Derivation of New Numerical Model Capable of Solving Second and Third Order Ordinary Differential Equations Directly

E. O. Adeyefa, and J. O. Kuboye

Abstract— Numerical methods are widely used for the numerical integration of initial value problems (IVPs) in ordinary differential equations (ODEs). Nevertheless, the block method is not normally used for the numerical integration of both p^{th} and $(p+1)^{th}$ order IVPs. This paper focuses on the formulation of a self-starting method capable of obtaining the numerical solution of second and third-order IVPs. The method is formulated from continuous schemes obtained via collocation and interpolation techniques and applied in a block-by-block manner as a numerical integrator for second and third-order ODEs. The convergence properties of this method are discussed via zero-stability and consistency. Numerical examples are included and comparisons are made with existing methods in the literature.

Keywords: Block Method, Convergence, Second and Third Order Ordinary Differential Equations, Zero-stability

I. INTRODUCTION

The demand for the solution of Differential Equations (DEs) is on the increase as the quest for numerical methods has increasingly been of much interest to researchers because most of these DEs are difficult to solve or their analytical solutions do not exist. Although it is possible to integrate p^{th} an initial value problem (IVP) using a numerical method but to use the same method to integrate two or more IVP $(p+1)^{th}$s of different has not been commonly reported. Thus, the focus of this paper is to develop a self-starting method for the numerical solution of the ODE of the form

$$\left.\begin{aligned} &y^{p}(x)= f(x,y,y^{'},\dots,y^{p-1}),\\ &y^{a}(x_0)=y_0^{a},\ a=0,1,\dots,k-1\end{aligned}\right\} \qquad (1a)$$

and

$$\left.\begin{aligned} &y^{(p+1)}(x)= f(x,y,y^{'},\dots,y^{p}),\\ &y^{b}(x_0)=y_0^{b},\ b=0,1,\dots,k-1\end{aligned}\right\} \qquad (1b)$$

The solution of (1a) and (1b) for p ranging from p = 1(1)3 has been extensively discussed by various researchers. Among them are [1]-[3] and [7]-[18].

The block method approach which simultaneously generates approximations at different grid points within the interval of integration without overlapping of sub-intervals has been reported to circumvent the setback commonly experience in reducing m>1 in (1a) to a system of first-order equations and the predictor-corrector approach, see [16]. Furthermore, this new method is superior to those mentioned above since it is equipped with handling both p^{th} and $(p+1)^{th}$ order IVPs.

The aim of developing new methods has always been to improve on the efficiency and convergence of existing methods with the ultimate aim of reducing the error of approximation. Thus, in what immediately follows in Section 2, we derive the proposed method for direct integration of p^{th} and $(p+1)^{th}$ order IVPs in ODEs where p = 2. The basic properties of the method are discussed in Section 3, numerical examples are given to show the efficiency of the methods in Section 4 and the discussion of results is given in Section 5. Finally, the conclusion of the paper is discussed in Section 6.

II. DERIVATION OF THE METHOD

This section examines the derivation of a new block method that can solve the second and third-order initial value problems of ODEs.

Let the power series

$$y(x)=\sum_{j=0}^{k+6} a_j x^j \qquad (2)$$

be considered as an approximate solution to second and third-order ODEs of the form

$$\left.\begin{aligned} &y''(x)= f(x,y(x),y'(x)),\\ &y(x_0)=y_0, y'(x_0)=y_0'\end{aligned}\right\} \qquad (3)$$

$$\left.\begin{aligned} &y'''(x)= f(x,y(x),y'(x),y''(x)),\quad y(x_0)=y_0,\\ &y'(x_0)=y_0',, y''(x_0)=y_0'''\end{aligned}\right\} \qquad (4)$$

Interpolating (2) at $x=x_{n+u}, u=0,1$, the second derivative of (2) is collocated at $x=x_{n+v}, v=0,\frac{1}{2},1$ and collocating the third derivative of (2) at $x=x_{n+w}, w=0,\frac{1}{2},1$.

Consequently, we have

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E. O. Adeyefa is with the Department of Mathematics, Federal University Oye-Ekiti, Ekiti State, Nigeria (+2348034150709; e-mail: adeoluman@yahoo.com).

