite Number of Terms and its Exact Solutions, "IAENG International Journal of Applied Mathematics, vol. 47, no.1, pp.89-96, 2017.
[10] N.A. Kudryashov, "Painlevé analysis and exact solutions of the Korteweg-de Vries equation with a source," Appl. Math. Lett., vol 41, pp. 41-45, 2015.
[11] M. L. Wang, "Solitary wave solution for variant Boussinesq equations," Phys. Lett. A, vol. 199, pp.169-72, 1995.
[12] E.G. Fan and H.Q. Zhang, "New exact solutions to a system of coupled KdV equations,"Phys. Lett. A, vol. 245, pp. 389-392, 1998.
[13] E.G. Fan and H.Q. Zhang, "A note on the homogeneous balance method,"Phys. Lett. A, vol. 246, pp. 403-406, 1998.
[14] E.G. Fan, "Two new applications of the homogeneous balance method,"Phys. Lett. A, vol. 265, pp. 353-357, 2000.
[15] E.G. Fan, "Auto-Bäcklund transformation and similarity reductions for general variable coefficient KdV equations,"Phys. Lett. A, vol. 294, pp. 26-30, 2002.
[16] X.G. Geng and Y.T. Wu, "Finite-band solutions of the classical Boussinesq-Burgers equations, "J. Math. Phys, vol. 40, pp. 2971-2982, 1999.
[17] A.H. Chen, "Multi-kink solutions and soliton fission and fusion of Sharma-Tasso-Olver equation, "Phys. Lett. A, vol. 374, pp. 2340-2345, 2010.

