# Graphics for Complex Polynomials in Jungck-SP Orbit 

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#### Abstract

The objective of this paper is to generate fractals for complex valued polynomials using Jungck-SP orbit. We present an algorithm to generate fractals. With the help of this algorithm, we generate such beautiful graphics which may be useful for graphic designers. In this paper, we extend and improve the corresponding results of Kang et al. (2015) and Kumari et al. (2017) existing in the literature. One can further generalize our results and derive a new escape criterion to generate some more beautiful fractals.


Index Terms-Julia set; Mandelbrot set; Jungck-SP orbit; Escape criterion; Complex polynomials.

## I. Introduction

Fractal theory is a popular branch of mathematical art. In mathematical visualization, fractals are the complex objects that are used to simulate naturally occurring objects. Mathematicians have been using computer programming to generate fractals via iterative procedures. Gaston Julia was the first, who used the iterative process and obtained the Julia set ( [7], p. 122). After that in 1975, Gaston's idea was extended by Benoit Mandelbrot and he introduced the Mandelbrot set. The geometry of fractals had been studied for quadratic [2], [7], [18], [25], cubic [1], [4], [5], [18], [24] and higher degree complex polynomials [35] using Picard orbit.
In the first decade of 21 st century, researchers used Superior orbits to generate fractals (see, [6], [8], [12]-[19], [22], [26], [27], [29]-[33], [36], [37] and references therein). In the sequel, Ashish et al. [3] and M. Kumari et al. [9] obtained further generalized form of Mandelbrot and Julia sets for four step feedback processes.
In 2015, Kang et al. [22] established new escape criterion for Mandelbrot and Julia sets under Jungck-Mann and Jungck-Ishikawa orbits and presented some graphics of Mandelbrot and Julia sets. Further, they presented the generalized form of Mandelbrot sets and Julia sets for complex valued polynomials using Jungck three-step orbit [23]. In 2011, Chugh and Kumar [19] defined Jungck-SP iterative scheme and with the help of examples, they proved that the rate of convergence of Jungck-SP iterative scheme is faster than that of Jungck-Mann, Jungck-Ishikawa and Jungck-Noor iterative schemes. Recently, authors used SP orbit in Fractal theory and generated beautiful fractals (see [10], [11], [20], [21], [28], [34]).
In this paper, firstly, we recall some basic definitions. In Section III, we derive escape criterion to generate fractals

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of complex valued polynomials. Moreover, an algorithm to compute fractals have been presented in Section IV. The beautiful graphics of Mandlbrot sets and Julia sets have been generated in Section V. In Section VI, we conclude our findings.


## II. Preliminaries

Definition 2.1: (Orbit) [18] The orbit of a point $x_{0} \in \mathbb{R}$ under a mapping $T$ is defined as a sequence of points

$$
x_{0}, x_{1}=T\left(x_{0}\right), x_{2}=T^{2}\left(x_{0}\right), \ldots, x_{n}=T^{n}\left(x_{0}\right), \ldots
$$

Definition 2.2: ( [18], p. 225) The Julia set of a function $g$ is the boundary of the set of points $z \in \mathbb{C}$ that tends to infinity under repeated iteration by $g(z)$, i.e., for a function $g$, the Julia set is given by

$$
J(g)=\partial\left\{z \in \mathbb{C}: g^{n}(z) \rightarrow \infty \text { as } n \rightarrow \infty\right\}
$$

where $g^{n}(z)$ is the $n^{t h}$ iterate of function $g$.
Definition 2.3: ( [18], p. 249) The Mandelbrot set $M$ is the collection of all complex numbers for which the Julia set is connected, i.e.,

$$
M=\{z \in \mathbb{C}: J(g) \text { is connected }\}
$$

Definition 2.4: Let $X$ be a subset of set of complex numbers and $T: X \rightarrow X$ be a mapping. For any initial point $z_{0} \in X$, consider a sequence $\left\{z_{n}\right\}$ of iterates such that

$$
\begin{align*}
S z_{n+1} & =\left(1-\alpha_{n}\right) S u_{n}+\alpha_{n} T u_{n} \\
S u_{n} & =\left(1-\beta_{n}\right) S v_{n}+\beta_{n} T v_{n}  \tag{1}\\
S v_{n} & =\left(1-\gamma_{n}\right) S z_{n}+\gamma_{n} T z_{n}
\end{align*}
$$

where $\alpha_{n}, \beta_{n}, \gamma_{n}$ are sequences of positive numbers in [0, 1] and $S z=b z$. Then, (1) is called Jungck-SP orbit [19] having five tuples $\left(T, z_{0}, \alpha_{n}, \beta_{n}, \gamma_{n}\right)$.

