On the Solvability of a Resonant $p$-Laplacian Third-order Integral m-Point Boundary Value Problem

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Abstract— In this work, we establish conditions for the existence of at least one solution for a $p$-Laplacian third order integral and m-point boundary value problem at resonance. The Ge and Ren extension of Mawhin’s coincidence theory will be used to obtain existence results for the $p$-Laplacian problem at resonance.

Index Terms— Coincidence degree, resonance, m-point, integral boundary value problem, $p$-Laplacian.

1 Introduction

This work deals with the following $p$-Laplacian third order integral and m-point boundary value problem at resonance

\[(\phi_p(u''(t)))' = w(t, u(t), u'(t), u''(t)), \quad t \in (0, 1), \quad (1)\]

subject to the boundary conditions

\[\phi_p(u''(0)) = \sum_{i=1}^{m} \alpha_i \int_{0}^{\xi_i} \phi_p(u''(t))dt,\]
\[u''(1) = 0, \quad u'(1) = \beta u'(\eta), \quad (2)\]

where the function $w : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, the inverse of $\phi_p^{-1}$ is $\phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $\beta > 0$, $\alpha_i (1 \leq i \leq m) \in \mathbb{R}$ and $\eta \in (0, 1)$. Since we require a nontrivial kernel for our quasi-linear operator, the condition $\sum_{i=1}^{m} \alpha_i \xi_i = 1$ is critical. The integral in (2) is the Riemann-Stieltjes integral.

A boundary value problem $Lu = u''(t) = 0$ is said to be at resonance if $L$ is non-vertible else it is a non-resonance problem where $L$ is a linear operator. Since the establishment of the coincidence degree theory by Mawhin, for boundary value problems at resonance [13], many authors have studied resonant problems when the differential operator is linear (see [1, 3, 5, 6, 8, 9, 12]). When the differential operator is nonlinear, like in $p$-Laplace boundary value problems the Mawhin coincidence degree theory fails while the extension of the theorem by Ge and Ren [4] is used (see [7, 2, 15, 10]).

Inspired by the above works, this paper uses the Ge and Ren extension of the coincidence degree theory [4] to establish the existence of solutions for the problems (1)-(2) at resonance.

The rest of the paper is organized as follows. Section 2 gives necessary definitions, lemmas and theorems that are needed for the work. In section 3, we obtain existence results for (1)-(2) while an example will be given in section 4 to corroborate our result.

2 Preliminaries

In this section, we will give necessary lemmas, definitions and theorems.

Definition 1. Given two Banach spaces, $U$ and $Z$ with norms $\| \cdot \|_U$ and $\| \cdot \|_Z$ respectively, a continuous operator $M : \text{dom } M \subset U \to Z$ is said to be quasi-linear if

(i) $\text{Im } M$ is a closed subset of $Z$;

(ii) $\ker M$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$.

Definition 2. ([10]) Let $\Omega \subset U$ be a bounded open set with the origin $\sigma \in \Omega$. The nonlinear operator $N_{\lambda} : \overline{\Omega} \to Z$, $\lambda \in [0, 1]$ is said to be $M$-compact in $\overline{\Omega}$ if there exist $Z_1 \subset Z$ with $\text{dim } Z_1 = \text{dim } \ker M$ and a continuous, compact operator $T : \overline{\Omega} \times [0, 1] \to U_2$ such that for $\lambda \in [0, 1]$,

(i) $(I - Q)N_{\lambda} \subset \text{Im } M \subset (I - Q)Z$;

(ii) $QN_{\lambda}u = 0$, $\lambda \in (0, 1) \Leftrightarrow QNu = 0$, $\forall u \in \Omega$;

(iii) $T(\cdot, 0) \equiv 0$ and $T(\cdot, \lambda)|_{\Sigma_{\lambda}} = (I - P)\Sigma_{\lambda}$;

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(iv) $M[P + T(\cdot, \lambda)] = (I - Q)N_{\lambda}$, $\lambda \in [0, 1]$.

where $U_2 \in U$ is a complement space of ker $M$, i.e. $U = \ker M \oplus U_2$; $P$, $Q$ are projectors such that ker $M = \text{Im } P$, Im $Q = Z_1$, $N = N_1$, and $\sum_{\lambda} = \{ u \in \Omega : M u = N_{\lambda} u \}$.

