Vertex-Distinguishing E-Total Coloring of Complete Bipartite Graph $K_{7,n}$ when $n \geq 978$

Xiang’en Chen

Abstract—Let $G$ be a simple graph. A total coloring $f$ of $G$ is called an E-total coloring if no two adjacent vertices of $G$ receive the same color, and no edge of $G$ receives the same color as one of its endpoints. For an E-total coloring $f$ of a graph $G$ and any vertex $x$ of $G$, let $C(x)$ denote the set of colors of vertex $x$ and of the edges incident with $x$, we call $C(x)$ the color set of $x$. If $C(u) \neq C(v)$ for any two different vertices $u$ and $v$ of $V(G)$, then we say that $f$ is a vertex-distinguishing E-total coloring of $G$ or a VDET coloring of $G$ for short. The minimum number of colors required for a VDET coloring of $G$ is denoted by $\chi_{vt}^{*}(G)$ and is called the VDET chromatic number of $G$. The VDET coloring of complete bipartite graph $K_{7,n}$ ($n \geq 978$) is discussed in this paper and the VDET chromatic number of $K_{7,n}$ ($n \geq 978$) has been obtained.

Index Terms—graph; complete bipartite graph, E-total coloring, vertex-distinguishing E-total coloring, vertex-distinguishing E-total chromatic number.

1. INTRODUCTION AND NOTATIONS

Graph theory is the historical foundation of the science of networks and the basis of information science. The problem in which we are interested is a particular case of the great variety of different ways of labeling a graph. The original motivation of studying this problem came from irregular networks. The idea was to weight the edges by positive integers such that the sum of the weights of edges incident with each vertex formed a set of distinct numbers.

For an edge coloring (proper or not) $g$ of $G$ and a vertex $x$ of $G$, let $S(x)$ be the set (not multiset) of colors of the edges incident with $x$ under $g$.

For a proper edge coloring, if $S(u) \neq S(v)$ for any two distinct vertices $u$ and $v$, then the coloring is a vertex-distinguishing proper edge coloring. The minimum number of colors required for a vertex-distinguishing proper edge coloring of $G$ is denoted by $\chi_{vt}(G)$. This coloring is proposed in [5] and [4] independently. Many scholars have studied this parameter in [2], [3], [4], [5], [20], [21], [22].

For an edge coloring which is not necessary proper, if $S(u) \neq S(v)$ for any two distinct vertices $u$ and $v$, then the coloring is a point distinguishing edge coloring. The minimum number of colors required for a point distinguishing edge coloring of $G$ is denoted by $\chi_0(G)$. This coloring is proposed in [15] by Harary et al. This parameter has been researched in many papers [6], [15], [16], [17], [18], [23], [24].

For a total coloring (proper or not) $f$ of $G$ and a vertex $x$ of $G$, let $C(x)$ be the set (not multiset) of colors of vertex $x$ and edges incident with $x$ under $f$.

Manuscript received October 26, 2018; revised January 12, 2019. This work was supported by the National Natural Science Foundation of China (Grant Nos. 11761064, 61163037).

Xiang’en Chen is with the College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, P R China. The corresponding author. Email: chenx@nwnu.edu.cn; xiangenchen@163.com.

For a proper total coloring, if $C(u) \neq C(v)$, for any two distinct vertices $u$ and $v$, then the coloring is called a vertex-distinguishing (proper) total coloring, or a VDT coloring of $G$ for short. The minimum number of colors required for a VDT coloring of $G$ is denoted by $\chi_{vt}(G)$.

The vertex distinguishing proper total colorings of graphs are introduced and studied by Zhongfu Zhang et al in [25]. After studying the vertex distinguishing proper total coloring of complete graph, star, complete bipartite graph, wheel, fan, path and cycle, a conjecture was proposed in [25]: let $\mu(G) = \min\{k \geq n_i, \delta \leq i \leq \Delta\}$, then $\chi_{vt}(G) = \mu(G)$ or $\mu(G) + 1$. In [7], the vertex-distinguishing total coloring of $n$-cube were discussed, respectively. In [11], the relations of vertex distinguishing total chromatic numbers between a subgraph and its supergraph had been studied.