Remark 2.5: The Jungck-SP orbit reduces to :

1. The Jungck Thaiwan orbit when $\gamma_{n}=0$, i.e.

$$
\begin{gathered}
S z_{n+1}=\left(1-\alpha_{n}\right) S u_{n}+\alpha_{n} T\left(u_{n}\right) \\
S u_{n}=\left(1-\beta_{n}\right) S z_{n}+\beta_{n} T\left(z_{n}\right)
\end{gathered}
$$

2. The Jungck Mann orbit [22] when $\gamma_{n}=\beta_{n}=0$, i.e.

$$
S z_{n+1}=\left(1-\alpha_{n}\right) S z_{n}+\alpha_{n} T\left(z_{n}\right)
$$

3. The Jungck Picard orbit when $\gamma_{n}=\beta_{n}=0$ and $\alpha_{n}=1$, i.e.

$$
S z_{n+1}=T\left(z_{n}\right)
$$

## III. Main Results

The escape criterion plays a vital role in the generation of fractals. In this section, we derive escape criterion to generate fractals for $n^{\text {th }}$ degree complex polynomials in Jungck-SP orbit. Throughout this paper, we assume that $z_{0}=z, u_{0}=$ $u, v_{0}=v$ and $\alpha_{n}=\alpha, \beta_{n}=\beta, \gamma_{n}=\gamma$. The Jungck-SP iteration scheme (1) can be written in the following manner:

$$
\begin{gathered}
S z_{n+1}=(1-\alpha) S u_{n}+\alpha Q_{c}\left(u_{n}\right) \\
S u_{n}=(1-\beta) S v_{n}+\beta Q_{c}\left(v_{n}\right) \\
S v_{n}=(1-\gamma) S z_{n}+\gamma Q_{c}\left(z_{n}\right)
\end{gathered}
$$

where $n=0,1,2, \ldots ; 0 \leq \alpha, \beta, \gamma \leq 1$ and $Q_{c}\left(z_{n}\right)$ is the $n^{\text {th }}$ degree complex polynomial for $\bar{n}=2,3, \ldots$.

## A. Escape Criterion for Quadratic Polynomials:

Let $P_{b, c}(z)=z^{2}-b z+c$ be a quadratic complex polynomial. We choose $Q_{c}(z)=z^{2}+c$ and $S z=b z$ where $b, c \in \mathbb{C}$.

Theorem 3.1: Suppose $|z| \geq|c|>2(1+|b|) / \alpha,|z| \geq$ $|c|>2(1+|b|) / \beta$ and $|z| \geq|c|>2(1+|b|) / \gamma$, where $0 \leq \alpha, \beta, \gamma \leq 1$ and $c \in \mathbb{C}$.
Define

$$
\begin{gathered}
S z_{1}=(1-\alpha) S u+\alpha Q_{c}(u) \\
S z_{2}=(1-\alpha) S u_{1}+\alpha Q_{c}\left(u_{1}\right) \\
\cdots \\
\cdots \\
\cdots \\
S z_{n}=(1-\alpha) S u_{n-1}+\alpha Q_{c}\left(u_{n-1}\right)
\end{gathered}
$$

where $Q_{c}(z)$ is a quadratic polynomial in terms of $\alpha$ and $n=1,2,3, \ldots$, then $\left|z_{n}\right| \rightarrow \infty$ as $n \rightarrow \infty$.

## Proof: Consider

$$
\begin{array}{rlrl}
|S v| & =\left|(1-\gamma) S z+\gamma Q_{c}(z)\right|, \text { where } Q_{c}(z)=z^{2}+c \\
|b v| & =\left|(1-\gamma) b z+\gamma\left(z^{2}+c\right)\right| & & \\
& \geq\left|\gamma z^{2}\right|-|(1-\gamma) b z|-|\gamma c| & & \\
& \geq\left|\gamma z^{2}\right|-|b z|+|\gamma b z|-|\gamma z|, & (\because|z| \geq|c| \\
& \geq\left|\gamma z^{2}\right|-|b z|-|\gamma z|, & & (\because|b| \geq 0) \\
\Rightarrow|b v| & \geq\left|\gamma z^{2}\right|-|b||z|-|z|, & & (\because \gamma<1) \\
& =\left|\gamma z^{2}\right|-|z|(|b|+1) & & \\
& =|z|\{\gamma|z|-(|b|+1)\} . & &
\end{array}
$$

