On $E$-Orlicz Theory

Abdulhameed Qahtan Abbood Altai and Nada Mohammed Abbas Alsafar

Abstract—In this paper, based on concepts of $E$-convex sets, $E$-convex functions and $E$-continuous, we establish the $E$-Orlicz theory which is a generalization to the Orlicz theory by relaxing the concepts of $N$-function, Young function, strong Young function and Orlicz function. In this theory, we introduce the definitions of $E$-Orlicz spaces, weak $E$-Orlicz spaces, $E$-Orlicz-Sobolev spaces, weak $E$-Orlicz-Sobolev spaces, $E$-Orlicz-Morrey spaces and weak $E$-Orlicz-Morrey spaces, $E$-Orlicz-Lorentz spaces and weak $E$-Orlicz-Lorentz spaces. However, we consider their implicit properties based on the effect of the operator $E$.

Index Terms—$E$-$N$-function, $E$-Young function, $E$-strong Young function, $E$-Orlicz function, $E$-Orlicz spaces, $E$-Orlicz-Sobolev space, $E$-Orlicz-Morrey Space, $E$-Orlicz-Lorentz Spaces.

I. INTRODUCTION

BIRNBAUM and Orlicz introduced the Orlicz spaces in 1931 as a generalization of the classical Lebesgue spaces, where the function $u^p$ is replaced by a more general convex function $\Phi$ [2]. The concept of $E$-convex sets and $E$-convex functions were introduced by Youness to generalize the classical concepts of convex sets and convex functions to extend the studying of the optimality for non-linear programming problems in 1999 [3]. Chen defined the semi-$E$-convex functions and studied its basic properties in 2002 [3]. The concepts of pseudo $E$-convex functions and $E$-quasiconvex functions and strictly $E$-quasiconvex functions were introduced by Syau and Lee in 2004 [4]. The concept of Semi strongly $E$-convex functions was introduced by Youness and Tarek Emam in 2005 [5]. Sheiba Grace and Thangavelu considered the algebraic properties of $E$-convex sets in 2009 [6]. $E$-differentiable convex functions was defined by Meghed, Gomma, Youness and El-Banna to transform a non-differentiable function to a differentiable function in 2013 [7]. Semi-$E$-convex function was introduced by Ayache and Khaled in 2015 [1].

The purpose behind this paper is to define the $E$-$N$-functions, $E$-Young functions, $E$-strong Young functions and $E$-Orlicz functions using the concepts of $E$-convex sets, $E$-convex functions and $E$-continuous functions to generalize and extend the studying of the classical Orlicz theory via defining a new class of Orlicz spaces equipped by the luxemburg norms and generated by non-Young functions but $E$-Young functions with a map $E$, like $E$-Orlicz spaces, weak $E$-Orlicz spaces, $E$-Orlicz-Sobolev spaces, weak $E$-Orlicz-Sobolev spaces, $E$-Orlicz-Morrey space, weak $E$-Orlicz-Morrey space, $E$-Orlicz-Lorentz spaces and weak $E$-Orlicz-Lorentz spaces.

II. PRELIMINARIES

The setting for this paper is $n$-dimensional Euclidean space $\mathbb{R}^n, n \geq 1$. Let $\Omega$ be a nonempty subset of $\mathbb{R}^n$ and $(\Omega, \mathcal{A}, \mu)$ be a measure space. A set $\Omega$ is said to be $E$-convex iff there is a map $E: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\lambda E(x) + (1 - \lambda)E(y) \in \Omega$, for each $x, y \in \Omega$, $0 \leq \lambda \leq 1$. A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be $E$-convex on a set $\Omega$ iff there is a map $E: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\Omega$ is an $E$-convex set and

$$f(\lambda E(x) + (1 - \lambda)E(y)) \leq \lambda f(E(x)) + (1 - \lambda)f(E(y)),$$

for each $x, y \in M$ and $0 \leq \lambda \leq 1$. And $f$ is called $E$-concave on a set $\Omega$ if

$$f(\lambda E(x) + (1 - \lambda)E(y)) \geq \lambda f(E(x)) + (1 - \lambda)f(E(y)),$$

for each $x, y \in \Omega$ and $0 \leq \lambda \leq 1$ (see [6]). A function $f: \Omega \rightarrow \mathbb{R}^n$ is said to be $E$-continuous at $a \in \Omega$ iff there is a map $E: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ implies

$$\|f(E(x)) - f(E(a))\| < \varepsilon$$

whenever

$$\|x - a\| < \delta$$

and $f$ is said to be $E$-continuous on $\Omega$ iff $f$ is $E$-continuous at every $x \in \Omega$.

Definition 1. A function $\Phi: \Omega \times [0, \infty) \rightarrow \mathbb{R}$ is called an $E$-$N$-function if there exists a map $E: \Omega \times [0, \infty) \rightarrow \Omega \times [0, \infty)$ such that for $\mu$-a.e. $t \in \Omega$, $[0, \infty)$ is an $E$-convex and $\Phi$ is an $E$-even, $E$-continuous, $E$-convex of $u$ on $[0, \infty)$, $\Phi(E(\mu, u)) > 0$ for any $u \in (0, \infty)$,

$$\lim_{u \rightarrow 0^+} \frac{\Phi(E(t, u))}{u} = 0, \lim_{u \rightarrow \infty} \frac{\Phi(E(t, u))}{u} = \infty$$

and for each $u \in [0, \infty)$, $\Phi(E(t, u))$ is an $\mu$-measurable function of $t$ on $\Omega$.
Remark 2. Every $N$-function is an $E$-$N$-function if the map $E$ is taken as the identity map. But not every $E$-$N$-function is an $N$-function.

Examples 3. We cite examples of $E$-$N$-function which is not $N$-function
i. Let $\Phi: \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be defined by $\Phi(t, u) = tu^2$ and let $E: \mathbb{R} \times [0, \infty) \to \mathbb{R} \times [0, \infty)$ be defined as $E(t, u) = (\lvert t \rvert, u)$. Then $\Phi$ is an $E$-$N$-function but it is not an $N$-function because, for $\mu$-a.e. $t \in \mathbb{R}$, $\Phi(t, u)$ is concave of $u$ for $t \in (-\infty, 0)$.

ii. Let $\Phi: \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be defined by $\Phi(t, u) = (1 - t)u^2 + t \exp(u)$ and let $E: \mathbb{R} \times [0, \infty) \to \mathbb{R} \times [0, \infty)$ be defined by $E(t, u) = (t, \ln u^2)$. Then, $\Phi$ is an $E$-$N$-function but it is not an $N$-function since, for $\mu$-a.e. $t \in \mathbb{R}, \Phi(t, u)$ is not even.

Definition 4. A function $\Phi: \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is called an $E$-Young function if there exists a map $E: \mathbb{R} \times [0, \infty) \to \mathbb{R} \times [0, \infty]$ such that for $\mu$-a.e. $t \in \mathbb{R}, [0, \infty)$ is an $E$-convex and $\Phi$ is an $E$-convex of $u$ on $[0, \infty)$,

$$\lim_{u \to 0^+} \Phi(E(t, u)) = 0,$$

$$\lim_{u \to 0^-} \Phi(E(t, u)) = \infty$$

and for each $u \in [0, \infty), \Phi(E(t, u))$ is a $\mu$-measurable function of $t$ on $\mathbb{R}$.

Remark 4. Every Young function is an $E$-Young function if the map $E$ is taken as the identity map. But not every $E$-Young function is a Young function.