J. O. Kuboye is with the Department of Mathematics, Federal University Oye-Ekiti, Ekiti State, Nigeria (john.kuboye@fuoye.edu.ng).

$$\left.\begin{aligned}
&\sum_{j=0}^{k+6} a_j x_{n+u}^{j} = y_{n+u} \\
&\sum_{j=1}^{k+5} j(j-1) a_j x_{n+v}^{j-2} = f_{n+v} \\
&\sum_{j=2}^{k+5} j(j-1)(j-2) a_j x_{n+w}^{j-3} = g_{n+w}
\end{aligned}\right\} \qquad (5)$$

In (5), Gaussian elimination is applied to find the unknown variables $a's$ which are then substituted to (2) to produce a continuous implicit scheme of the form:

$$\left.\begin{aligned}
&\alpha_0(t) y_n + \alpha_{\frac{1}{2}}(t) y_{n+\frac{1}{2}} = h^2 \sum_{j=0}^{k} \beta_j(t) f_{n+j} \\
&+ h^2 \beta_{\frac{1}{2}}(t) f_{n+\frac{1}{2}} + h^3 \left(\sum_{j=0}^{k} \delta_j(t) g_{n+j} + \delta_{\frac{1}{2}}(t) g_{n+\frac{1}{2}} \right)
\end{aligned}\right\} (6)$$

where $t = \dfrac{x - x_n}{h}$,

$$\begin{pmatrix} \alpha_0 \\ \alpha_{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} t^0 \\ t \end{pmatrix}$$

$$\begin{pmatrix} \beta_0 \\ \beta_{\frac{1}{2}} \\ \beta_1 \\ \delta_0 \\ \delta_{\frac{1}{2}} \\ \delta_1 \end{pmatrix} = \begin{pmatrix}
0 & -\frac{13}{84} & \frac{1}{2} & 0 & -\frac{23}{12} & \frac{33}{10} & -\frac{34}{15} & \frac{4}{7} \\
0 & -\frac{1}{12} & 0 & 0 & \frac{4}{3} & -\frac{8}{5} & \frac{8}{15} & 0 \\
0 & 0 & -\frac{1}{84} & 0 & \frac{7}{12} & -\frac{17}{10} & \frac{26}{15} & -\frac{4}{7} \\
0 & 0 & -\frac{59}{6720} & \frac{1}{6} & -\frac{1}{2} & \frac{13}{20} & -\frac{2}{5} & \frac{2}{21} \\
0 & \frac{2}{105} & 0 & 0 & -\frac{2}{3} & \frac{8}{5} & -\frac{4}{3} & \frac{8}{21} \\
0 & \frac{11}{6720} & 0 & 0 & -\frac{1}{12} & \frac{1}{4} & -\frac{4}{15} & \frac{2}{21}
\end{pmatrix} \begin{pmatrix} t^0 \\ t^1 \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \\ t^7 \end{pmatrix}$$

Equation (6) is differentiated once to give

$$\left.\begin{aligned}
&\alpha_0'(t) y_n + \alpha_{\frac{1}{2}}'(t) y_{n+\frac{1}{2}} = h \left(\sum_{j=0}^{k} \beta_j'(t) f_{n+j} + \beta_{\frac{1}{2}}'(t) f_{n+\frac{1}{2}} \right) \\
&+ h^2 \left(\sum_{j=0}^{k} \delta_j'(t) g_{n+j} + \delta_{\frac{1}{2}}'(t) g_{n+\frac{1}{2}} \right)
\end{aligned}\right\} (7)$$

where

$$\begin{pmatrix} \alpha_0' \\ \alpha_{\frac{1}{2}}' \end{pmatrix} = \begin{pmatrix} \frac{-2}{h} & 0 \\ \frac{2}{h} & 0 \end{pmatrix} \begin{pmatrix} t^0 \\ t \end{pmatrix},$$

$$\begin{pmatrix} \beta_0' \\ \beta_{\frac{1}{2}}' \\ \beta_1' \\ \delta_0' \\ \delta_{\frac{1}{2}}' \\ \delta_1' \end{pmatrix} = \begin{pmatrix}
-\frac{13}{84} & 1 & 0 & -\frac{23}{3} & \frac{33}{2} & -\frac{68}{5} & 4 \\
-\frac{1}{12} & 0 & 0 & \frac{16}{3} & -8 & \frac{16}{5} & 0 \\
\frac{1}{84} & 0 & 0 & -\frac{7}{3} & \frac{17}{2} & -\frac{52}{5} & 4 \\
-\frac{59}{6720} & 0 & \frac{1}{2} & -2 & \frac{13}{4} & -\frac{12}{5} & \frac{2}{3} \\
\frac{2}{105} & 0 & 0 & -\frac{8}{3} & 8 & -8 & \frac{8}{3} \\
\frac{11}{6720} & 0 & 0 & -\frac{1}{3} & \frac{5}{4} & -\frac{8}{5} & \frac{2}{3}
\end{pmatrix} \begin{pmatrix} t^0 \\ t^1 \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \end{pmatrix}$$