Thus,

$$
\begin{aligned}
|b||v| & \geq|z|\{\gamma|z|-(|b|+1)\} \\
|v| & \geq|z|(1+1 /|b|)\{\gamma|z| /(|b|+1)-1\} \\
& \geq|z|\{\gamma|z| /(|b|+1)-1\},
\end{aligned}
$$

i.e.,

$$
\begin{equation*}
|v| \geq|z|\{\gamma|z| /(|b|+1)-1\} . \tag{2}
\end{equation*}
$$

Also,

$$
|S u|=\left|(1-\beta) S v+\beta Q_{c}(v)\right|
$$

$$
\begin{aligned}
|b u|= & \left|(1-\beta) b v+\beta\left(v^{2}+c\right)\right| \\
& \geq \mid(1-\beta) b\{|z|(\gamma|z| /(|b|+1)-1)\} \\
& +\beta\left[\{|z|(\gamma|z| /(|b|+1)-1)\}^{2}+c\right] \mid .
\end{aligned}
$$

Since $|z|>2(1+|b|) / \gamma$, we have $|z|(\gamma|z| /(|b|+1)-1)>1$. This gives

$$
\begin{array}{rlrl}
|b u| & \geq|(1-\beta) b| z\left|+\beta\left(|z|^{2}+c\right)\right| & & \\
& \geq \beta|z|^{2}-|(1-\beta) b z|-\beta|z|, & (\because|z| \geq|c| \\
& \geq \beta|z|^{2}-|b z|+\beta b|z|-\beta|z| & & \\
& \geq \beta|z|^{2}-|b z|-|z|, & & \\
& =\beta|z|^{2}-|z|(|b|+1) & \\
\Rightarrow|b||u| & \geq|z|\{\beta|z|-(1+|b|)\}, & \\
\text { i.e., } & & \\
|u| & \geq|z|\{\beta|z| /(|b|+1)-1\} . & & \tag{3}
\end{array}
$$

Now, for $S z_{n}=(1-\alpha) u_{n-1}+\alpha Q_{c}\left(u_{n-1}\right)$, we have

$$
\begin{aligned}
\left|S z_{1}\right| & =\left|(1-\alpha) S u+\alpha Q_{c}(u)\right| \\
\left|b z_{1}\right| & =\left|(1-\alpha) b u+\alpha\left(u^{2}+c\right)\right| .
\end{aligned}
$$

Using (3), we have

$$
\begin{aligned}
\left|b z_{1}\right| & \geq|(1-\alpha) b| z\left|+\alpha\left(|z|^{2}+c\right)\right| \\
& \geq \alpha|z|^{2}-|(1-\alpha) b z|-|\alpha z| \\
& \geq \alpha|z|^{2}-|b z|-|z|, \quad(\because \alpha<1) \\
& =|z|\{\alpha|z|-(1+|b|)\}, \\
\text { i.e., }\left|z_{1}\right| & \geq|z|(\alpha|z| /(1+|b|)-1) .
\end{aligned}
$$

Since $|z| \geq|c|>2(1+|b|) / \alpha,|z| \geq|c|>2(1+|b|) / \beta$ and $|z| \geq|c|>2(1+|b|) / \gamma$ exist. Therefore, we have $\alpha|z| /(1+$ $|b|)-1>1$. Hence, there exists $\delta>0$ such that $\alpha|z| /(1+$ $|b|)-1>\delta+1>1$. Consequently, we have

$$
\left|z_{1}\right|>(1+\delta)|z|
$$

In particular, $\left|z_{n}\right|>|z|$. So, repeating the same argument $n$ times, we obtain,

$$
\left|z_{n}\right|>(1+\delta)^{n}|z| .
$$

Thus, the orbit of $z$ tends to infinity as $n$ tends to infinity. Hence, the result.

## B. Escape Criterion for Cubic Polynomials:

Now, we prove the following theorem for a cubic complex polynomial $P_{b, c}(z)=z^{3}-b z+c$, which is equivalent to all other cubic polynomials. We consider $Q_{c}(z)=z^{3}+c$ and $S z=b z$ where $b, c \in \mathbb{C}$.