Examples 5. We cite examples of $E$-Young function which is not Young function
i. Let $\Phi: \mathbb{R} \to \mathbb{R}$ be defined by $\Phi(t, u) = e^{tu} - 1$ and let $E: \mathbb{R} \times [0, \infty) \to \mathbb{R} \times [0, \infty)$ be defined by $E(t, u) = (u, u)$. Then, $\Phi$ is an $E$-Young function but it is not a Young function because for $\mu$-a.e. $t \in \mathbb{R}$, $\Phi(t, 0) = e^t - 1 \neq 0$.

ii. Let $\Phi: \mathbb{C} \times [0, \infty) \to \mathbb{R}$ be defined by

$$\Phi(t, u) = \begin{cases} \ln(u), & u > 1 \\ 0, & 0 \leq u \leq 1 \\ -(\lvert t \rvert, u). & \end{cases}$$

and let $E: \mathbb{C} \times [0, \infty) \to \mathbb{C} \times [0, \infty)$ be defined by $E(t, u) = (\lvert t \rvert, u)$. So, $\Phi$ is an $E$-Young function but it is not a Young function because, for $\mu$-a.e. $t \in \mathbb{C}$, $\Phi(t, u)$ is not convex because for $t \in (0, \infty), \frac{\partial^2 \Phi}{\partial u^2} = -\frac{t}{u^2} < 0$.

Definition 6. A function $\Phi: \Omega \times [0, \infty) \to \mathbb{R}$ is called an $E$-strong Young function if there exists a map $E: \Omega \times [0, \infty) \to \Omega \times [0, \infty]$ such that for $\mu$-a.e. $t \in \Omega, [0, \infty)$ is an $E$-convex and $\Phi$ is an $E$-convex $E$-continuous of $u$ on $[0, \infty)$,

$$\lim_{u \to 0^+} \Phi(E(t, u)) = 0,$$

$$\lim_{u \to 0^-} \Phi(E(t, u)) = \infty$$

and for each $u \in [0, \infty), \Phi(E(t, u))$ is a $\mu$-measurable function of $t$ on $\Omega$.

Remark 7. Every strong Young function is an $E$-strong Young function if the map $E$ is taken as the identity map. But not every $E$-strong Young function is a strong Young function.

Example 8. We cite examples of $E$-strong Young function which is not strong Young function
i. Let $\Phi: \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be defined by $\Phi(t, u) = e^{tu} - 1$ and let $E: \mathbb{R} \times [0, \infty) \to \mathbb{R} \times [0, \infty)$ be defined by $E(t, u) = (\lvert t \rvert, u)$. Then $\Phi$ is an $E$-strong Young function but it is not a strong Young function, where $\Phi(t, u) = e^{tu} - 1$ is not convex because for $t \in (-\infty, 0), u^t$ is not convex.

ii. Let $\Phi: [0, \infty) \times [0, \infty) \to \mathbb{R}$ be defined by $\Phi(t, u) = \cosh(tu^{p}) - 1$ and let $E: [0, \infty) \times [0, \infty) \to [0, \infty) \times [0, \infty)$ be defined by $E(t, u) = (u, 0)$. Then $\Phi$ is an $E$-strong Young function but it is not a strong Young function since for $\mu$-a.e. $t \in [0, \infty), \Phi(t, 0) = \cosh(t) - 1 \neq 0$.

Definition 9. A function $\Phi: \Omega \times [0, \infty) \to \mathbb{R}$ is called an $E$-Orlicz function if there exists a map $E: \Omega \times [0, \infty) \to \Omega \times [0, \infty]$ such that for $\mu$-a.e. $t \in \Omega, [0, \infty)$ is an $E$-convex and $\Phi$ is an $E$-convex of $u$ on $[0, \infty)$,

$$\lim_{u \to 0^+} \Phi(E(t, u)) = 0,$$

$$\lim_{u \to 0^-} \Phi(E(t, u)) = \infty$$

and for each $u \in [0, \infty), \Phi(E(t, u))$ is a $\mu$-measurable function of $t$ on $\Omega$.

Remark 10. Every Orlicz function is an $E$-Orlicz function if the map $E$ is taken as the identity map. But not every $E$-Orlicz function is an Orlicz function.

Examples 11. We cite examples of $E$-Orlicz function which is not Orlicz function
i. Let $\Phi: \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be defined by $\Phi(t, u) = -t + u$ and let $E: \mathbb{R} \times [0, \infty) \to \mathbb{R} \times [0, \infty)$ be defined by $E(t, u) = (0, u^p), p \geq 1$. Then $\Phi$ is an $E$-Orlicz function but it is not an Orlicz function because for $\mu$-a.e. $t \in \mathbb{R}, \Phi(t, 0) = -t \neq 0$.

ii. Let $\Phi: \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be defined by $\Phi(t, u) = t + u^{\frac{p}{1-t^p}}, p \geq 1$ and let $E: \mathbb{R} \times [0, \infty) \to \mathbb{R} \times [0, \infty)$ be defined by $E(t, u) = (u, 0)$. Then $\Phi$ is an $E$-Orlicz function but it is not an Orlicz function because for $\mu$-a.e. $t \in \mathbb{R}, \Phi(t, 0) = t \neq 0$.

III. ELEMENTARY PROPERTIES

A. Properties of $E$-$N$-Functions

Theorem 12. Let $\Phi_1, \Phi_2: \Omega \times [0, \infty) \to \mathbb{R}$ be $E$-$N$-functions with respect to $E: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi_1 + \Phi_2$ and $c\Phi_1, c \geq 0$ are $E$-$N$-functions with respect to $E$.

Theorem 13. Let $\Phi: \Omega \times [0, \infty) \to \mathbb{R}$ be a linear $E$-$N$-function with respect to $E_1, E_2: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi$ is an $E$-$N$-function with respect to $E_1 + E_2$ and $cE_1, c \geq 0$. 
Theorem 14. Let $\Phi: \Omega \times [0, \infty) \to \mathbb{R}$ be a linear $E$-$N$-function with respect to $E_1, E_2: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi$ is an $E$-$N$-function with respect to $E_1 \circ E_2$ and $E_2 \circ E_1$.

Theorem 15. Let $\Phi_i: \Omega \times [0, \infty) \to \mathbb{R}$ for $i = 1, \ldots, n$ be $E$-$N$-functions with respect to $E: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi = \max_i \Phi_i$ is an $E$-$N$-function with respect to $E$.

Theorem 16. Let $\Phi: \Omega \times [0, \infty) \to \mathbb{R}$ be an $E$-$N$-function with respect to $E_1, E_2: \Omega \times [0, \infty) \to \Omega \times [0, \infty), i = 1, \ldots, n$. Then $\Phi$ is an $E$-$N$-function with respect to $E_M = \max_i E_i$ and $E_m = \min_i E_i$.

Theorem 17. Let $(\Phi_n)_{n \in \mathbb{N}}$ be a sequence of continuous $E$-$N$-functions defined on a compact set $\Omega \times [0, \infty)$ with respect to $E: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$ such that $(\Phi_n)_{n \in \mathbb{N}}$ converges uniformly to a continuous function $\Phi: \Omega \times [0, \infty) \to \mathbb{R}$ and $\Phi$ is continuous on $\Omega \times [0, \infty)$. Then $\Phi$ is an $E$-$N$-function with respect to $E_M = \max_i E_i$ and $E_m = \min_i E_i$.