The discrete scheme and its derivatives in (8) are derived by evaluating (6) at $x = x_{n+1}(t=1)$ and (7) at $x = x_{n+i}, i = 0, \frac{1}{2}, 1 (t = 0, \frac{1}{2}, 1)$.

2.0

$$\left.\begin{pmatrix} y_{n+1} \\ y_n' \\ y_{n+\frac{1}{2}}' \\ y_{n+1}' \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ \frac{-2}{h} & \frac{2}{h} \\ \frac{-2}{h} & \frac{2}{h} \\ \frac{-2}{h} & \frac{2}{h} \end{pmatrix} \begin{pmatrix} y_n \\ y_{n+\frac{1}{2}} \end{pmatrix} + h^2 Q \begin{pmatrix} f_n \\ f_{n+\frac{1}{2}} \\ f_{n+1} \end{pmatrix} + h^3 R \begin{pmatrix} g_n \\ g_{n+\frac{1}{2}} \\ g_{n+1} \end{pmatrix}\right\} (8)$$

for

$$Q = \begin{pmatrix}
\frac{1}{30} & \frac{11}{60} & \frac{1}{30} \\
-\frac{13}{84h} & \frac{1}{12h} & \frac{1}{84h} \\
\frac{187}{3360h} & \frac{11}{60h} & \frac{37}{3360h} \\
\frac{11}{140h} & \frac{9}{20h} & \frac{31}{140h}
\end{pmatrix}$$

and

$$R = \begin{pmatrix}
\frac{1}{320} & 0 & -\frac{1}{320} \\
-\frac{59}{6720h} & \frac{2}{105h} & \frac{11}{6720h} \\
\frac{1}{210h} & -\frac{19}{840h} & -\frac{1}{672h} \\
\frac{53}{6720h} & \frac{2}{105h} & \frac{-101}{6720h}
\end{pmatrix}$$

To get the block, (8) can be rewritten in the form

$$\begin{aligned}
&A y_{N+1} = A^0 y_N + h B y_{N-1}' + h^2 [D^0 f_N + D^1 f_{N+1}] \\
&+ h^3 [E^0 g_N + E^1 g_{N+1}] \qquad (9)
\end{aligned}$$

3.0 where

$y_{N+1} = [y_{n+\frac{1}{2}}, y_{n+1}]^T$, $y_{N-1} = [y_{n-\frac{1}{2}}, y_n]^T$,

$y'_{N-1} = [y'_{n-\frac{1}{2}}, y'_n]^T$,

$f_N = [f_{n-\frac{1}{2}}, f_n]^T$,

$f_{N+1} = [f_{n+\frac{1}{2}}, f_{n+1}]^T$, $g_N = [g_{n-\frac{1}{2}}, g_n]^T$, and

$g_{N+1} = [g_{n+\frac{1}{2}}, g_{n+1}]^T$

$A, A^0, B, D^0, D^1, E^0, E^1$ are $n \; x \; n$ matrices.

Therefore, A^{-1} is multiplied by (9) and this gives

$$\begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y'_{n+\frac{1}{2}} \\ y'_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{h}{2} \\ 1 & h \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_n \\ y'_n \end{pmatrix} + h^2 S \begin{pmatrix} f_n \\ f_{n+\frac{1}{2}} \\ f_{n+1} \end{pmatrix} + h^3 T \begin{pmatrix} g_n \\ g_{n+\frac{1}{2}} \\ g_{n+1} \end{pmatrix}$$

(10)

where

$$S = \begin{pmatrix} \frac{13}{168} & \frac{7}{168} & \frac{1}{168} \\ \frac{79}{420} & \frac{112}{420} & \frac{19}{420} \\ \frac{101}{480h} & \frac{128}{480h} & \frac{11}{480h} \\ \frac{7}{30h} & \frac{16}{30h} & \frac{7}{30h} \end{pmatrix},$$