Theorem 3.2: Suppose that $|z| \geq|c|>(2(1+|b|) / \alpha)^{1 / 2}$, $|z| \geq|c|>(2(1+|b|) / \beta)^{1 / 2}$ and $|z| \geq|c|>(2(1+$ $|b|) / \gamma)^{1 / 2}$ where $0<\alpha, \beta, \gamma \leq 1$ and $b, c$ are complex numbers. Define

$$
\begin{gathered}
S z_{1}=(1-\alpha) S u+\alpha Q_{c}(u) \\
S z_{2}=(1-\alpha) S u_{1}+\alpha Q_{c}\left(u_{1}\right)
\end{gathered}
$$

$$
S z_{n}=(1-\alpha) S u_{n-1}+\alpha Q_{c}\left(u_{n-1}\right) ; n=1,2,3, \ldots,
$$

where $Q_{c}(u)$ is a cubic polynomial in terms of $\alpha$, then $z_{n} \rightarrow \infty$ as $n \rightarrow \infty$.

## Proof: Consider

$$
\begin{array}{rlrl}
|S v| & =\left|(1-\gamma) S z+\gamma Q_{c}(z)\right|, \text { where } & Q_{c}(z)=z^{3}+c \\
|b v| & =\left|(1-\gamma) b z+\gamma\left(z^{3}+c\right)\right| & & \\
& \geq\left|\gamma z^{3}\right|-|(1-\gamma) b z|-|\gamma c| & & \\
& \geq\left|\gamma z^{3}\right|-|b z|+|\gamma b z|-|\gamma z|, & (\because|z| \geq|c| \\
& \geq\left|\gamma z^{3}\right|-|b z|-|\gamma z|, & & (\because|b| \geq 0) \\
\Rightarrow|b v| & \geq\left|\gamma z^{3}\right|-|b||z|-|z|, & & (\because \gamma<1) \\
& =\left|\gamma z^{3}\right|-|z|(|b|+1) & & \\
& =|z|\left\{\gamma\left|z^{2}\right|-(|b|+1)\right\} . & &
\end{array}
$$

Thus,

$$
\begin{aligned}
|b||v| & \geq|z|\left\{\gamma\left|z^{2}\right|-(|b|+1)\right\} \\
|v| & \geq|z|(1+1 /|b|)\left\{\gamma\left|z^{2}\right| /(|b|+1)-1\right\} \\
& \geq|z|\left\{\gamma\left|z^{2}\right| /(|b|+1)-1\right\},
\end{aligned}
$$

i.e.,

$$
\begin{equation*}
|v| \geq|z|\left\{\gamma\left|z^{2}\right| /(|b|+1)-1\right\} \tag{4}
\end{equation*}
$$

Now, take

$$
\begin{aligned}
|S u|= & \left|(1-\beta) S v+\beta Q_{c}(v)\right| \\
|b u|= & \left|(1-\beta) b v+\beta\left(v^{3}+c\right)\right| \\
\geq & \mid(1-\beta) b\left\{|z|\left(\gamma\left|z^{2}\right| /(|b|+1)-1\right)\right\} \\
& +\beta\left[\left\{|z|\left(\gamma\left|z^{2}\right| /(|b|+1)-1\right)\right\}^{2}+c\right] \mid .
\end{aligned}
$$

Since $|z|>(2(1+|b|) / \gamma)^{1 / 2}$, we have $\left(\gamma\left|z^{2}\right| /(|b|+1)-1\right)>$ 1. This gives

$$
\begin{array}{rlrl}
|b u| & \geq|(1-\beta) b| z\left|+\beta\left(|z|^{3}+c\right)\right| \\
& \geq \beta|z|^{3}-|(1-\beta) b z|-\beta|z|, & \\
& \geq \beta|z|^{3}-|b z|+\beta b|z|-\beta|z| \\
& \geq \beta|z|^{3}-|b z|-|z|, & \\
& =\beta|z|^{3}-|z|(|b|+1) & \\
\Rightarrow|b||u| & \geq|z|\left\{\beta\left|z^{2}\right|-(1+|b|)\right\}, & \\
\text { i.e., } & & \\
|u| & \geq|z|\left\{\beta\left|z^{2}\right| /(|b|+1)-1\right\} . & \tag{5}
\end{array}
$$

Now, for $S z_{n}=(1-\alpha) u_{n-1}+\alpha Q_{c}\left(u_{n-1}\right)$, we have

$$
\begin{aligned}
\left|S z_{1}\right| & =\left|(1-\alpha) S u+\alpha Q_{c}(u)\right| \\
\left|b z_{1}\right| & =\left|(1-\alpha) b u+\alpha\left(u^{3}+c\right)\right| .
\end{aligned}
$$