Theorem 18. Let $\Phi$ be a continuous $E$-$N$-function defined on a compact set $\Omega \times [0, \infty)$ with respect to a sequence of maps $(E_n)_{n \in \mathbb{N}}, E_n: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$ such that $(E_n)_{n \in \mathbb{N}}$ converges uniformly to a map $E: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$ and $\Phi$ is continuous on $\Omega \times [0, \infty)$. Then $\Phi(E(t, u)) \to E(t, u)$ uniformly on $\Omega \times [0, \infty)$ and for $\mu$-$a.e.$ $t \in \Omega$,

$$\Phi(E(t, u)) = \lim_{n \to \infty} \Phi_n(E(t, u))$$

is even continuous convex of $u$ on $[0, \infty)$, $\Phi(E(t, u)) > 0$ for any $u \in (0, \infty)$,

$$\lim_{u \to 0^+} \frac{\Phi(E(t, u))}{u} = \lim_{n \to \infty} \frac{\Phi_n(E(t, u))}{u} = 0,$$

$$\lim_{u \to \infty} \frac{\Phi(E(t, u))}{u} = \lim_{n \to \infty} \frac{\Phi_n(E(t, u))}{u} = \infty$$

and for each $u \in [0, \infty), \Phi(E(t, u))$ is an $\mu$-measurable function of $t$ on $\Omega$.

Theorem 19. Let $\Phi$ be a continuous $E$-$N$-function defined on a compact set $\Omega \times [0, \infty)$ with respect to a sequence of maps $(E_n)_{n \in \mathbb{N}}, E_n: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$, such that $(\Phi_n)_{n \in \mathbb{N}}$ converges uniformly to a continuous function $\Phi: \Omega \times [0, \infty) \to \mathbb{R}$ and $(E_n)_{n \in \mathbb{N}}$ converges uniformly to a continuous map $E: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi$ is an $E$-$N$-function with respect to $E$.

Proof. Assume that $(\Phi_n)_{n \in \mathbb{N}}$ is a sequence of continuous $E$-$N$-functions with respect to a sequence of continuous maps $(E_n)_{n \in \mathbb{N}}$ such that $\Phi_n \to \Phi$ uniformly and $E_n \to E$ uniformly on a compact set $\Omega \times [0, \infty)$ and $\Phi$ and $E$ are continuous on $\Omega \times [0, \infty)$. So $\Phi_n(E_n) \to \Phi(E)$ uniformly on $\Omega \times [0, \infty)$ and for $\mu$-$a.e.$ $t \in \Omega$, that

$$\Phi(E(t, u)) = \lim_{n \to \infty} \Phi_n(E_n(t, u))$$

is even continuous convex of $u$ on $[0, \infty)$, $\Phi(E(t, u)) > 0$, $u \in (0, \infty)$,

$$\lim_{u \to 0^+} \frac{\Phi(E(t, u))}{u} = \lim_{n \to \infty} \frac{\Phi_n(E_n(t, u))}{u} = 0,$$

$$\lim_{u \to \infty} \frac{\Phi(E(t, u))}{u} = \lim_{n \to \infty} \frac{\Phi_n(E_n(t, u))}{u} = \infty$$

and for each $u \in [0, \infty), \Phi(E(t, u))$ is an $\mu$-measurable function of $t$ on $\Omega$.

B. Properties of $E$-Young Functions

Theorem 20. Let $\Phi_1, \Phi_2: \Omega \times [0, \infty) \to \mathbb{R}$ be $E$-Young functions with respect to $E: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi_1 + \Phi_2$ and $c\Phi_1, c \geq 0$ are $E$-Young functions with respect to $E$.

Theorem 21. Let $\Phi: \Omega \times [0, \infty) \to \mathbb{R}$ be a linear $E$-Young function with respect to $E_1, E_2: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi$ is an $E$-Young function with respect to $E_1 + E_2$.

Theorem 22. Let $\Phi_1, \Phi_2: \Omega \times [0, \infty) \to \mathbb{R}$ be $E$-Young functions with respect to $E_1, E_2: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi$ is an $E$-Young function with respect to $E_1 + E_2$.

Theorem 23. Let $\Phi_1, \Phi_2: \Omega \times [0, \infty) \to \mathbb{R}$ be $E$-Young functions with respect to $E_1, E_2: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi$ is an $E$-Young function with respect to $E_1 + E_2$.

Theorem 24. Let $\Phi_1, \Phi_2: \Omega \times [0, \infty) \to \mathbb{R}$ be $E$-Young functions with respect to $E_1, E_2: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi$ is an $E$-Young function with respect to $E_1 + E_2$.

Theorem 25. Let $\Phi_1, \Phi_2: \Omega \times [0, \infty) \to \mathbb{R}$ be $E$-Young functions with respect to $E_1, E_2: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi$ is an $E$-Young function with respect to $E_1 + E_2$. 


Theorem 26. Let $\Phi: \Omega \times [0, \infty) \to \mathbb{R}$ be a continuous $E$-Young function defined on a compact set $\Omega \times [0, \infty)$ with respect to a sequence of maps $(E_n)_{n \in \mathbb{N}}, E_n: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$, such that $(E_n)_{n \in \mathbb{N}}$ converges uniformly to a map $E: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi$ is an $E$-Young function with respect to $E$.

Proof. Suppose that $\Phi$ is a continuous $E$-Young function with respect to a sequence of maps $(E_n)_{n \in \mathbb{N}}$ such that $E_n \to E$ uniformly on a compact set $\Omega \times [0, \infty)$. Then $\Phi(E_n) \to \Phi(E)$ uniformly on $\Omega \times [0, \infty)$ and for $\mu$-a.e. $t \in \Omega$, $\Phi(E(t, u)) = \lim_{n \to \infty} \Phi(E_n(t, u))$ is convex of $u$ on $[0, \infty)$.

Theorem 27. Let $(\Phi_n)_{n \in \mathbb{N}}$ be a sequence of continuous $E$-Young functions defined on a compact set $\Omega \times [0, \infty)$ with respect to a sequence of continuous maps $(E_n)_{n \in \mathbb{N}}, E_n: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$, such that $(\Phi_n)_{n \in \mathbb{N}}$ converges uniformly to a continuous function $\Phi: \Omega \times [0, \infty) \to \mathbb{R}$ and $(E_n)_{n \in \mathbb{N}}$ converges uniformly to a continuous map $E: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi$ is an $E$-Young function with respect to $E$.

Proof. Assume that $(\Phi_n)_{n \in \mathbb{N}}$ is a sequence of continuous $E$-Young functions with respect to a sequence of maps $(E_n)_{n \in \mathbb{N}}$ such that $\Phi_n \to \Phi$ and $E_n \to E$ uniformly on a compact set $\Omega \times [0, \infty)$ and $\Phi$ and $E$ are continuous on $\Omega \times [0, \infty)$. Then $\Phi_n(E_n) \to \Phi(E)$ uniformly on $\Omega \times [0, \infty)$ and for $\mu$-a.e. $t \in \Omega$, $\Phi(E(t, u)) = \lim_{n \to \infty} \Phi_n(E_n(t, u))$ is convex of $u$ on $[0, \infty)$.

C. Properties of $E$-Young Functions

Theorem 28. Let $\Phi_1, \Phi_2: \Omega \times [0, \infty) \to \mathbb{R}$ be $E$-strong Young functions with respect to $E: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi_1 + \Phi_2 + c \Phi_1, c \geq 0$ are $E$-strong Young functions with respect to $E$.