**Lemma 3.** [16] The following are true for $\phi_p$:

1. (i) $\phi_p$ is continuous, invertible and monotonically increasing. In addition, $\phi_p$ and for $q > 1$ then $\frac{1}{p} + \frac{1}{q} = 1$;

(ii) For all $y, z, \geq 0$,

$$\phi_p(y + z) \leq \phi_p(y) + \phi_p(z), \quad \text{if } 1 < p < 2,$$

$$\phi_p(y + z) \leq 2^{p-2}(\phi_p(y) + \phi_p(z)), \quad \text{if } p \geq 2.$$

**Theorem 1.** (4) Let $U$ and $Z$ be Banach spaces, and $\Omega \subset U$ a bounded open nonempty set. Also $M : dom \ M \subset U \rightarrow Z$ is quasi-linear and $N_{\lambda} : \Omega \rightarrow \lambda$ is compact in $\Omega$. Assume the following conditions are satisfied

(i) $M u \neq N_{\lambda} u$ for every $(u, \lambda) \in \{(\text{dom } M \setminus \text{ker } M) \cap \partial \Omega \} \times (0, 1)$;

(ii) $Q N u \neq 0$ for every $u \in \ker M \cap \partial \Omega$;

(iii) deg$(JQ N, \Omega \cap \ker M, 0) \neq 0$, where $J : \text{Im } Q \rightarrow \ker M$ is a homeomorphism.

Then, the abstract equation $M u = N u$ has at least one solution in $\Omega$.

Let

$U = \{ u \in C^2[0, 1] : \phi_p(u''(t)) \in C^1[0, 1], u(t) \text{ satisfies (2)} \}$

where the norms $\|z\|_{\infty} = \max_{t \in [0, 1]} |z(t)|$ and $\|u\| = \max\{\|u\|_{\infty}, \|u''\|_{\infty}, \|u''\|_{\infty} \}$ are defined on $U$.

Let $Z = L^1[0, 1]$ with the norm on $Z$ denoted by $\| \cdot \|_1$. The quasi-linear operator $M : \text{dom } M \subset U \rightarrow Z$ will be defined by

$M : u \mapsto M u = (\phi_p(u''(t)))', \ t \in [0, 1],$

where dom $M = \{ u \in U \cap C^2[0, +\infty) :$

$\phi_p(u''(0)) = \sum_{i=1}^{m} \alpha_i \int_0^\xi \phi_p(u''(t)) \, dt, u''(1) = 0, u'(1) = \beta u'(\eta) \}$.

Also, the nonlinear operator $N_{\lambda} : U \rightarrow Z$, $\lambda \in [0, 1]$ will be defined by

$(N_{\lambda} u)t = \lambda q(t, u(t), u'(t), u''(t)), \ t \in [0, 1],$

thus problem (1)-(2) may be written in the form

$M u = N_{\lambda} u.$

**Lemma 2.** If $\sum_{i=1}^{m} \alpha_i \xi_i = 1$ then there exists $r \in \{ 1, 2, \ldots, m - 1 \}$, such that

$$\sum_{i=1}^{m} \alpha_i \xi_i^{r+2} \neq 0.$$

**Proof.** Since $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, and $\sum_{i=1}^{m} \alpha_i \xi_i = 1$ then there exists $i \in [1, m]$ such that $\alpha_i \neq 0$, hence $\sum_{i=1}^{m} \alpha_i \neq 0$. Assuming

$$\sum_{i=1}^{m} \alpha_i \xi_i^{r+2} = 0, \ r = 0, 1, \ldots, m - 2,$$

we have

$$\begin{pmatrix}
\xi_1^2 & \xi_2^2 & \cdots & \xi_m^2 \\
\xi_1^3 & \xi_2^3 & \cdots & \xi_m^3 \\
\vdots & \vdots & \ddots & \vdots \\
\xi_1^m & \xi_2^m & \cdots & \xi_m^m
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_m
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}.$$

Since

$$\det \begin{pmatrix}
\xi_1^2 & \cdots & \xi_m^2 \\
\xi_1^3 & \cdots & \xi_m^3 \\
\vdots & \ddots & \vdots \\
\xi_1^m & \cdots & \xi_m^m
\end{pmatrix} = \prod_{i=1}^{m} (\xi_i - \xi_i) \prod_{1 \leq i < j \leq m} (\xi_i - \xi_j) \neq 0,$$

then $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$, which contradicts $\sum_{i=1}^{m} \alpha_i \neq 0$. Hence, Lemma 2 holds.