Using (5), we have

$$
\begin{aligned}
\left|b z_{1}\right| & \geq|(1-\alpha) b| z\left|+\alpha\left(|z|^{3}+c\right)\right| \\
& \geq \alpha|z|^{3}-|(1-\alpha) b z|-|\alpha z| \\
& \geq \alpha|z|^{3}-|b z|-|z|, \quad(\because \alpha<1) \\
& =|z|\left\{\alpha\left|z^{2}\right|-(1+|b|)\right\}, \\
\text { i.e., }\left|z_{1}\right| & \geq|z|\left(\alpha\left|z^{2}\right| /(1+|b|)-1\right) .
\end{aligned}
$$

Since $|z| \geq|c|>(2(1+|b|) / \alpha)^{1 / 2},|z| \geq|c|>(2(1+$ $|b|) / \beta)^{1 / 2}$ and $|z| \geq|c|>(2(1+|b|) / \gamma)^{1 / 2}$ exist. Therefore, we have $\alpha\left|z^{2}\right| /(1+|b|)-1>1$. Hence, there exists $\delta>0$ such that $\alpha\left|z^{2}\right| /(1+|b|)-1>\delta+1>1$. Consequently, we have

$$
\left|z_{1}\right|>(1+\delta)|z|
$$

Particularly, $\left|z_{n}\right|>|z|$. So, using the same argument $n$ times, we have,

$$
\left|z_{n}\right|>(1+\delta)^{n}|z| .
$$

Thus, the orbit of $z$ tends to infinity as $n$ tends to infinity. Hence, the result.

## C. A General Escape Criterion

Now, we obtain a general escape criterion for higher degree complex polynomials.

Theorem 3.3: Let $G_{b, c}(z)=z^{n}-b z+c, n=2,3, \ldots$ be a higher degree complex polynomial. Choose $Q_{c}(z)=z^{n}+c$ and $S z=b z$ where $b, c \in \mathbb{C}$. Define

$$
\begin{gathered}
S z_{1}=(1-\alpha) S u+\alpha Q_{c}(u) \\
S z_{2}=(1-\alpha) S u_{1}+\alpha Q_{c}\left(u_{1}\right)
\end{gathered}
$$

$$
S z_{n}=(1-\alpha) S u_{n-1}+\alpha Q_{c}\left(u_{n-1}\right) ; n=1,2, \ldots
$$

Then, the general escape criterion is $|z|>\max \{|c|,(2(1+$ $\left.|b|) / \alpha)^{1 /(n-1)},(2(1+|b|) / \beta)^{1 /(n-1)},(2(1+|b|) / \gamma)^{1 /(n-1)}\right\}$.

Proof: We shall prove the theorem by using the method of induction. For $n=1$, we have $Q_{c}(z)=z+c$ and this implies

$$
|z|>\max \{|c|, 0,0,0\} .
$$

For $n=2$, we have $Q_{c}(z)=z^{2}+c$, so by Theorem 3.1, the escape criterion is

$$
|z|>\max \{|c|, 2(1+|b|) / \alpha, 2(1+|b|) / \beta, 2(1+|b|) / \gamma\} .
$$

Similarly, for $n=3$, we get $Q_{c}(z)=z^{3}+c$. Then, from Theorem 3.2, the escape criterion is given by

$$
\begin{gathered}
|z|>\max \left\{|c|,(2(1+|b|) / \alpha)^{1 / 2},(2(1+|b|) / \beta)^{1 / 2},\right. \\
\left.(2(1+|b|) / \gamma)^{1 / 2}\right\} .
\end{gathered}
$$

Thus, the theorem is true for $n=1,2,3$. Now, suppose that theorem is true for any $n$. We shall prove that, the result holds for $n+1$. Let $Q_{c}(z)=z^{n+1}+c$ and $|z| \geq|c|>$ $(2(1+|b|) / \alpha)^{1 / n},|z| \geq|c|>(2(1+|b|) / \beta)^{1 / n}$ and $|z| \geq$ $|c|>(2(1+|b|) / \gamma)^{1 / n}$ be escape criterion.
Then, consider

$$
\begin{array}{rlrl}
|S v| & =\left|(1-\gamma) S z+\gamma Q_{c}(z)\right|, & \\
|b v| & =\left|(1-\gamma) b z+\gamma\left(z^{n+1}+c\right)\right| & & \\
& \geq\left|\gamma z^{n+1}\right|-|(1-\gamma) b z|-|\gamma c| & & \\
& \geq\left|\gamma z^{n+1}\right|-|b z|+|\gamma b z|-|\gamma z|, & (\because|z| \geq|c| \\
& \geq\left|\gamma z^{n+1}\right|-|b z|-|\gamma z|, & & (\because|b| \geq 0) \\
\Rightarrow|b v| & \geq\left|\gamma z^{n+1}\right|-|b||z|-|z|, & & (\because \gamma<1) \\
& =\left|\gamma z^{n+1}\right|-|z|(|b|+1) & & \\
& =|z|\left\{\gamma\left|z^{n}\right|-(|b|+1)\right\} . & &
\end{array}
$$