Theorem 29. Let $\Phi: \Omega \times [0, \infty) \to \mathbb{R}$ be a linear $E$-strong Young function with respect to $E_1, E_2: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi$ is an $E$-strong Young function with respect to $E_1 + E_2$ and $c E_1, c \geq 0$.
[0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi$ is an $E$-strong Young function with respect to $E$.

**Proof.** Assume that $(\Phi_n)_{n \in \mathbb{N}}$ is a sequence of continuous $E$-strong Young functions with respect to a sequence of continuous maps $(E_n)_{n \in \mathbb{N}}$ such that $\Phi_n \to \Phi$ and $E_n \to E$ uniformly on a compact set $\Omega \times [0, \infty)$ and $\Phi$ and $E$ are continuous on $\Omega \times [0, \infty)$. So, $\Phi(E_n) \to \Phi(E)$ uniformly on $\Omega \times [0, \infty)$ and for $\mu$-a.e. $t \in \Omega$,

$$
\Phi(E(t, u)) = \lim_{n \to \infty} \Phi_n(E_n(t, u))
$$

is convex continuous of $u$ on $[0, \infty)$,

$$
\Phi(E(t, 0)) = \lim_{n \to \infty} \Phi_n(E_n(t, 0)) = 0 \iff u = 0,
$$

$$
\lim_{u \to 0^+} \Phi(E(t, u)) = \lim_{n \to \infty} \lim_{u \to 0^+} \Phi_n(E_n(t, u)) = \infty,
$$

and for each $u \in [0, \infty)$, $\Phi(E(t, u))$ is a $\mu$-measurable function of $t$ on $\Omega$.

**D. Properties of $E$-Orlicz Functions**

**Theorem 36.** Let $\Phi_1, \Phi_2: \Omega \times [0, \infty) \to \mathbb{R}$ be $E$-Orlicz functions with respect to $E_{1, 2}: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi_1 \circ \Phi_2$ and $c \Phi_1, c \geq 0$ are $E$-Orlicz functions with respect to $E$.

**Theorem 37.** Let $\Phi: \Omega \times [0, \infty) \to \mathbb{R}$ be a linear $E$-Orlicz function with respect to $E_{1, 2}: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi$ is an $E$-Orlicz function with respect to $E_{1, 2} \circ E_1$ and $E_{2, 1} \circ E_1$.

**Theorem 38.** Let $\Phi: \Omega \times [0, \infty) \to \mathbb{R}$ be an $E$-Orlicz function with respect to $E_{1, 2}: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi$ is an $E$-Orlicz function with respect to $E_{1, 2} \circ E_1$ and $c E_1, c \geq 0$.

**Theorem 39.** Let $\Phi_i: \Omega \times [0, \infty) \to \mathbb{R}, i = 1, \ldots, n$ be $E$-Orlicz functions with respect to $E_i: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi = \max\{\Phi_i\}$ is an $E$-Orlicz function with respect to $E$.

**Theorem 40.** Let $\Phi: \Omega \times [0, \infty) \to \mathbb{R}$ be an $E$-Orlicz function with respect to $E_i: \Omega \times [0, \infty) \to \Omega \times [0, \infty], i = 1, \ldots, n$. Then $\Phi$ is an $E$-Orlicz function with respect to $E_m = \max_i E_i$ and $E_m = \min_i E_i$.

**Theorem 41.** Let $(\Phi_n)_{n \in \mathbb{N}}$ be a sequence of continuous $E$-Orlicz functions with respect to $E: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$ such that $(\Phi_n)_{n \in \mathbb{N}}$ converges uniformly to a continuous function $\Phi: \Omega \times [0, \infty) \to \mathbb{R}$. Then $\Phi$ is an $E$-Orlicz function with respect to $E$.

**Proof.** Assume that $(\Phi_n)_{n \in \mathbb{N}}$ is a sequence of continuous $E$-Orlicz functions with respect to a map $E$ such that $\Phi_n \to \Phi$ uniformly on a compact set $\Omega \times [0, \infty)$ and $\Phi$ is continuous on $\Omega \times [0, \infty)$. Then $\Phi(E_n) \to \Phi(E)$ uniformly on $\Omega \times [0, \infty)$ and for $\mu$-a.e. $t \in \Omega$,

$$
\Phi(E(t, u)) = \lim_{n \to \infty} \Phi_n(E_n(t, u))
$$

is convex of $u$ on $[0, \infty)$,

$$
\Phi(E(t, 0)) = \lim_{n \to \infty} \Phi_n(E_n(t, 0)) = 0,
$$

$$
\lim_{u \to 0^+} \Phi(E(t, u)) = \lim_{n \to \infty} \lim_{u \to 0^+} \Phi_n(E_n(t, u)) = \infty,
$$

$0 < \Phi(E(t, u)) < \infty$ for any $u \in (0, \infty)$, $\Phi(E(t, u))$ is left continuous at $u = 0$ for every $u > 0: \Phi(E(t, u)) < +\infty$. 

and for each $u \in [0, \infty)$, $\Phi(E(t, u))$ is a $\mu$-measurable function of $t$ on $\Omega$.

**Theorem 42.** Let $\Phi: \Omega \times [0, \infty) \to \mathbb{R}$ be a continuous $E$-Orlicz function defined on a compact set $\Omega \times [0, \infty)$ with respect to a sequence of maps $(E_n)_{n \in \mathbb{N}}, E_n: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$ such that $(E_n)_{n \in \mathbb{N}}$ converges uniformly to a map $E: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi$ is an $E$-Orlicz function with respect to $E$.

**Proof.** Suppose that $\Phi$ is a continuous $E$-Orlicz function with respect to a sequence of continuous maps $(E_n)_{n \in \mathbb{N}}$ such that $E_n \to E$ uniformly on a compact set $\Omega \times [0, \infty)$ and $E$ is continuous on $\Omega \times [0, \infty)$. Then $\Phi(E_n) \to \Phi(E)$ uniformly on $\Omega \times [0, \infty)$ and for $\mu$-a.e. $t \in \Omega$,

$$
\Phi(E(t, u)) = \lim_{n \to \infty} \Phi(E_n(t, u))
$$

is convex of $u$ on $[0, \infty)$,

$$
\Phi(E(t, 0)) = \lim_{n \to \infty} \Phi(E_n(t, 0)) = 0,
$$

$$
\lim_{u \to 0^+} \Phi(E(t, u)) = \lim_{n \to \infty} \lim_{u \to 0^+} \Phi_n(E_n(t, u)) = \infty,
$$

$0 < \Phi(E(t, u)) < \infty$ for any $u \in (0, \infty)$ and $\Phi(E(t, u))$ is left continuous at $u = 0$ for every $u > 0: \Phi(E(t, u)) < +\infty$. 

and for each $u \in [0, \infty)$, $\Phi(E(t, u))$ is a $\mu$-measurable function of $t$ on $\Omega$.

**Theorem 43.** Let $(\Phi_n)_{n \in \mathbb{N}}$ be a sequence of continuous $E$-Orlicz functions defined on a compact set $\Omega \times [0, \infty)$ with respect to a sequence of continuous maps $(E_n)_{n \in \mathbb{N}}, E_n: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$ such that $(\Phi_n)_{n \in \mathbb{N}}$ converges uniformly to a continuous function $\Phi: \Omega \times [0, \infty) \to \mathbb{R}$ and $(E_n)_{n \in \mathbb{N}}$ converges uniformly to a continuous map $E: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then $\Phi$ is an $E$-Orlicz function with respect to $E$.