**Lemma 3.** If $\sum_{i=1}^{m} \alpha_i \xi_i = 1$, then, the operator $M : \text{dom } M \subset U \rightarrow Z$ is quasi-linear.

**Proof.** By simple calculation, we see that

$$\ker M = \{ u \in \text{dom } M : u = d, d \in \mathbb{R} \}.$$

We will now show that

$$\text{Im } M = \{ y \in Z : \sum_{i=1}^{m} \alpha_i \int_0^\xi \int_0^x y(v) \, dv \, dx = 0 \}.$$ (3)

The p-Laplacian problem

$$\phi_p(u''(t)) = y(v)$$ (4)

has a solution $u(t)$ that satisfies (2) when

$$\sum_{i=1}^{m} \alpha_i \int_0^\xi \int_0^x y(v) \, dv \, dx = 0.$$ (5)
Applying the boundary condition (2) and \( t \) to (6) we obtain
\[
\sum_{i=1}^{m} \alpha_i \int_{0}^{\xi_i} \int_{0}^{x} y(v) dv dx = 0,
\]
which satisfies (3) and
\[
u(t) = d + \frac{\beta(t-1)}{1 - \beta} \int_{\eta}^{1} \phi_0 \int_{x}^{1} y(v) dv dxs - \int_{t}^{1} \int_{s}^{1} \phi_0 \left( \int_{x}^{1} y(v) dv \right) dx ds,
\]
where \( d \) is an arbitrary constant and \( u(t) \) is the solution to (4) satisfying (2). Since \( \ker M = 1 < \infty \) and \( M(U \cap \text{dom} M) \subset Z \) is closed, the operator \( M \) is quasi-linear.

**Lemma 4.** The nonlinear operator \( N_\lambda \) is \( M \)-compact, if \( w \in C([0, 1] \times \mathbb{R}^3, \mathbb{R}) \).

**Proof.** We define projectors \( P : U \rightarrow U_1 \) as \( Pu = u(1) \) for all \( u \in U \) and \( Q : Z \rightarrow Z_1 \) as
\[
Qy = \frac{r + 1)(r + 2)}{\sum_{i=1}^{m}} \alpha_i \xi_i + 2 \left( \sum_{i=1}^{m} \alpha_i \int_{0}^{\xi_i} \int_{0}^{x} y(v) dv dx \right) t^r,
\]
\( t \in [0, 1], \forall y \in Z, \) where \( Z_1 \) is the complement space of \( \text{Im} M \) in \( Z \). Let \( \overline{\Omega} \subset \overline{U} \) be bounded, then we will define \( T : \overline{\Omega} \times [0, 1] \rightarrow \ker P \) as
\[
T(u, \lambda)(t) = \frac{\beta(t-1)}{1 - \beta} \int_{\eta}^{1} \phi_0 \int_{x}^{1} ([Q - Q_\lambda u] (v) dv) dxs - \int_{t}^{1} \int_{s}^{1} \phi_0 \left( \int_{x}^{1} [Q - Q_\lambda u] (v) dv \right) dx ds, t \in [0, 1].
\]

(7)

\( T(\cdot, \lambda) \) is continuous and relatively compact since \( w \in C([0, 1] \times \mathbb{R}^3, \mathbb{R}) \), and \( \lambda \in [0, 1] \). We will now show in the following four steps that \( N_\lambda \) is \( M \)-compact.