We will consider a kind of not necessarily proper total coloring which is vertex distinguishing. A total coloring $f$ of $G$ is called an E-total coloring if no two adjacent vertices of $G$ receive the same color, and no edge of $G$ receives the same color as one of its endpoints. If $f$ is an E-total coloring of graph $G$ and for any $u, v \in V(G), u \neq v$, we have $C(u) \neq C(v)$, then $f$ is called a vertex-distinguishing E-total coloring, or a VDT coloring briefly. The minimum number of colors required for a VDT coloring of $G$ is called the vertex-distinguishing E-total chromatic number of $G$ and is denoted by $\chi_{vt}^{*}(G)$.

The VDET colorings of complete graph, complete bipartite graph $K_{2,n}$, star, wheel, fan, path and cycle were discussed in [14].

A parameter was introduced in [14]: $\eta(G) = \min\{l : \frac{l}{2} + \cdots + \frac{l}{i} \geq n_k + n_{k+1} + \cdots + n_1, 1 \leq i \leq \Delta\}$. If $\eta(G)$ denote the number of vertices with degree $i$, $\delta \leq i \leq \Delta$. At the end of the paper [14], a Vizing-like conjecture was proposed.

Conjecture 1 ([14]) For a graph $G$ with no isolated vertices and chromatic number at most 5, we have $\chi_{vt}^{*}(G) = \eta(G)$ or $\eta(G) + 1$.

We have studied the vertex-distinguishing E-total colorings of $mc_5$ and $mc_4$ in article [13] and confirmed Conjecture 1 for these two kinds of graphs.

The VDET chromatic numbers of complete bipartite graphs $K_{7,n}$ ($7 \leq n \leq 977$) had been determined and Conjecture 1 is confirmed for $K_{7,n}$ ($7 \leq n \leq 977$) in [8], [9], [10]. In this paper, we will consider the VDET coloring of complete bipartite graph $K_{7,n}$ ($n \geq 978$) and confirm Conjecture 1 for $K_{7,n}$ ($n \geq 978$).

For a vertex distinguishing E-total coloring $f$ of a graph $G$ and an element $z \in V(G) \cup E(G)$, we use $f(z)$ to denote the color of $z$ under $f$.

Let $X = \{u_1, u_2, \ldots, u_7\}$, $Y = \{v_1, v_2, \ldots, v_n\}$, $V(K_{7,n}) = X \cup Y$ and $E(K_{7,n}) = \{u_iv_j : 1 \leq i \leq 7, 1 \leq$
Given a vertex distinguishing E-total coloring of $K_{7,n}$, let $C(X) = \{C(u_1), C(u_2), \ldots, C(u_7)\}$, $C(Y) = \{C(v_1), C(v_2), \ldots, C(v_7)\}$.

For a positive integer l, we use \([l]\) to denote the set \([1, 2, \ldots, l]\). If we mention an l-VDET coloring, then the colors which we have used are 1, 2, \ldots, l. And for \(c \in [1, 2, \ldots, l]\), we use \(C_c\) to denote \([1, 2, \ldots, l] \setminus \{c\}\), i.e., \(\{c\} = [l] \setminus \{c\}\). Generally for subset \(A\) of \([l]\), we use \(\bar{A}\) to denote the complementary subset of \(A\) in \([l]\), i.e., \(\bar{A} = [l] \setminus A\).

A subset \(A = \{1, 2, \ldots, l\}\) is called i-subset if \(A\) contain \(i\) elements, i.e., \(|A| = i\).