**Proof.** Assume that $(\Phi_n)_{n \in \mathbb{N}}$ is a sequence of continuous $E$-Orlicz functions with respect to a sequence of continuous maps $(\Phi_n)_{n \in \mathbb{N}}$ such that $\Phi_n \to \Phi$ and $E_n \to E$ uniformly on a compact set $\Omega \times [0, \infty)$ and $\Phi$ and $E$ are continuous on $\Omega \times [0, \infty)$. Then $\Phi_n(E_n) \to \Phi(E)$ uniformly on $\Omega \times [0, \infty)$ and for $\mu$-a.e. $t \in \Omega$,

$$
\Phi(E(t, u)) = \lim_{n \to \infty} \Phi_n(E_n(t, u))
$$

is convex of $u$ on $[0, \infty)$,

$$
\Phi(E(t, 0)) = \lim_{n \to \infty} \Phi_n(E_n(t, 0)) = 0,
$$

$$
\lim_{u \to 0^+} \Phi(E(t, u)) = \lim_{n \to \infty} \lim_{u \to 0^+} \Phi_n(E_n(t, u)) = \infty,
$$

$0 < \Phi(E(t, u)) < \infty$ for any $u \in (0, \infty)$, $\Phi(E(t, u))$ is left continuous at $u = 0$ for every $u > 0: \Phi(E(t, u)) < +\infty$. 

and for each $u \in [0, \infty)$, $\Phi(E(t, u))$ is a $\mu$-measurable function of $t$ on $\Omega$.

**IV. RELATIONSHIPS BETWEEN $E$-CONVEX FUNCTIONS**

In this section, we generalize the theorems in [9] to consider the relationships between $E$-convex functions, $E$-Young functions, $E$-strong Young functions and $E$-Orlicz functions.
Theorem 44. If $\Phi$ is an $E$-$N$-function, then $\Phi$ is an $E$-strong Young function.

Proof. Assume $\Phi: \Omega \times [0, \infty) \to \mathbb{R}$ is an $E$-$N$-function with a map $E: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. So, for $\mu$-a.e. $t \in \Omega$, $\Phi(E(t, u))$ is convex continuous of $u$ on $[0, \infty)$ satisfying

$$\forall \varepsilon > 0, \exists \delta > 0, 0 < u < \delta \implies \left| \frac{\Phi(E(t, u))}{u} - \right| < \varepsilon$$

because

$$\lim_{u \to 0^+} \frac{\Phi(E(t, u))}{u} = 0.$$ 

Letting $\delta < 1$, we get

$$0 \leq \Phi(E(t, u)) \leq \frac{\Phi(E(t, u))}{\delta} \leq \frac{\Phi(E(t, u))}{u} < \varepsilon.$$

By the squeeze theorem for functions, we get $\Phi(E(t, 0)) = 0$ if $u = 0$ because $\Phi$ is continuous at $u = 0$ and $\Phi(E(t, u)) > 0$ for any $u \in (0, \infty)$. Moreover,

$$\forall M \in \mathbb{R}, \exists u_M > 0, u > u_M \implies \Phi(E(t, u)) > M$$

because

$$\lim_{u \to \infty} \frac{\Phi(E(t, u))}{u} = \infty.$$ 

Taking $u_M > 1$, we have that

$$\Phi(E(t, u)) > Mu > M u_M > M.$$ 

That is,

$$\lim_{u \to \infty} \Phi(E(t, u)) = \infty.$$ 

Furthermore, for each $u \in [0, \infty), \Phi(E(t, u))$ is an $\mu$-measurable function of $t$ on $\Omega$ which completes the proof.

Remark 45. The converse of theorem 44 is not correct. That is, an $E$-strong Young function may not be an $E$-$N$-function. For example, let the function $\Phi: \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be defined as $\Phi(t, u) = e^{\sqrt{t}} - 1$ with the map $E: \mathbb{R} \times [0, \infty) \to \mathbb{R} \times [0, \infty)$ defined by $E(t, u) = (1, u)$. Then $\Phi$ is an $E$-strong Young function, but it is not an $E$-$N$-function because for $\mu$-a.e. $t \in \mathbb{R},$

$$\lim_{u \to 0^+} \frac{e^{\sqrt{u}} - 1}{u} = 1 \neq 0.$$ 

Theorem 46. If $\Phi$ is an $E$-strong Young function, then $\Phi$ is an $E$-Orlicz function.

Proof. Suppose that $\Phi: \Omega \times [0, \infty) \to \mathbb{R}$ is an $E$-strong Young function with a map $E: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$. Then for $\mu$-a.e. $t \in \Omega$, $\Phi(E(t, u))$ is convex continuous of $u$ on $[0, \infty)$ satisfying $\Phi(E(t, 0)) = 0$, $\Phi(E(t, u)) > 0$ for any $u \in (0, \infty)$ because $\Phi(E(t, 0)) = 0$ if $u = 0$ and

$$\lim_{u \to 0^+} \Phi(E(t, u)) = \infty$$

and $\Phi(E(t, u))$ is left continuous at $u_\Phi = +\infty$ because

$$\lim_{u \to \infty} \Phi(E(t, u)) = +\infty.$$ 

Moreover, for each $u \in [0, \infty), \Phi(E(t, u))$ is an $\mu$-measurable function of $t$ on $\Omega$. Hence, $\Phi$ is an $E$-Orlicz function.

Remark 47. The converse of theorem 46 is not correct. That is, not every $E$-strong Young function is an $E$-Orlicz function. For instance, let the function $\Phi: \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be defined as

$$\Phi(t, u) = \left\{ \begin{array}{ll} \log(u + [t]^1/2^1 + 1), 0 \leq u < 1 \\ +\infty, & 1 \leq u \end{array} \right.$$ 

with a map $E: [0, \infty) \times [0, \infty) \to [0, \infty) \times [0, \infty)$ defined by $E(t, u) = (u^p, u), p \geq 1$. Then $\Phi$ is an $E$-Young function but it is not an $E$-Orlicz function because $\Phi(E(t, u))$ is not left continuous at $u_\Phi = 1$, where

$$\lim_{u \to 1^+} \Phi(E(t, u)) = -\log(3) \neq +\infty = \Phi(E(t, 1)).$$ 

Corollary 50. $E$-$N$-function $\Rightarrow$ $E$-strong Young function $\Rightarrow$ $E$-Orlicz function $\Rightarrow$ $E$-Young function.
Corollary 51. E-N-function ≠ E-strong Young function ≠ E-Orlicz function ≠ E-Young function.

V. MAIN RESULTS

In this section, we are going to study a class of Orlicz spaces equipped by E-luxemburg norms and generated by E-Young functions and then we establish their inclusion properties.

Lemma 52. Let \( \Phi: \Omega \times [0, \infty) \to \mathbb{R} \) be an increasing E-Young function with respect to \( E_1, E_2: \Omega \times [0, \infty) \to \Omega \times [0, \infty) \) such that, for \( \mu \)-a.e. \( t \in \Omega \), \( E_1(t, x) \leq E_2(t, x) \). Then, for \( \mu \)-a.e. \( t \in \Omega \), \( \Phi(E_1(t, x)) \leq \Phi(E_2(t, x)) \).