**Step 1:** Let \( y \in \Omega \), then
\[
Q^2 y = Q(Qy) = Qy(Q) = Qy \left[ \sum_{i=1}^{m} \alpha_i \int_{0}^{\xi_i} \int_{0}^{x} v^r dv dx \right] = Qy, \quad t \in [0, 1],
\]
hence \( Q^2 = Q \). Therefore \( Q(I - Q)N_\lambda(\overline{\Omega}) = (Q - Q_\lambda N_\lambda(\overline{\Omega}) \subset \ker Q = \text{Im} M \). Now, if \( g \in \text{Im} M \), then \( Qg = 0 \). We can write \( g = g - Qg = (I - Q)g \), thus \( g \in (I - Q)Z \). Therefore (i) of definition 2.2 is satisfied.

**Step 2:** If \( QN_\lambda = 0 \), then \( N_\lambda = Nu - QNu = (I - Q)Nu = 0 \). Since \( Nu \neq 0 \), \( (I - Q) \) is a zero operator. Hence \( (I - Q)N_\lambda u = 0 \), same logic it can also be shown that when \( QN_\lambda u = 0 \), \( QNu = 0 \). Hence (ii) of definition 2.1 is satisfied.

**Step 3:** Here we show that (iii) of definition 2 holds. From (7), we have
\[
T(u, \lambda)(t) = \frac{\beta(t-1)}{1 - \beta} \int_{\eta}^{1} \phi_0 \int_{x}^{1} ([I - Q]Nu(v) dv) dxs - \lambda \int_{t}^{1} \int_{s}^{1} \phi_0 \left( \int_{x}^{1} [I - Q]Nu(v) dv \right) dx ds,
\]
hence \( T(\cdot, 0) = 0 \).

Also for \( u \in \sum_{\lambda} \{ u \in \overline{\Omega} : Mu = N_\lambda u \} \) or \( \{ u \in \overline{\Omega} : (\phi_p(u''(v))) = \lambda u(t, u'(t), u''(t)) \} \), we have
\[
T(u, \lambda)(t) = \frac{\beta(t-1)}{1 - \beta} \int_{\eta}^{1} \phi_0 \int_{x}^{1} ([I - Q]Nu(v) dv) dxs - \int_{t}^{1} \int_{s}^{1} \phi_0 \left( \int_{x}^{1} [Q - Q_\lambda u] (v) dv \right) dx ds
\]
\[
= \frac{\beta(t-1)}{1 - \beta} \int_{\eta}^{1} \phi_0 \int_{x}^{1} ([I - Q]Nu(v) dv) dxs - \int_{t}^{1} \int_{s}^{1} \phi_0 \left( \int_{x}^{1} [Q - Q_\lambda u] (v) dv \right) dx ds
\]
\[
= \frac{\beta(t-1)}{1 - \beta} [u'(t) - u'(1)] + u'(1)(1 - t) - u(1) + u(t)
\]
\[
= u'(1)(t - 1) + u'(1)(1 - t) - u(1) + u(t)
\]
\[
= ([I - P]u)(t).
\]

**Step 4:** Now for all \( u \in U \cap \text{dom} M \), we have
\[
M[P + T(\cdot, \lambda)]u = u(1) + \frac{\beta(t-1)}{1 - \beta} \int_{\eta}^{1} \phi_0 \int_{x}^{1} ([I - Q]Nu(v) dv) dxs - \int_{t}^{1} \int_{s}^{1} \phi_0 \left( \int_{x}^{1} [Q - Q_\lambda u] (v) dv \right) dx ds
\]
\[
= (I - Q)N_\lambda u(t).
\]
Since conditions (i) - (iv) of Definition 2 are satisfied in \( \overline{\Omega} \), then \( N_\lambda \) is \( M \)-compact.