Thus,

$$
\begin{aligned}
|b||v| & \geq|z|\left\{\gamma\left|z^{n}\right|-(|b|+1)\right\} \\
|v| & \geq|z|(1+1 /|b|)\left\{\gamma\left|z^{n}\right| /(|b|+1)-1\right\} \\
& \geq|z|\left\{\gamma\left|z^{n}\right| /(|b|+1)-1\right\},
\end{aligned}
$$

i.e.,

$$
\begin{equation*}
|v| \geq|z|\left\{\gamma\left|z^{n}\right| /(|b|+1)-1\right\} . \tag{6}
\end{equation*}
$$

Now, take

$$
\begin{aligned}
|S u|= & \left|(1-\beta) S v+\beta Q_{c}(v)\right| \\
|b u|= & \left|(1-\beta) b v+\beta\left(v^{n+1}+c\right)\right| \\
\geq & \mid(1-\beta) b\left\{|z|\left(\gamma\left|z^{n}\right| /(|b|+1)-1\right)\right\} \\
& +\beta\left[\left\{|z|\left(\gamma\left|z^{n}\right| /(|b|+1)-1\right)\right\}^{n+1}+c\right] \mid .
\end{aligned}
$$

Since $|z|>(2(1+|b|) / \gamma)^{1 / n}$, we have $|z|\left(\gamma\left|z^{n}\right| /(|b|+1)\right.$ $-1)>1$.
This gives

$$
\begin{array}{rlr}
|b u| & \geq|(1-\beta) b| z\left|+\beta\left(|z|^{n+1}+c\right)\right| & \\
& \geq \beta|z|^{n+1}-|(1-\beta) b z|-\beta|z|, & (\because|z| \geq|c| \\
& \geq \beta|z|^{n+1}-|b z|+\beta b|z|-\beta|z| \\
& \geq \beta|z|^{n+1}-|b z|-|z|, & \\
& =\beta|z|^{n+1}-|z|(|b|+1), &
\end{array} \quad(\because \beta<1)
$$

i.e., $|b||u| \geq|z|\left\{\beta\left|z^{n}\right|-(1+|b|)\right\}$

$$
\begin{equation*}
\Rightarrow|u| \geq|z|\left\{\beta\left|z^{n}\right| /(|b|+1)-1\right\} \tag{7}
\end{equation*}
$$

Now, for $S z_{n}=(1-\alpha) u_{n-1}+\alpha Q_{c}\left(u_{n-1}\right)$, we have

$$
\begin{aligned}
\left|S z_{1}\right| & =\left|(1-\alpha) S u+\alpha Q_{c}(u)\right| \\
\left|b z_{1}\right| & \geq\left|(1-\alpha) b u+\alpha\left(u^{n+1}+c\right)\right| .
\end{aligned}
$$

Using (7), we have

$$
\begin{aligned}
\left|b z_{1}\right| & \geq|(1-\alpha) b| z\left|+\alpha\left(|z|^{n+1}+c\right)\right| \\
& \geq \alpha|z|^{n+1}-|(1-\alpha) b z|-|\alpha z| \\
& \geq \alpha|z|^{n+1}-|b z|-|z|,(\because \alpha<1) \\
& =|z|\left\{\alpha\left|z^{n}\right|-(1+|b|)\right\}, \\
\text { i.e., }\left|z_{1}\right| & \geq|z|\left(\alpha\left|z^{n}\right| /(1+|b|)-1\right) .
\end{aligned}
$$

Since $|z| \geq|c|>(2(1+|b|) / \alpha)^{1 / n},|z| \geq|c|>(2(1+$ $|b|) / \beta)^{1 / n}$ and $|z| \geq|c|>(2(1+|b|) / \gamma)^{1 / n}$ exist. Therefore, we have $\alpha\left|z^{n}\right| /(1+|b|)-1>1$. Hence, there exists $\delta>0$ such that $\alpha\left|z^{n}\right| /(1+|b|)-1>\delta+1>1$. Consequently, we have

$$
\left|z_{1}\right|>(1+\delta)|z|
$$

So, repeating the same argument $n$ times, we have

$$
\left|z_{n}\right|>(1+\delta)^{n}|z|
$$

Thus, the orbit of $z$ tends to infinity as $n$ tends to infinity.
From the above theorem, we obtain the following results in the form of corollaries:

Corollary 3.4: Assume that $|c|>\quad>\quad(2(1+$ $|b|) / \alpha)^{1 / n-1},|c|>(2(1+|b|) / \beta)^{1 / n-1} \quad$ and $|c|>(2(1+|b|) / \gamma)^{1 / n-1}$ exists. Then, the orbit $S P\left(Q_{c}, 0, \alpha, \beta, \gamma\right)$ escapes to infinity.