For adjacent vertex distinguishing proper edge colorings of bicyclic graphs we may see [12]. For the hamiltonicity and hamiltonian connectivity of L-shaped supergraph graphs we may see [19]. For a particle swarm optimization (PSO) approach: shortest path planning algorithm, we may see [1].

II. PRELIMINARIES

Lemma 1 [8](8) Let \(g\) be an l-VDET coloring of $K_{7,n}$. Suppose that there are two distinct colors among \(g(u_1), g(u_2), \ldots, g(u_7)\), and \(g(v_1), g(v_2), \ldots, g(v_7)\). If there exists a color \(a \in \{3, 4, \ldots, l\}\) such that \(\{1, 2, a\} \subset C(Y)\), i.e., \(\{1, 2, a\}\) is a color set of some vertex in \(Y\), then \([2] \subset C(u_i), i = 1, 2, \ldots, 7\).

Lemma 2[8] Let \(g\) be an l-VDET coloring of $K_{7,n}$. Suppose that there are two distinct colors among \(g(u_1), g(u_2), \ldots, g(u_7)\), and \(g(v_1), g(v_2), \ldots, g(v_7)\). If \(a_1, a_2, \ldots, a_r\) are \(r \geq 2\) distinct colors in \([3, 4, \ldots, l]\). If each 2-subset of \(\{a_1, a_2, \ldots, a_r\}\) is a color set of a vertex in \(Y\), then there exist \(r - 1\) distinct colors in \(\{a_1, a_2, \ldots, a_r\}\) such that \(r - 1\) distinct colors are contained in each set \(C(u_i), i = 1, 2, \ldots, 7\).

Lemma 3[8] Let \(g\) be an l-VDET coloring of $K_{7,n}$. Suppose that there are two distinct colors among \(g(u_1), g(u_2), \ldots, g(u_7)\), and \(g(v_1), g(v_2), \ldots, g(v_7)\). If \(a_1, a_2, \ldots, a_r\) be \(r\) distinct colors in \([3, 4, \ldots, l]\).

(i) If \(\{a_1, a_2\} \subset C(Y)\), then each color set in \(C(X)\) contains \(a_1\) or \(a_2\), i.e., \(a_1 \in C(u_i)\) or \(a_2 \in C(u_i), i = 1, 2, \ldots, 7\).

(ii) Given \(j \in [1, 2]\), if every 3-subset of \(\{a_1, a_2, \ldots, a_r\}\) which contains color \(j\) belongs to \(C(Y)\), then there exist \(r - 1\) distinct colors in \(\{a_1, a_2, \ldots, a_r\}\) such that these \(r - 1\) distinct colors are contained in each set \(C(u_i)\) with \(g(u_i) = j\).

(iii) If every 3-subset of \(\{a_1, a_2, \ldots, a_r\}\) which contains color 1 or 2 but not both belongs to \(C(Y)\), then each color set \(C(u_i)\) contains at least \(r - 1\) colors in \(\{a_1, a_2, \ldots, a_r\}\) for \(i = 1, 2, \ldots, 7\).

Lemma 4[9] $K_{7,472}$ has a 9-VDET coloring \(h_{472}\) such that (i) the color of \(u_i\) is 1 \((i = 1, 2, 3)\) and the color of \(u_j\) is 2 \((j = 4, 5, 6)\); (ii) the color sets of \(u_1, u_2, \ldots, u_7\) are \([9] \setminus \{3, 4\}, [9] \setminus \{3, 9\}, [4], [9] \setminus \{4\}, [9] \setminus \{5, 6\}, [9] \setminus \{5\}\) and \([9] \setminus \{6\}\) respectively; (iii) the color set of each vertex in \(Y\) is one of the following sets:

\[
\{3, 7\}, \{3, 8\}, \{3, 9\}, \{4, 7\}, \{4, 8\}, \{4, 9\}, \{5, 7\}, \{5, 8\}, \{5, 9\}, \{6, 7\}, \{6, 8\}, \{6, 9\}, \{7, 8\}, \{7, 9\}, \{8, 9\};
\]


The resulting coloring \(h_{477}\) is obviously an 11-VDET coloring of $K_{7,477}$.