4.0

$$T = \begin{pmatrix} \frac{59}{13440} & -\frac{128}{13440} & -\frac{11}{13440} \\ \frac{105}{8820} & -\frac{168}{8820} & -\frac{42}{8820} \\ \frac{13}{960h} & -\frac{40}{960h} & -\frac{3}{960h} \\ \frac{1}{60h} & 0 & -\frac{1}{60h} \end{pmatrix}$$

III. PROPERTIES OF THE METHOD

The basic properties of this method such as order, error constant, zero stability and consistency are analyzed below.

3.1 Order

Equation (10) derived is a discrete scheme belonging to the class of LMMs of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j} + h^3 \sum_{j=0}^{k} \gamma_j g_{n+j} \quad (10)$$

Following [4] and [8], we define the local truncation error associated with (11) by the difference operator

$$L[y(x):h] = \sum_{j=0}^{k} \begin{bmatrix} \alpha_j y(x_n + jh) - h^2 \beta_j f(x_n + jh) \\ - h^3 \gamma_j g(x_n + jh) \end{bmatrix} \quad (11)$$

where $y(x)$ is an arbitrary function, continuously differentiable on [a, b].

Expanding (11) in Taylor series about the point x, we obtain the expression

$$L[y(x);h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \ldots + C_{p+2} h^{p+2} y^{p+2}(x)$$

where the C_0 , C_1 , $C_2 \ldots C_p \ldots C_{p+2}$ are obtained as

$$C_0 = \sum_{j=0}^{k} \alpha_j, \; C_1 = \sum_{j=1}^{k} j\alpha_j, \; C_2 = \frac{1}{2!} \sum_{j=1}^{k} j^2 \alpha_j,$$

$$C_q = \frac{1}{q!} \begin{bmatrix} \sum_{j=1}^{k} j^q \alpha_j - q(q-1) \sum_{j=1}^{k} \beta_j j^{q-2} \\ - q(q-1)(q-2) \sum_{j=1}^{k} \gamma_j j^{q-3} \end{bmatrix}.$$

In the spirit of [11], (11) is of order p if $C_0 = C_1 = C_2 = \ldots C_p = C_{p+1} = 0$ and $C_{p+2} \neq 0$.

The $C_{p+2} \neq 0$ is called the error constant and $C_{p+2} h^{p+2} y^{p+2}(x_n)$ is the principal local truncation error at the point x_n.

Thus, the block (10) is of order $p = 6$ and error constant

$$C_{p+2} = \left[\frac{1}{4423680}, \frac{1}{1209600}, \frac{1}{1209600}, \frac{1}{604800} \right]^T.$$

3.2 Zero Stability of the Method

The linear multistep method (10) is said to be zero-stable if no root of the first characteristic polynomial $\rho(R)$ has modulus greater than one and if every root of modulus one has multiplicity not greater than the order of the differential equation.

To analyze the zero-stability of the method, we present (10) in vector notation form of column vectors $e = (e_1 \ldots e_r)^T$, $d = (d_1 \ldots d_r)^T$, $y_m = (y_{n+1} \ldots y_{n+r})^T$, $F(y_m) = (f_{n+1} \ldots f_{n+r})^T$, $G(y_m) = (g_{n+1} \ldots g_{n+r})^T$ and matrices $A = (a_{ij})$, $B = (b_{ij})$.

Thus, (10) forms the block formula

$$A^0 y_m = hBF(y_m) + A^1 y_n + hbf_n + hDG(y_m) + hdg_n \quad (12)$$

where h is a fixed mesh size within a block.

In line with (12),

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A^1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, b = \begin{pmatrix} \frac{13}{168} \\ \frac{79}{420} \end{pmatrix}, d = \begin{pmatrix} \frac{59}{13440} \\ \frac{105}{8820} \end{pmatrix},$$

$$B = \begin{pmatrix} \frac{7}{168} & \frac{1}{168} \\ \frac{112}{420} & \frac{19}{420} \end{pmatrix} \text{ and } D = \begin{pmatrix} \frac{-128}{13440} & \frac{-11}{13440} \\ \frac{-168}{8820} & \frac{-42}{8820} \end{pmatrix}$$

The first characteristic polynomial of the block hybrid method is given by

$$\rho(R) = \det(RA^0 - A^1) \qquad (13)$$

where

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } A^1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

3.3 CONSISTENCY AND CONVERGENCE OF THE METHOD

The linear multistep method (10) is said to be consistent if it has order $p \geq 1$. Equation (10) is of order 6.