Corollary 3.5: (Escape Criterion) Let us suppose that for some $k \geq 0,\left|z_{k}\right|>\max \left\{|c|,(2(1+|b|) / \alpha)^{1 / k-1},(2(1+\right.$ $\left.|b|) / \beta)^{1 / k-1},(2(1+|b|) / \gamma)^{1 / k-1}\right\}$, then $\left|z_{k}\right|>\delta\left|z_{k-1}\right|$ and $\left|z_{n}\right| \rightarrow \infty$ as $n \rightarrow \infty$.

Using this corollary, we obtain an algorithm for computing the connected Julia sets of $n^{t h}$ degree complex polynomials of the form $G_{c}(z)=z^{n}-b z+c, n=2,3, \ldots$. For any point $z_{k}$ satisfying $\left|z_{k}\right|>\max \left\{|c|,(2(1+|b|) / \alpha)^{1 / k-1},(2(1+\right.$ $\left.|b|) / \beta)^{1 / k-1},(2(1+|b|) / \gamma)^{1 / k-1}\right\}$, we obtain the orbit of $z_{k}$. If for some $n,\left|z_{n}\right|$ lies outside the circle of radius $\max \left\{|c|,(2(1+|b|) / \alpha)^{1 / k-1},(2(1+|b|) / \beta)^{1 / k-1},(2(1+\right.$ $\left.|b|) / \gamma)^{1 / k-1}\right\}$, then we observe that the orbit escapes to infinity, i.e., $z_{k}$ does not lie in the connected Julia set. If $\left|z_{n}\right|$ never exceeds this bound, then according to definition $z_{k}$ lies in the connected Julia set.

## IV. Algorithm for generating Mandelbrot sets and Julia sets

Using the general escape criterion which is obtained in Theorem 3.3, we provide an algorithm for generating Mandelbrot sets and Julia sets. This algorithm demonstrate the significance of our results obtained in Section III. This algorithm includes the following steps:

1) Setup :

Choose a complex number $c=l+m \iota$.
Initialize values to parameters $\alpha, \beta, \gamma, b$.
Take $z_{0}=x+y \iota$ as first iteration.
2) Iterate :

$$
\begin{aligned}
S z_{n+1} & =(1-\alpha) S u_{n}+\alpha Q_{c}\left(u_{n}\right) \\
S u_{n} & =(1-\beta) S v_{n}+\beta Q_{c}\left(v_{n}\right) \\
S v_{n} & =(1-\gamma) S z_{n}+\gamma Q_{c}\left(z_{n}\right) ; n=0,1,2, \ldots,
\end{aligned}
$$

where $Q_{c}\left(z_{n}\right)=z^{n}+c, n=2,3, \ldots$ and $S z=b z$.
3) Stop :
$\left|z_{n}\right|>$ escape radius

$$
\begin{aligned}
& =\max \left\{|c|,(2(1+|b|) / \alpha)^{1 / n-1},(2(1+|b|) / \beta)^{1 / n-1},\right. \\
& \left.\quad(2(1+|b|) / \gamma)^{1 / n-1}\right\} .
\end{aligned}
$$

4) Count : number of iterations to escape.
5) Color : point depends on number of iterations required to escape.

Note: To generate Mandelbrot set, we take $z_{0}=0$ as our first iteration while in case of Julia set $z_{0}$ is taken non-zero, i.e., $z_{0} \neq 0$.

With the help of this algorithm, we make a program in Mathematica 11.0 and generate fractals of $n^{t h}$ degree complex polynomials of the form $G_{b, c}(z)=z^{n}-b z+c$, $n=2,3, \ldots$.