III. MAIN RESULT

Theorem 1 Suppose \(k \geq 11, n \geq 978\). If \(\sum_{i=2}^{n} (k^i - 2k - 3) < n \leq \sum_{i=2}^{n} (k^i - 2k) - 5\), then \(\chi_{v}(K_{7,n}) = k\).

Proof Firstly, we prove that \(K_{4,n}\) does not have a \((k - 1)\)-VDET coloring. Assume that \(K_{4,n}\) has a \((k - 1)\)-VDET coloring \(g\). There are three cases to consider.

Case 1 \(u_1, u_2, \ldots, u_7\) receive the same color under \(g\). We may suppose that \(g(u_i) = 1, i = 1, 2, \ldots, 7\). So none of the \(C(v_i)\) include color 1, and each \(C(v_i)\) is one of the subsets of \([2, 3, \ldots, k - 1]\). Let \(A\) be the set composed by the 2-7, 6-5, 4-3, 2-subsets of \([2, 3, \ldots, k - 1]\). Then \(A\) contains \(\sum_{i=2}^{n} (k^i - 2k - 3)\) members and \(C(Y) \subseteq A\). As \(n > \sum_{i=2}^{n} (k^i - 2k - 3) = \sum_{i=2}^{n} (k^i - 2k) + \sum_{i=2}^{n} (i - 2k - 3) \geq \sum_{i=2}^{n} (k^i - 2k - 3) + 4(k - 2) - 2k - 3 = \sum_{i=2}^{n} (k^i - 2k - 3) + 2n > \sum_{i=2}^{n} (k^i - 2k) - 5\), this is a contradiction.
Case 2 \( u_1, u_2, \ldots, u_7 \) receive two different colors under \( g \). We may suppose that \( g(u_i) \in \{1, 2\} \), \( i = 1, 2, \ldots, 7 \). Each \( C(v_j) \) does not include color \( i \) when \( |C(v_j)| = 2, i = 1, 2 \).

Let \( \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \), where
\[
\mathcal{B}_1 = \{a, b, c) \mid a, b \in \{3, 4, \ldots, k-1\}, a < b \};
\mathcal{B}_2 = \{1, 2, c) \mid c = 3, 4, \ldots, k-1 \};
\mathcal{B}_3 = \{ \text{the set composed by the } 8, 7, 6, 5, \text{ 4-subsets of} \{k-1\} \text{ and 3-subsets of} \{k-1\} \text{ which are not in} \mathcal{B}_2 \}.
\]
Then \( C(Y) \subseteq \mathcal{B} \) and
\[
|\mathcal{B}| = \sum_{i=3}^8 (k-1) + (k-3) = 8.
\]

By simple calculation we have that \(|\mathcal{B}| - (\sum_{i=2}^8 (k-1) - 2k - 3) = 8\). So we assume that \(|\mathcal{B}| - 8 < n \leq |\mathcal{B}| \). At most seven subsets in \( \mathcal{B} \) are not in \( C(Y) \). Thus \( \mathcal{B}_2 \cap C(Y) \neq \emptyset \) and, by Lemma 1, \( 1, 2 \in C(u_i), i = 1, 2, \ldots, 7 \).

1) If \( C(u_1) \cap C(u_2) \cap \cdots \cap C(u_7) \cap \{3, 4, \ldots, k-1\} \) contains at most \( k - 8 \) colors, there exist five colors \( a_1, a_2, a_3, a_4, a_5 \in \{3, 4, 5, \ldots, k-1\} \), such that \( \{3, 4, \ldots, k-1\} \setminus \{a_1, a_2, a_3, a_4, a_5\} \subseteq C(u_1) \cap C(u_2) \cap \cdots \cap C(u_7) \cap \{3, 4, \ldots, k-1\} \).