Lemma 53. Let \( \Phi_1, \Phi_2: \Omega \times [0, \infty) \to \mathbb{R} \) be E-Young functions with respect to \( E: \Omega \times [0, \infty) \to \Omega \times [0, \infty) \) such that, for \( \mu \)-a.e. \( t \in \Omega \), \( \Phi_1(t, x) \leq \Phi_2(t, x) \). So, for \( \mu \)-a.e. \( t \in \Omega \), \( \Phi_1(E(t, x)) \leq \Phi_2(E(t, x)) \).

A. E-Orlicz Spaces and Weak E-Orlicz Spaces

Let \( \Phi: \Omega \times [0, \infty) \to \mathbb{R} \) be an E-Young function with respect to a map \( E: \Omega \times [0, \infty) \to \Omega \times [0, \infty) \). The E-Orlicz space generated by \( \Phi \) is defined by

\[
E_{\Phi(E)}(\Omega, \Sigma, \mu) = \{ f \in BS_{\Omega}: \| f \|_{\Phi(E)} < \infty \},
\]

and the weak E-Orlicz space generated by \( \Phi \) is

\[
E_{\Phi(E), weak}(\Omega, \Sigma, \mu) = \{ f \in BS_{\Omega}: \| f \|_{\Phi(E), weak} < \infty \},
\]

where \( BS_{\Omega} \) is the set of all \( \mu \)-measurable functions \( f \) from \( \Omega \) to \( BS \) such that \( BS, \| \cdot \|_{BS} \) is a Banach space and \( m(\Omega, f, u) = \mu(\Omega): \| f \|_{BS} \), where \( \mu(\Omega) = m(\Omega) \).

Example 54. We have seen from example 8-i that \( \Phi(t, u) = e^{e^u} - 1 \) is an E-Young function with respect to the map \( E(t, u) = (u, u) \). Then the E-Orlicz space and the weak E-Orlicz space generated by \( \Phi(E(t, u)) = e^{e^u} - 1 \) are equipped with the norm

\[
\| f \|_{\Phi(E)} = \inf \left\{ \lambda > 0: \int_\Omega \left( e^{\frac{2 \| f(t) \|_{BS}}{\lambda}} - 1 \right) \, d\mu \right\}
\]

for all \( f \in E_{\Phi(E)}(\Omega, \Sigma, \mu) \) and

\[
\| f \|_{\Phi(E), weak} = \inf \left\{ \lambda > 0: \sup_u (e^{e^u} - 1) \, m(\Omega, f, \lambda, u) \right\}
\]

for all \( f \in E_{\Phi(E), weak}(\Omega, \Sigma, \mu) \).

If \( \Phi_1(E(t, u)) = u^p, p \geq 1 \), we get

\[
E_{p}(\Omega, \Sigma, \mu) = E_{\Phi_1(E)(\Omega, \Sigma, \mu)} = \{ f \in X_\Omega: \| f \|_p < \infty \},
\]

\[
\| f \|_p = \inf \left\{ \lambda > 0: \int_\Omega \left( \frac{| f(t) |}{\lambda} \right)^p \, d\mu \right\}^{\frac{1}{p}}
\]

for all \( f \in E_{p}(\Omega, \Sigma, \mu) \) and

\[
E_{\Phi_1(E), weak}(\Omega, \Sigma, \mu) = E_{\Phi_1(E), weak}(\Omega, \Sigma, \mu) = \{ f \in X_\Omega: \| f \|_{\Phi_1(E), weak} < \infty \},
\]

\[
\| f \|_{\Phi_1(E), weak} = \inf \left\{ \lambda > 0: \sup_u u^p \, m(\Omega, f, \lambda, u) \right\}
\]

for all \( f \in E_{\Phi_1(E), weak}(\Omega, \Sigma, \mu) \).

Example 55. Let \( \Phi: \mathbb{C} \times [0, \infty) \to \mathbb{R} \) be defined as

\[
\Phi(t, u) = \begin{cases} \ln(u), & u > 1 \\ 0, & 0 \leq u \leq 1 \\ (1, e^{u^p}), & 1 \leq p, \end{cases}
\]

for \( E(t, u) = \begin{cases} (1,0), & 1 < u, p = +\infty, \\ (0,0), & 0 \leq u \leq 1, p = +\infty. \end{cases} \)

Then, for \( \mu \)-a.e. \( t \in \mathbb{C} \), that

\[
\Phi(E(t, u)) = \begin{cases} u^p, & 1 \leq p, \\ +\infty, & 0 \leq u \leq 1, p = +\infty, \end{cases}
\]

is an E-Young function and the obtained spaces are \( E_{\Phi}(\Omega, \Sigma, \mu) \) and \( E_{\Phi, weak}(\Omega, \Sigma, \mu) \) for \( 1 \leq p \leq \infty \).

Theorem 56. If \( \Phi: \Omega \times [0, \infty) \to \mathbb{R} \) is an increasing E-Young function with respect to \( E_1, E_2: \Omega \times [0, \infty) \to \Omega \times [0, \infty) \) such that, for \( \mu \)-a.e. \( t \in \Omega \), \( \Phi_1(E(t, x)) \leq \Phi_2(E(t, x)) \). Then

\[
E_{\Phi_1(E)}(\Omega, \Sigma, \mu) \subseteq E_{\Phi_2(E)}(\Omega, \Sigma, \mu)
\]

and \( E_{\Phi_1(E), weak}(\Omega, \Sigma, \mu) \subseteq E_{\Phi_2(E), weak}(\Omega, \Sigma, \mu) \).

Theorem 57. If \( \Phi_1, \Phi_2: \Omega \times [0, \infty) \to \mathbb{R} \) are E-Young functions with respect to \( E: \Omega \times [0, \infty) \to \Omega \times [0, \infty) \) such that, for \( \mu \)-a.e. \( t \in \Omega \), \( \Phi_1(E(t, x)) \leq \Phi_2(E(t, x)) \). Then

\[
E_{\Phi_2(E)}(\Omega, \Sigma, \mu) \subseteq E_{\Phi_1(E)}(\Omega, \Sigma, \mu)
\]

and \( E_{\Phi_2(E), weak}(\Omega, \Sigma, \mu) \subseteq E_{\Phi_1(E), weak}(\Omega, \Sigma, \mu) \).

Theorem 58. If \( \Phi: \Omega \times [0, \infty) \to \mathbb{R} \) is an increasing E-Young function with respect to \( E_1, E_2: \Omega \times [0, \infty) \to \Omega \times [0, \infty) \) such that, for \( \mu \)-a.e. \( t \in \Omega \), \( E_1(t, x) \leq E_2(t, x) \). Then

\[
E_{\Phi_2(E)}(\Omega, \Sigma, \mu) \subseteq E_{\Phi_1(E)}(\Omega, \Sigma, \mu)
\]

and \( E_{\Phi_2(E), weak}(\Omega, \Sigma, \mu) \subseteq E_{\Phi_1(E), weak}(\Omega, \Sigma, \mu) \).

Proof. Let \( f \in E_{\Phi_2(E)}(\Omega, \Sigma, \mu) \) and let \( \Phi \) be an increasing E-Young function. Then, by Lemma 52, we have

\[
\int_{\Omega} \left( \frac{\| f(t) \|_{BS}}{\lambda} \right)^p \, d\mu \leq \int_{\Omega} \Phi(E(t, u)) \, d\mu \leq \int_{\Omega} \Phi(E(t, u)) \, d\mu.
\]

Since \( u \) is arbitrary, we have

\[
\sup_u \Phi(E(t, u)) \leq \| f \|_{\Phi_1(E), weak} \leq 1
\]

and \( f \in E_{\Phi_1(E), weak}(\Omega, \Sigma, \mu) \) with \( \| f \|_{\Phi_1(E), weak} \leq \| f \|_{\Phi_2(E)} \).