According to the theorem of [5], the necessary and sufficient condition for a LMM to be convergent is to be consistent and zero stable. Since the method satisfies the two conditions hence it is convergent.

substituting A^0 and A^1 in (13) and solving for R, the values of R are obtained as 0 and 1.

According to [6] and [7], the block method (10) is zero-stable, since from (13), $\rho(R) = 0$, satisfy $|R_j| \leq 1$, $j = 1$ and for those roots with $|R_j| = 1$, the multiplicity does not exceed two.

IV NUMERICAL EXPERIMENTS

In examining the efficiency of the newly developed block method, it is applied to the following second and third-order initial value problems of ordinary differential equations.

Problem1:

$$y'' - x(y')^2 = 0, \; y(0) = 1, y'(0) = \frac{1}{2}, h = 0.003125$$

Exact Solution: $y(x) = 1 + \frac{1}{2} In\left(\frac{2+x}{2-x}\right)$

Table Ia: Comparing the solutions of the exact and the new block for Problem 1

X	Exact Solution	Numerical Solution
0.1	1.050041729278491400	1.050041729285235300
0.2	1.100335347731075300	1.100335347786803200
0.3	1.151140435936466500	1.151140436133040400
0.4	1.202732554054081600	1.202732554548837200
0.5	1.255412811882994600	1.255412812926617300
0.6	1.309519604203111900	1.309519606185874500
0.7	1.365443754271397100	1.365443757799182000
0.8	1.423648930193603500	1.423648936214441100
0.9	1.484700278594054600	1.484700288613987300
1.0	1.549306144334058600	1.549306160797821400

Table Ib: Comparing the errors of the new block and existing methods for Problem 1

X	Error in the new method, *k*=1	Error in[9], *k*=3	Error in [10], *k*=6	Error in [1], *k*=6
0.1	6.743939E-012	5.850875E-13	9.577668E-10	0.1329867326E-09
0.2	5.572787E-011	2.848832E-12	2.368709E-09	0.5872691257E-08
0.3	1.965739E-010	6.328715E-12	3.732243E-09	0.1327845616E-07
0.4	4.947556E-010	6.756392E-09	5.475119E-09	0.2317829012E-07
0.5	1.043623E-009	1.380119E-08	1.142189E-08	0.3218793564E-07
0.6	1.982763E-009	2.174817E-08	4.567944E-08	0.6871246012E-07
0.7	3.527785E-009	1.073052E-07	2.055838E-06	0.1012728156E-06
0.8	6.020838E-009	2.001340E-07	4.248299E-06	0.1231093271E-06
0.9	1.001993E-008	3.088383E-07	6.660458E-06	0.2019286712E-06
1.0	1.646376E-008	9.805074E-07	9.445166E-06	0.2990871645E-06

Problem 2: $y'' = y', \quad y(0) = 0, y'(0) = -1, \; h = 0.01$

Exact Solution: $y(x) = 1 - e^{x}$

Table IIa: Comparing the solutions of the exact and the new block for Problem 2

X	Exact Solution	Numerical Solution
0.1	-0.105170918075647710	-0.105170918285230290
0.2	-0.221402758160169850	-0.221402760252888020
0.3	-0.349858807576003180	-0.349858815418549070
0.4	-0.491824697641270570	-0.491824717736273730
0.5	-0.648721270700128640	-0.648721312697835510
0.6	-0.822118800390509550	-0.822118877678934210
0.7	-1.013752707470477500	-1.013752837854869500
0.8	-1.225540928492468800	-1.225541134976360700
0.9	-1.459603111156951200	-1.459603422838633400
1.0	-1.718281828459047300	-1.718282281559109400

Table IIb : Comparing the errors of the new block and existing methods for Problem 2