By Lemma 2, any 2-subsets of \( \{a_1, a_2, a_3, a_4, a_5\} \) is not in \( C(Y) \). Thus
\[
C(Y) \subseteq \mathcal{B} \setminus \{(a_i, a_j) \mid 1 \leq i < j \leq 5 \}
\]
So \( n \leq |\mathcal{B}| - 10 \). This is a contradiction.

2) If \( C(u_1) \cap C(u_2) \cap \cdots \cap C(u_7) \cap \{3, 4, 5, \ldots, k-1\} \) contains \( k - 7 \) colors, there exist four colors \( a, b, c, d \in \{3, 4, 5, \ldots, k-1\} \), such that
\[
\{3, 4, 5, \ldots, k-1\} \setminus \{a, b, c, d\} \subseteq C(u_1) \cap C(u_2) \cap \cdots \cap C(u_7) \cap \{3, 4, 5, \ldots, k-1\}.
\]
By Lemma 2, any 2-subsets of \( \{a, b, c, d\} \) is not in \( C(Y) \). Thus
\[
C(Y) \subseteq \mathcal{B} \setminus \{(a, b), (a, c), (a, d), (b, c), (b, d), (c, d)\}.
\]
So \( n \leq |\mathcal{B}| - 6 \) and \( n = |\mathcal{B}| - 6 \) or \( n = |\mathcal{B}| - 7 \). Consider the following 12 subsets:
\[
\{1, a, b\}, \{1, a, c\}, \{1, a, d\}, \{1, b, c\}, \{1, b, d\}, \{1, c, d\}.
\]
Therefore at least 11 subsets (possibly except for one, say \( 2, c, d \)) are in \( C(Y) \). From \( \{1, a, b\}, \{1, a, c\}, \{1, a, d\}, \{1, b, c\}, \{1, b, d\}, \{1, c, d\} \) in \( C(Y) \), by Lemma 3, we know that there exist three colors in \( \{a, b, c\} \) which are contained in each \( C(u_i) \) for color 1 vertex \( u_i \), say \( a, b, c \in C(u_i) \) for color 1 vertex \( u_i \). So there are two choices for color 1 vertex \( u_i \); \( C(u_i) = \{d\} \), or \( k-1 \). From \( \{2, a, b\}, \{2, a, c\}, \{2, a, d\}, \{2, b, c\}, \{2, b, d\}, \{2, c, d\} \) in \( C(Y) \), by Lemma 3, we know that there exist two colors in \( \{a, b, c\} \) which are contained in each \( C(u_i) \) for color 2 vertex \( u_i \), say \( a, b \in C(u_i) \) for color 2 vertex \( u_i \). So there are four choices for color 2 vertex \( u_i \), \( C(u_i) \) in \( \{k-1 \\setminus \{c\}, \{c\}, \{d\}, \{d\}, \{k-1\}\} \). This is a contradiction.

3) If \( C(u_1) \cap C(u_2) \cap \cdots \cap C(u_7) \cap \{3, 4, 5, \ldots, k-1\} \) contains \( k - 6 \) colors, there exist three colors \( a, b, c \in \{3, 4, 5, \ldots, k-1\} \), such that
\[
\{3, 4, 5, \ldots, k-1\} \setminus \{a, b, c\} \subseteq C(u_1) \cap C(u_2) \cap \cdots \cap C(u_7) \cap \{3, 4, 5, \ldots, k-1\}.
\]
Then \( \{3, 4, 5, \ldots, k-1\} \setminus \{a, b, c\} \subseteq C(u_i), i = 1, 2, \ldots, 7 \).

By Lemma 2, we have the following Claim 1.

Claim 1 any 2-subsets of \( \{a, b, c\} \) is not in \( C(Y) \).

Claim 2 \( \{a, b, c\} \not\in C(Y) \).