3 Existence Results

**Theorem 2** Let \( w : [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R} \) be a continuous function. The \( p \)-Laplacian boundary value problem (1)-(2) with \( \sum_{i=1}^{m} \alpha_i \xi_i = 1 \),
\[
\phi_p(2)^{2p - 4}((\|x\|^q + \|y\|^q + \|z\|^q) < 1 \quad \text{for} \quad p < 2
\]
and
\[
\phi_p(2)((\|x\|^q + \|y\|^q + \|z\|^q) < 1 \quad \text{for} \quad p \geq 2
\]
has at least one solution in \( C^2[0, 1] \), if the following conditions hold.
(C1) There exist function $x, y, z, h \in C([0, 1], [0, \infty))$ such that for all $(a, b, c) \in \mathbb{R}^3$, $t \in [0, 1]$
\[
|w(t, a, b, c)| \leq x(t)\phi_p(|a|) + y(t)\phi_p(|b|) + z(t)\phi_p(|c|) + h(t).
\]
\[
(10)
\]
(C2) There exists a constant $D > 0$, such that for any $u$ in $\text{dom } M$, $|u(t)| > D$, $|u'(t)| > D$, or $|u''(t)| > D$, for every $t \in [0, 1]$ then
\[
QNu(t) \neq 0, \ t \in [0, 1].
\]
\[
(11)
\]
(C3) There exists a constant $F > 0$ such that for $d \in \mathbb{R}$, if $|d| > F$, then either
\[
d \cdot \sum_{i=1}^{m} \alpha_i \int_{0}^{\xi_i} \int_{0}^{t} w(v, d, 0, 0) dv dt < 0,
\]
\[
(12)
\]
or
\[
d \cdot \sum_{i=1}^{m} \alpha_i \int_{0}^{\xi_i} \int_{0}^{t} w(v, d, 0, 0) dv dt > 0.
\]
\[
(13)
\]
Proof. We set
\[
\Omega_1 = \{ u \in \text{dom } M : M u = N\lambda u, \ \lambda \in [0, 1] \}.
\]
If $u \in \Omega_1$, then $M u = N\lambda u$ and $\lambda \neq 0$, then $Nu \in \text{Im } M = \ker Q$ and $QN\lambda u(0) = 0$. From (C2), it follows that there exists $t_1, t_2, \xi_1, \xi_2 \in [0, 1]$ such that $|u(t_0)| \leq D$, $|u'(t_1)| \leq D$ and $|u''(t_2)| \leq D$. By the absolute continuity of $u, u'$, we have $u(t) = u(t_1) + \int_{t_1}^{t} u'(v) dv$, i.e.,
\[
|u(t)| = |u(t_0) + \int_{0}^{t} u'(v) dv| \leq D + \int_{0}^{t} |u'(v)| dv.
\]
Hence, $|u|_{\infty} \leq D + |u'\parallel_{\infty}$. Also, since $u'(t) = u(t_1) + \int_{t_1}^{t} u''(v) dv$, then
\[
|u'(t)| = |u(t_1) + \int_{0}^{t} u''(v) dv| \leq D + \int_{0}^{t} |u''(v)| dv
\]
Hence, $|u'\parallel_{\infty} \leq D + |u''\parallel_{\infty}$. Thus,
\[
|u|_{\infty} \leq 2D + |u''\parallel_{\infty}
\]
Therefore,
\[
|u| = \max \{ |u|_{\infty}, |u'\parallel_{\infty}, |u''\parallel_{\infty} \}
\]
\[
\leq 2D + |u''\parallel_{\infty}
\]
\[
(14)
\]
\[
|u''(t)| = \phi_p|\phi_p(|u''(t)|)| + \int_{t_2}^{t} u'''(v) dv t_2
\]
\[
\leq \phi_p|\phi_p(|u''(t_2)|)| + \int_{t_2}^{t} |Nu(v)| dv t_2
\]
\[
\leq \phi_p|\phi_p(D) | + \| Nu\parallel_{1},
\]
\[
\]
Suppose $\| Nu\parallel_{1} \leq \phi_p(D)$, then
\[
|u''|_{\infty} \leq \phi_p(\| Nu\parallel_{1}).
\]
For $1 < p < 2$, considering (10) and lemma 3, we have