Let \( f \in E_{\Phi_2(E), weak}(\Omega, \Sigma, \mu) \) and assume that \( \Omega \times [0, \infty) \) is compact. Then...
Theorem 59. If $\Phi_1, \Phi_2 : \Omega \times [0, \infty) \rightarrow \mathbb{R}$ are E-Young functions with respect to $E : \Omega \times [0, \infty) \rightarrow \Omega \times [0, \infty]$ such that, for $\mu$-a.e. $t \in \Omega$, $\Phi_1(E(t, x)) \leq \Phi_2(E(t, x))$. Then $E \Phi_2(\Omega, \Sigma, \mu) \subseteq E \Phi_2(\Omega, \Sigma, \mu)$.

Proof. Let $f \in E \Phi_2(\Omega, \Sigma, \mu)$. Then
\[
\Phi_1(E(t, u)) m(\Omega, f/\lambda, u) \leq \Phi_2(E(t, u)) m(\Omega, f/\lambda, u)
\]
for all $u$ and $f \in E \Phi_2(\Omega, \Sigma, \mu)$ with $\|f\|_{\Phi_1(\Omega, \Sigma, \mu)} \leq 1$.

B. E-Orlicz-Sobolev Space and Weak E-Orlicz-Sobolev Space

Let $\Phi : \Omega \times [0, \infty) \rightarrow \mathbb{R}$ be an E-Young function with respect to $E : \Omega \times [0, \infty) \rightarrow \Omega \times [0, \infty]$. The E-Orlicz-Sobolev space $E \Phi(E)(\Omega, \Sigma, \mu)$ generated by $\Phi(E)$ is

\[
E \Phi(E)(\Omega, \Sigma, \mu) = \left\{ f \in E \Phi(\Omega, \Sigma, \mu) : D^\alpha f \in E \Phi(\Omega, \Sigma, \mu), \forall |\alpha| \leq k \right\},
\]
for all $f \in E \Phi(E)(\Omega, \Sigma, \mu)$ and the weak E-Orlicz-Sobolev space is

\[
E \Phi(E)(\Omega, \Sigma, \mu) = \left\{ f \in E \Phi(\Omega, \Sigma, \mu) : D^\alpha f \in E \Phi(\Omega, \Sigma, \mu), \forall |\alpha| \leq k \right\},
\]
for all $f \in E \Phi(E)(\Omega, \Sigma, \mu)$. If $\Phi_\mu(E(t, u)) = u^\mu, \mu \geq 1$, we get the E-Sobolev space

\[
E \Phi(E)(\Omega, \Sigma, \mu) = E \Phi(\Omega, \Sigma, \mu) = \left\{ f \in E \Phi(\Omega, \Sigma, \mu) : D^\alpha f \in E \Phi(\Omega, \Sigma, \mu), \forall |\alpha| \leq k \right\},
\]
equipped with the norm

\[
\|f\|_{\Phi(E)} = \left( \sum_{|\alpha| \leq k} \|D^\alpha f\|_{\Phi(\Omega, \Sigma, \mu)} \right)^{1/\mu}
\]
for all $f \in E \Phi(E)(\Omega, \Sigma, \mu)$ and the weak E-Sobolev space

\[
E \Phi(E)(\Omega, \Sigma, \mu) = E \Phi(\Omega, \Sigma, \mu) = \left\{ f \in E \Phi(\Omega, \Sigma, \mu) : D^\alpha f \in E \Phi(\Omega, \Sigma, \mu), \forall |\alpha| \leq k \right\},
\]
equipped with the norm

\[
\|f\|_{\Phi(E)} = \left( \sum_{|\alpha| \leq k} \|D^\alpha f\|_{\Phi(\Omega, \Sigma, \mu)} \right)^{1/\mu}
\]
for all $f \in E \Phi(E)(\Omega, \Sigma, \mu)$ and the weak E-Sobolev space

\[
E \Phi(E)(\Omega, \Sigma, \mu) = E \Phi(\Omega, \Sigma, \mu) = \left\{ f \in E \Phi(\Omega, \Sigma, \mu) : D^\alpha f \in E \Phi(\Omega, \Sigma, \mu), \forall |\alpha| \leq k \right\},
\]
equipped with the norm

\[
\|f\|_{\Phi(E)} = \left( \sum_{|\alpha| \leq k} \|D^\alpha f\|_{\Phi(\Omega, \Sigma, \mu)} \right)^{1/\mu}
\]
for all $f \in E \Phi(E)(\Omega, \Sigma, \mu)$ and the weak E-Sobolev space

\[
E \Phi(E)(\Omega, \Sigma, \mu) = E \Phi(\Omega, \Sigma, \mu) = \left\{ f \in E \Phi(\Omega, \Sigma, \mu) : D^\alpha f \in E \Phi(\Omega, \Sigma, \mu), \forall |\alpha| \leq k \right\},
\]
equipped with the norm

\[
\|f\|_{\Phi(E)} = \left( \sum_{|\alpha| \leq k} \|D^\alpha f\|_{\Phi(\Omega, \Sigma, \mu)} \right)^{1/\mu}
\]
for all $f \in E \Phi(E)(\Omega, \Sigma, \mu)$ and the weak E-Sobolev space

\[
E \Phi(E)(\Omega, \Sigma, \mu) = E \Phi(\Omega, \Sigma, \mu) = \left\{ f \in E \Phi(\Omega, \Sigma, \mu) : D^\alpha f \in E \Phi(\Omega, \Sigma, \mu), \forall |\alpha| \leq k \right\},
\]
equipped with the norm

\[
\|f\|_{\Phi(E)} = \left( \sum_{|\alpha| \leq k} \|D^\alpha f\|_{\Phi(\Omega, \Sigma, \mu)} \right)^{1/\mu}
\]
for all $f \in E \Phi(E)(\Omega, \Sigma, \mu)$ and the weak E-Sobolev space

\[
E \Phi(E)(\Omega, \Sigma, \mu) = E \Phi(\Omega, \Sigma, \mu) = \left\{ f \in E \Phi(\Omega, \Sigma, \mu) : D^\alpha f \in E \Phi(\Omega, \Sigma, \mu), \forall |\alpha| \leq k \right\},
\]
equipped with the norm

\[
\|f\|_{\Phi(E)} = \left( \sum_{|\alpha| \leq k} \|D^\alpha f\|_{\Phi(\Omega, \Sigma, \mu)} \right)^{1/\mu}
\]
the weak \( E \)-Orlicz-Morrey space is

\[
E_{\Phi(E),\phi,weak}(\Omega,\Sigma,\mu) = \{ f \in X_\Omega; \|f\|_{\Phi(E),\phi,weak} < \infty \},
\]

and the weak \( E \)-Orlicz-Morrey space is

\[
E_{\Phi(E),\phi,weak}(\Omega,\Sigma,\mu) = \{ f \in X_\Omega; \|f\|_{\Phi(E),\phi,weak} < \infty \}.
\]