Otherwise if \( \{a, b, c\} \in C(Y) \), and \( C(v_{m+1}) = \{a, b, c\}, g(v_{m+1}) = a \), then each \( C(u_i) \) contains \( b \) or \( c \). So \( C(X) \subseteq \{k-1\} \{a, b\}, \{k-1\} \{a, c\}, \{a\}, \{b\}, \{c\}, \{k-1\}\}. This is a contradiction.
4-subsets of $[11]$ which contains 11, the 3-subsets of $[11]$ which contains 11, and $[5, 11], [6, 11], [7, 11], [8, 11], [9, 11], [10, 11]$. And then color vertex $v_j$ $(980 \leq j \leq 1942)$ and its incident edges according to the manner listed in Table 2 where let $k = 11$. In such a way we have obtained an 11-VDET coloring $h_{1942}$ of $K_{7,1942}$. The restriction of 11-VDET coloring $h_{1942}$ of $K_{7,1942}$ on its subgraph induced by \( \{v_1, v_2, \ldots , v_7, v_{11}, v_{12}, \ldots , v_{1942} \} \) is obviously an 11-VDET coloring $h_j$ when $978 \leq j \leq 1941$.

For $k = 12, 13, 14, \ldots$, we will execute the following algorithm recursively and then give a $k$-VDET coloring of $K_{k,n}$ when $k \geq 12$.

Let $s = \sum_{i=2}^{k-1} \binom{k-1}{i} - 2k - 3, t = \sum_{i=2}^{k} \binom{k}{i} - 2k - 5$. Note that $s$ and $t$ depend on $k$. Assume that $(k - 1)$-VDET coloring $h_k$ of $K_{k,s}$ has been constructed according to the method given in this proof. We can create all $2k$-subsets, 3-subsets, 4-subsets, 5-subsets, 6-subsets, 7-subsets, 8-subsets of $\{1, 2, \ldots , k \}$ which contain $k$, except for $\{1, k \}$ and $\{2, k \}$, into a sequence $S_k$. Then $S_k$ has $\binom{k-1}{i} - 2$ terms. Let the terms in $S_k$ be corresponded to vertices $v_{s+1}, v_{s+2}, \ldots , v_t$. Then the subgraph of $K_{7,n}$ induced by $X \cup \{ v_1, v_2, \ldots , v_{11} \}$ be colored using the $(k - 1)$-VDET coloring $h_k$ given in this proof, and then color each vertex $v_j$ $(s+1 \leq j \leq t)$ and its incident edges in the manner listed in Table 5 of reference [9]. The $k$-VDET coloring $h_k$ of $K_{k,t}$ has been constructed.

The restriction of $k$-VDET coloring $h_k$ of $K_{k,t}$ on its subgraph induced by $\{v_1, v_2, \ldots , v_{11}, v_{12}, \ldots , v_{978} \}$ is obviously an $k$-VDET coloring $h_j$, where $s+1 \leq j \leq t$.

The proof of Theorem 1 is completed.

IV. CONCLUSION

By simple computation, we may give the value of $\eta(K_{7,n})$ (see Table II)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\eta(K_{7,n})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[7, 19]</td>
<td>5</td>
</tr>
<tr>
<td>[20, 50]</td>
<td>6</td>
</tr>
<tr>
<td>[51, 113]</td>
<td>7</td>
</tr>
<tr>
<td>[114, 240]</td>
<td>8</td>
</tr>
<tr>
<td>[241, 495]</td>
<td>9</td>
</tr>
<tr>
<td>[496, 1002]</td>
<td>10</td>
</tr>
</tbody>
</table>

| $\sum_{i=2}^{l} \binom{l}{i} + 1, \sum_{i=2}^{l} \binom{l}{i}, l \geq 11$ | $l$ |

ACKNOWLEDGMENT

The authors would like to thank the editor and anonymous reviewers for their valuable suggestions.

REFERENCES