\[
|u''|_{\infty} \leq \phi_p(\| Nu\parallel_{1})
\]
\[
\leq \phi_p(2)^{2p^{-q}}(\| u\parallel_{\infty}^p + \| u'\parallel_{\infty}^p + \| u''\parallel_{\infty}^p)
\]
\[
+ \phi_p(\| |z|_{\infty}^q + |h|_{\infty}^q)
\]
\[
\leq \phi_p(2)^{2p^{-q}}(\| u\parallel_{\infty}^p + \| u'\parallel_{\infty}^p + \| u''\parallel_{\infty}^p)
\]
\[
+ \| |z|_{\infty}^q \parallel_{\infty} \| h \parallel_{\infty}^q
\]
\[
+ \| |z|_{\infty}^q \parallel_{\infty} \| h \parallel_{\infty}^q
\]
\[
\leq \phi_p(2)^{2p^{-q}}(\| u\parallel_{\infty}^p + \| u'\parallel_{\infty}^p + \| u''\parallel_{\infty}^p)
\]
\[
+ \| |z|_{\infty}^q \parallel_{\infty} \| h \parallel_{\infty}^q + \| |z|_{\infty}^q \parallel_{\infty} \| h \parallel_{\infty}^q.
\]
From (14), we have
\[
|u| \leq 2D + |u''|_{\infty}
\]
\[
= 2D + \phi_p(2)^{2p^{-q}}(\| u\parallel_{\infty}^p + \| u'\parallel_{\infty}^p + \| u''\parallel_{\infty}^p)
\]
or
\[
|u| \leq \frac{2D + \phi_p(2)^{2p^{-q}}(\| h \parallel_{\infty}^p)}{1 - \phi_p(2)^{2p^{-q}}(1)}
\]
Let $D_1 = \frac{2D + \phi_p(2)^{2p^{-q}}(\| h \parallel_{\infty}^p)}{1 - \phi_p(2)^{2p^{-q}}(1)}$, in view of (8), we see that $D_1 > 0$ and $|u| \leq D_1$. Hence, $\Omega_1$ is bounded.
\[
\]
For $p \geq 2$,
\[
|u''|_{\infty} \leq \phi_p(\| Nu\parallel_{1})
\]
\[
\leq \phi_p(2)^{2p^{-q}}(\| u\parallel_{\infty}^p + \| u'\parallel_{\infty}^p + \| u''\parallel_{\infty}^p)
\]
\[
+ \| |z|_{\infty}^q \parallel_{\infty} \| h \parallel_{\infty}^q
\]
\[
\leq \phi_p(2)(\| u\parallel_{\infty}^p + \| u'\parallel_{\infty}^p + \| u''\parallel_{\infty}^p)
\]
\[
\leq \phi_p(2)(\| u\parallel_{\infty}^p + \| u'\parallel_{\infty}^p + \| u''\parallel_{\infty}^p)
\]
\[
(15)
\]
From (14), we have
\[
|u| \leq 2D + |u''|_{\infty}
\]
\[
= 2D + \phi_p(2)^{2p^{-q}}(\| u\parallel_{\infty}^p + \| u'\parallel_{\infty}^p + \| u''\parallel_{\infty}^p)
\]
or
\[
|u| \leq \frac{2D + \phi_p(2)^{2p^{-q}}(\| h \parallel_{\infty}^p)}{1 - \phi_p(2)^{2p^{-q}}(1)}
\]
Let $D_1 = \frac{2D + \phi_p(2)^{2p^{-q}}(\| h \parallel_{\infty}^p)}{1 - \phi_p(2)^{2p^{-q}}(1)}$, in view of (9), we see that $D_1 > 0$ and $|u| \leq D_1$. Hence, $\Omega_1$ is bounded.
\[
\]
We next let
\[
\Omega_2 = \{ u \in \ker M : Nu \in \text{Im } M \}.
\]
If $u \in \Omega_2$, then $u \in \ker M$ and $u$ can be defined as $u(t) = \omega, \ t \in [0, 1], \ \omega$ is an arbitrary constant.
\[
\]
Since $QN\lambda u = 0$, then
\[
\sum_{i=1}^{m} \alpha_i \int_{0}^{\xi_i} \int_{0}^{\xi_i} w(v, d, 0, 0) dv dt = 0.
\]
From \((C_3)\), it follows that \(\|u\| = \omega \leq F\). Hence, \(\Omega_2\) is bounded.