X	Error in new method, *k*=1	Error in [10], *k*=5	Error in [13], *k*=5	Error in [12], *k*=5
0.1	2.095826E-010	2.508826E-13	2.004000000E-07	2.198000000E-05
0.2	2.092718E-009	6.493175E-11	5.386000000E-07	6.070400000E-06
0.3	7.842546E-009	1.683146E-09	8.840000000E-07	1.005100000E-05
0.4	2.009500E-008	1.700635E-08	1.229700000E-06	1.402530000E-05
0.5	4.199771E-008	1.025454E-07	1.575200000E-06	1.799340000E-05
0.6	7.728842E-008	2.558711E-06	1.920400000E-06	2.161620000E-05
0.7	1.303844E-007	5.273300E-06	2.506000000E-06	2.799300000E-05
0.8	2.064839E-007	8.275935E-06	3.106000000E-06	3.456100000E-05
0.9	3.116817E-007	1.161667E-05	3.705000000E-06	4.111400000E-05
1.0	4.531001E-007	1.542187E-05	4.304000000E-06	4.765600000E-05

Problem 3: $y''' = e^x \quad y(0) = 3, y'(0) = 1, y''(0) = 5, h = 0.1$

Exact Solution: $y(x) = 2 + 2x^2 + e^x$

Table IIIa: Comparing the solutions of the exact and the new block for Problem 3

X	Exact Solution	Numerical Solution
0.1	3.125170918075647700	3.125170918075638800
0.2	3.301402758160169700	3.301402758160134200
0.3	3.529858807576003300	3.529858807575920300
0.4	3.811824697641270600	3.811824697641117900
0.5	4.148721270700128200	4.148721270699882200
0.6	4.542118800390508900	4.542118800390142000
0.7	4.993752707470476600	4.993752707469958800
0.8	5.505540928492466800	5.505540928491764200
0.9	6.079603111156949100	6.079603111156023600
1.0	6.718281828459044600	6.718281828457857200

Table IIIb : Comparing the errors of the new block and existing methods for Problem 3

X	Error in new method, k=1	Error in [1], k=5	Error in [14], k=5
0.1	8.881784E-015	3.369305E-12	9.24352E-10
0.2	3.552714E-014	2.160050E-11	8.3983E-10
0.3	8.304468E-014	5.333245E-11	4.23997E-10
0.4	1.527667E-013	9.988632E-11	3.58729E-10
0.5	2.460254E-013	1.598988E-10	2.99872E-10
0.6	3.668177E-013	2.511404E-10	3.90509E-10
0.7	5.178080E-013	3.961489E-10	1.47048E-09
0.8	7.025491E-013	5.926823E-10	2.49247E-09
0.9	9.254819E-013	8.429168E-10	0.15695E-09
1.0	1.187495E-012	1.144603E-09	3.54096E-09

Problem 4: $y''' - y'' + y' - y = 0, \quad y(0) = 1, y'(0) = 0, y''(0) = -1, h = 0.01$

Exact Solution: $y(x) = \cos x$

Table IV: Comparing the new block method with [15] for solving Problem 4

X	Exact Solution	Numerical Solution	Error in new method, k=1	Error in [15], k=3
0.1	0.999950000416665260	0.999950000414032480	2.632783E-012	1.9990E-07
0.2	0.999800006666577760	0.999800006605354950	6.122280E-011	1.9560E-07
0.3	0.999550033748987540	0.999550033315882550	4.331050E-010	1.3651E-07
0.4	0.999200106660977920	0.999200104881007410	1.779971E-009	2.5210E-07
0.5	0.998750260394966280	0.998750255022231250	5.372735E-009	1.3039E-06
0.6	0.998200539935204190	0.998200526636915390	1.329829E-008	3.0280E-06
0.7	0.997551000253279590	0.997550971585185910	2.866809E-008	3.3453E-06
0.8	0.996801706302619440	0.996801650474031580	5.582859E-008	1.2405E-06
0.9	0.995952733011994270	0.995952632438632860	1.005734E-007	1.3290E-06
1.0	0.995004165278025820	0.995003994920954820	1.703571E-007	1.7180E-05

Problem 5: Vanderpol's oscillator Problem

$$y'' = 2\cos x - \cos^3 x - y' - y - y^2 y', \quad y(0) = 0, y'(0) = 1$$

Exact Solution: $y(x) = \sin x$

Table V: Comparison of the new method with [3]