## V. Generation of Mandelbrot sets and Julia Sets IN Jungck-SP orbit

In this section, using above algorith, we generate several Mandelbrot sets and Julia sets for quadratic, cubic and higher degree complex polynomials by running a program in Mathematica 11.
A. Mandelbrot and Julia sets for quadratic complex polynomial $P_{b, c}(z)=z^{2}-b z+c$ :

We choose $Q_{c}(z)=z^{2}+2$ and $S z=b z$. Considering different values of parameters $\alpha, \beta, \gamma$ and $b$, we generate the following graphics :


Fig. 1: Mandelbrot set for $\alpha=0.3, \beta=0.1, \gamma=0.95, b=2$


Fig. 2: Mandelbrot set for $\alpha=\beta=\gamma=0.9, b=2$


Fig. 3: Mandelbrot set for $\alpha=\beta=0.1, \gamma=0.9, b=3$


Fig. 4: Mandelbrot set for $\alpha=0.9, \beta=\gamma=0.1, b=3$


Fig. 5: Mandelbrot set for $\alpha=\beta=\gamma=0.5, b=4$


Fig. 6: Julia set for $\alpha=0.3, \beta=0.1, \gamma=0.7, c=-6.15+4.9 \iota, b=2$


Fig. 7: Julia set for $\alpha=\beta=0.1, \gamma=0.9, c=-5.05-5.05 \iota, b=2$


Fig. 8: Julia set for $\alpha=\beta=\gamma=0.5, c=-1.0-1.0 \iota, b=3$


Fig. 9: Julia set for $\alpha=0.3, \beta=0.1, \gamma=0.6, c=-6.8-5.05 \iota, b=3$


Fig. 10: Julia set for $\alpha=0.3, \beta=0.1, \gamma=0.9, c=-1.15+4.15 \iota, b=4$
B. Mandelbrot and Julia sets for cubic complex polynomial $P_{b, c}(z)=z^{3}-b z+c:$

We choose $Q_{c}(z)=z^{3}+2$ and $S z=b z$. Using different values of parameters, we generate the following fractals :


Fig. 11: Mandelbrot set for $\alpha=\beta=\gamma=0.9, b=2$


Fig. 12: Mandelbrot set for $\alpha=0.9, \beta=\gamma=0.1, b=3$


Fig. 13: Mandelbrot set for $\alpha=\beta=\gamma=0.5, b=3$


Fig. 14: Mandelbrot set for $\alpha=\beta=\gamma=0.5, b=4$


Fig. 15: Julia set for $\alpha=0.03, \beta=0.1, \gamma=0.6, c=-1.05 \iota, b=1 / 2$


Fig. 16: Julia set for $\alpha=0.1, \beta=0.6, \gamma=0.9, c=-0.7+1.05 \iota, b=2$


Fig. 17: Julia set for $\alpha=0.1, \beta=0.5, \gamma=0.8, c=-0.75+1.05 \iota, b=3$


Fig. 18: Julia set for $\alpha=0.3, \beta=0.8, \gamma=0.1, c=-0.85+0.2 \iota, b=4$
C. Mandelbrot and Julia sets for higher degree complex polynomials $P_{b, c}(z)=z^{n}-b z+c, n=4,5, \ldots$ :

We choose $Q_{c}(z)=z^{n}+c$ and $S z=b z$. We have the following figures by using different values of parameters :


Fig. 19: Mandelbrot set for $\alpha=0.9, \beta=0.1, \gamma=0.1, b=3, n=4$


Fig. 20: Mandelbrot set for $\alpha=\beta=\gamma=0.9, b=2, n=5$


Fig. 21: Mandelbrot set for $\alpha=\beta=\gamma=0.5, b=3, n=5$


Fig. 22: Mandelbrot set for $\alpha=0.5, \beta=0.5, \gamma=0.5, b=4, n=10$


Fig. 23: Julia set for $\alpha=0.1, \beta=0.5, \gamma=0.8, c=-0.75+1.05 \iota$, $b=3, n=4$


Fig. 24: Julia set for $\alpha=0.1, \beta=0.3, \gamma=0.9, c=-0.1+0.5 \iota$, $b=4, n=15$

## VI. Conclusion

In this paper, we have established the new escape criterion for complex quadratic, cubic and $n^{t h}$ degree complex polynomials and present an algorithm to generate fractals via Jungck-SP orbit. Despite the theoretical development of fractal theory, we have the following observations:

1) Due to high rate of convergence of Jungck-SP orbit, we observe that $20-30$ iterations are enough to generate these beautiful graphics.
2) Julia set obtained in Fig. 23 resembles with a vegetable Turnip (Shalajam in Hindi).
3) We notice that the Mandelbrot sets and Julia sets vary with the variation of parameters.
4) Some Graphics generated by us may be useful for graphics designer (see, Figs. 20-24).
5) Our results and algorithm can be further generalized to obtain some new fractals.

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