If \( \phi(E(t,u)) = u^p, p \geq 1 \), then

\[ E_{\Phi(E),\phi}(\Omega,\Sigma,\mu) = E_{\Phi(E),\phi}(\Omega,\Sigma,\mu) = \{ f \in X_\Omega; \|f\|_{\Phi(E),\phi,weak} < \infty \}, \]

\[
\|f\|_{\Phi(E),\phi} = \sup_{u>0} \left( \frac{1}{|B|} \int_B \frac{\Phi(E(t,u))}{|B|^p} \int_B \frac{d\mu}{|x|^p} \right)^{\frac{1}{p}}.
\]

Theorem 64. If \( \Phi; \Omega \times [0,\infty) \rightarrow \mathbb{R} \) is an increasing \( E \)-Young function with respect to \( E_1, E_2; \Omega \times [0,\infty) \rightarrow \Omega \times [0,\infty) \) such that, for \( \mu \)-a.e. \( t \in \Omega, \Phi_1(E(t,x)) \leq \Phi_2(E(t,x)) \). Then

\[
E_{\Phi(E_1),\phi}(\Omega,\Sigma,\mu) \subseteq E_{\Phi(E_2),\phi}(\Omega,\Sigma,\mu),
\]

and

\[
E_{\Phi(E_2),\phi,weak}(\Omega,\Sigma,\mu) \subseteq E_{\Phi(E_1),\phi,weak}(\Omega,\Sigma,\mu).
\]

Theorem 65. If \( \Phi_1, \Phi_2; \Omega \times [0,\infty) \rightarrow \mathbb{R} \) are \( E \)-Young functions with respect to \( E_1, E_2; \Omega \times [0,\infty) \rightarrow \Omega \times [0,\infty) \) such that, for \( \mu \)-a.e. \( t \in \Omega, \Phi_1(E(t,x)) \leq \Phi_2(E(t,x)) \). Then

\[
E_{\Phi(E_1),\phi}(\Omega,\Sigma,\mu) \subseteq E_{\Phi(E_2),\phi}(\Omega,\Sigma,\mu)
\]

and

\[
E_{\Phi(E_2),\phi,weak}(\Omega,\Sigma,\mu) \subseteq E_{\Phi(E_1),\phi,weak}(\Omega,\Sigma,\mu).
\]

Theorem 66. If \( \Phi; \Omega \times [0,\infty) \rightarrow \mathbb{R} \) is an increasing \( E \)-Young function with respect to \( E_1, E_2; \Omega \times [0,\infty) \rightarrow \Omega \times [0,\infty) \) such that, for \( \mu \)-a.e. \( t \in \Omega, E_1(t,x) \leq E_2(t,x) \). Then

\[
E_{\Phi(E_1),\phi}(\Omega,\Sigma,\mu) \subseteq E_{\Phi(E_2),\phi}(\Omega,\Sigma,\mu)
\]

and

\[
E_{\Phi(E_2),\phi,weak}(\Omega,\Sigma,\mu) \subseteq E_{\Phi(E_1),\phi,weak}(\Omega,\Sigma,\mu).
\]

Theorem 67. If \( \Phi_1, \Phi_2; \Omega \times [0,\infty) \rightarrow \mathbb{R} \) are \( E \)-Young functions with respect to \( E_1, E_2; \Omega \times [0,\infty) \rightarrow \Omega \times [0,\infty) \) such that, for \( \mu \)-a.e. \( t \in \Omega, \Phi_1(E(t,x)) \leq \Phi_2(E(t,x)) \). Then

\[
E_{\Phi(E_1),\phi}(\Omega,\Sigma,\mu) \subseteq E_{\Phi(E_2),\phi}(\Omega,\Sigma,\mu).
\]

D. \( E \)-Orlicz-Lorentz Spaces
Let $\Phi: [0, \infty) \times [0, \infty) \to \mathbb{R}$ be an $E$-convex function with respect to $E: [0, \infty) \times [0, \infty) \to [0, \infty)$ and let $\omega: [0, \infty) \to [0, \infty)$ be a weight function and $W(t) = \int_0^t \omega(s) ds$. The $\omega$-Orlicz-Lorentz space is:

$$L_{\omega, \Phi(E)} = \{ f \in X_{[0, \infty)} : \| f \|_{\omega, \Phi(E)} < \infty \},$$

and the weak $\omega$-Orlicz-Lorentz space is

$$L_{\omega, \Phi(E), \text{weak}} = \{ f \in X_{[0, \infty)} : \| f \|_{\omega, \Phi(E), \text{weak}} < \infty \},$$

for all $f \in L_{\omega, \Phi(E)}$. If $\omega(t) = 1$ for $t \in [0, \infty)$, then

$$L_{\omega, \Phi(E)}(\Omega, \Sigma, \mu) = E_{\Phi(E)}(\Omega, \Sigma, \mu),$$

$$L_{\omega, \Phi(E), \text{weak}}(\Omega, \Sigma, \mu) = E_{\Phi(E), \text{weak}}(\Omega, \Sigma, \mu).$$

And if $\omega(t) = 1$ for $t \in [0, \infty)$ and $\Phi(E(t, u)) = u^p$ for $1 \leq p < \infty$, we get the Lorentz space

$$L_{\omega, \Phi(E)}(\Omega, \Sigma, \mu) = E_{\omega, p}(\Omega, \Sigma, \mu),$$

and the weak Lorentz space

$$L_{\omega, \Phi(E), \text{weak}}(\Omega, \Sigma, \mu) = E_{\omega, p, \text{weak}}(\Omega, \Sigma, \mu).$$

Theorem 68. If $\Phi: \Omega \times [0, \infty) \to \mathbb{R}$ is an increasing $E$-Young function with respect to $E_1, E_2: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$ such that, for $\mu$-a.e. $t \in \Omega, E_1(t, x) \leq E_2(t, x)$. Then

$$L_{\omega, \Phi(E_2)}(\Omega, \Sigma, \mu) \subseteq L_{\omega, \Phi(E_1)}(\Omega, \Sigma, \mu)$$

and

$$L_{\omega, \Phi(E_2), \text{weak}}(\Omega, \Sigma, \mu) \subseteq L_{\omega, \Phi(E_1), \text{weak}}(\Omega, \Sigma, \mu).$$

Theorem 69. If $\Phi_1, \Phi_2: \Omega \times [0, \infty) \to \mathbb{R}$ are $E$-Young functions with respect to $E: \Omega \times [0, \infty) \to \Omega \times [0, \infty)$ such that, for $\mu$-a.e. $t \in \Omega, \Phi_1(E(t, x)) \leq \Phi_2(E(t, x))$. Then

$$L_{\omega, \Phi_1(E)}(\Omega, \Sigma, \mu) \subseteq L_{\omega, \Phi_2(E)}(\Omega, \Sigma, \mu)$$

and

$$L_{\omega, \Phi_1(E), \text{weak}}(\Omega, \Sigma, \mu) \subseteq L_{\omega, \Phi_2(E), \text{weak}}(\Omega, \Sigma, \mu).$$

VI. CONCLUSION

We have shown that the non $N$-functions, non Young functions, non strong Young functions and non Orlicz functions can be transferred using the $E$-convex theory to $E$-functions, $E$-Young functions, $E$-strong Young functions and $E$-Orlicz functions respectively. We also have shown that the Orlicz spaces can be generated by non-Young functions but $E$-Young functions with an appropriate map $E$ to extend and generalize studying the classical Orlicz theory. Moreover, we have considered the inclusion properties of $E$-Orlicz spaces based on effects of the map $E$.

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