X	Exact Solution	Approximate solution	Error in new method	Error in [3]
0.1	0.099833416646828155	0.099833416316099016	3.307291E-010	4.16719627E-13
0.2	0.198669330795061220	0.198669328479548270	2.315513E-009	3.54860749E-12
0.3	0.295520206661339600	0.295520200499645890	6.161694E-009	9.0472212E-12
0.4	0.389418342308650520	0.389418330384836750	1.192381E-008	1.650241042E-11
0.5	0.479425538604203010	0.479425519259765230	1.934444E-008	2.544360932E-11
0.6	0.564642473395035370	0.564642445640541220	2.775449E-008	3.535590072E-11
0.7	0.644217687237691020	0.644217651157492410	3.608020E-008	4.570838971E-11
0.8	0.717356090899522680	0.717356047921081210	4.297844E-008	5.598981142E-11
0.9	0.783326909627483300	0.783326862546561360	4.708092E-008	6.574464284E-11
1.0	0.841470984807896390	0.841470937523322140	4.728457E-008	7.460291389E-11

Problem 6: We consider the second-order system equations

$$y_1'' = -4t^2 y_1 - \frac{2y_2}{\sqrt{y_1^{\,2} + y_2^{\,2}}}, \quad y_1\left(\sqrt{\frac{\pi}{2}}\right) = 0, \quad y_1'\left(\sqrt{\frac{\pi}{2}}\right) = -2\sqrt{\frac{\pi}{2}}, \quad h = 0.01$$

$$y_2'' = -4t^2 y_2 - \frac{2y_1}{\sqrt{y_1^{\,2} + y_2^{\,2}}}, \quad y_2\left(\sqrt{\frac{\pi}{2}}\right) = 1, \quad y_2'\left(\sqrt{\frac{\pi}{2}}\right) = 0, \quad \sqrt{\frac{\pi}{2}} \le t \le 10$$

Exact Solutions: $y_1(t) = \cos(t^2),\ y_2(t) = \sin(t^2)$

Table VI: Result generated when the new method was applied to system of second order ODEs

t- values	Exact Solution	Numerical Solution y_2	Error
1.2583141	0.999921147578931800	0.999921117529356310	3.004958E-008
1.2633141	0.999683345819422750	0.999683285585503570	6.023392E-008
1.2683141	0.999284741752009320	0.999284651012086630	9.073992E-008
1.2733141	0.998723501579551810	0.998723380268963120	1.213106E-007
1.2783141	0.997997811695503390	0.997997659663992120	1.520315E-007
1.2833141	0.997105879715436850	0.997105696969847740	1.827456E-007
1.2883141	0.996045935521627570	0.996045722087244840	2.134344E-007
1.2933141	0.994816232320477560	0.994815988276771760	2.440437E-007
1.2983141	0.993415047712551740	0.993414773263781360	2.744488E-007

t- values	Exact Solution	Numerical Solution y_1	Error
1.2583141	-0.012557811291468633	-0.012557811291308991	1.596414E-013
1.2633141	-0.025163626354015697	-0.025163626353447266	5.684307E-013
1.2683141	-0.037815405612265290	-0.037815405667311876	5.504659E-011
1.2733141	-0.050511062082270429	-0.050511062224706874	1.424364E-010
1.2783141	-0.063248461253903210	-0.063248461592840324	3.389371E-010
1.2833141	-0.076025420991302939	-0.076025421591992062	6.006891E-010
1.2883141	-0.088839711451837056	-0.088839712475807853	1.023971E-009
1.2933141	-0.101689055024075960	-0.101689056571485010	1.547409E-009
1.2983141	-0.114571126285240580	-0.114571128569164710	2.283924E-009
1.3033141	-0.127483551978623600	-0.127483555135529060	3.156905E-009

V. DISCUSSION OF RESULTS

Tables I – IV above show the tabular display of the numerical solutions on the implementation of the newly developed method. It is evident that the block method is more efficient in terms of error when compared with existing methods in spite of high step number *k* considered.

VI. CONCLUSION

In this paper, the derivation of the new block method for solving second and third-order ordinary differential equations directly is examined. The method is of order six which shows that it is consistent. The major advantage of the method over the existing numerical methods is its ability to solving effectively two different orders of differential equations namely second and third-order ordinary differential equations. To prove the efficiency of the new method, it is applied to some differential equations of order two and three, the results generated outperform the existing methods in terms of error as shown in Tables I – IV.

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