Let the isomorphism \(J : \text{Im} Q \to \ker L\) be defined as

\[
J(dt^r) = d, \quad d \in \mathbb{R}.
\]

If \(d \cdot \sum_{i=1}^{m} \alpha_i \int_0^x w(v, d, 0, 0) dv dt < 0\), we define

\[
\Omega_3 = \{u \in \ker M : \lambda J^{-1} u = (1 - \lambda) Q Nu, \lambda \in [0, 1]\}.
\]

For \(u \in \Omega_3\), we have

\[
\lambda dt^r
= t'(1 - \lambda)^{(r + 1)(r + 2)} \sum_{i=1}^{m} \alpha_i \int_0^x \int_0^x w(v, d, 0, 0) dv dt.
\]

When \(\lambda = 1, d = 0\). However, when \(|d| > F\), in view of (11), we obtain

\[
\lambda dt^r
= t'd(1 - \lambda)^{(r + 1)(r + 2)} \sum_{i=1}^{m} \alpha_i \int_0^x \int_0^x w(v, d, 0, 0) dv dt < 0,
\]

which contradicts \(\lambda dt^r > 0\). Therefore \(|u| = |d| \leq F\), implying that \(|u| \leq F\). Hence \(\Omega_2\) is bounded.

If \(d \cdot \sum_{i=1}^{m} \alpha_i \int_0^x w(v, d, 0, 0) dv dt > 0\), we define

\[
\Omega_3 = \{u \in \ker M : \lambda J^{-1} u = -(1 - \lambda) Q Nu, \lambda \in [0, 1]\}.
\]

Similar arguments can be used to show that \(\Omega_3\) is bounded. This concludes the proof of Theorem 2.

Finally, we will show that all the conditions of Theorem 1 are satisfied. Take an open bounded set \(\Omega \subseteq U\) such that \(U_{i=1}^{3 \cdot \Omega_2} \subseteq \Omega\). Lemma 3 shows that \(M\) is a quasi-linear operator while Lemma 4 shows that \(N_\lambda\) is \(M\)-compact on \(\Omega\). Thus conditions (i) and (ii) of Theorem 1 are satisfied. Finally, we show that (iii) also holds. Set \(E(u, \lambda) = \pm u + (1 - \lambda)Q Nu, J(dt^r) = d\). When \(\lambda = 0, JQ Nu \neq 0\), for \(\lambda = 1, E(u, 1) = \pm Id u \neq 0\). For \(\lambda \in (0, 1)\), from \((C_3)\), we see that \(E(u, 0) \neq 0\). Then based on the above argument, for every \(u \in \ker M \cap \partial \Omega, E(u, \lambda) \neq 0\). Therefore, the homotopy property of degree gives

\[
\deg(JQ Nu |_{ker M} \cap \ker M, 0) = \deg(E(\cdot, 0) \cap \ker M, 0) = \deg(E(\cdot, 1) \cap \ker M, 0) = \deg(\pm Id u, \cap \ker M, 0) = \pm 1 \neq 0.
\]

Therefore condition (iii) of Theorem 1 holds and problem (1)-(2) has at least one solution in \(\Omega\).

4 Example

We will consider the following \(p\)-Laplacian boundary value problem

\[
(\phi_3(u''(t)))' = t + 5u(t)^2 + 12\cos(u(t)^2) + 12u''(t)^2, \quad t \in (0, 1),
\]

\[
(17)
\]

\[
\phi_3(u'(0)) = 6 \int_0^t \phi_3(u''(t)) dt - 2 \int_0^t \phi_3(u''(t)) dt,
\]

\[
u''(1) = 0, \quad u'(1) = 3u' \left(\frac{1}{2}\right),
\]

where \(p = 3 > 2, q = \frac{2}{5}, \alpha_1 = 6, \alpha_2 = -2, \xi_1 = \frac{2}{3}, \xi_2 = \frac{2}{3}, \eta = \frac{1}{2}, \beta = 3\). Also,

\[
w(t, a, b, c) = t + 5a^2 + 12(\cos b^2) + 12c^2.
\]

The resonance condition is fulfilled since, \(\alpha_1 + \alpha_2 = 4 - 2 = 2 \neq 0\) and \(\alpha_1 \xi_1 + \alpha_2 \xi_2 = (4) \left(\frac{1}{2}\right) + (-2) \left(\frac{1}{2}\right) = 1\). Now

\[
|w(t, a, b, c)| \leq |t| + 5|a|^2 + 12|\cos b^2| + 12|c|^2
= 1 + 5|a|^2 + 12 + 12|c|^2
= 13 + 5|a|^2 + 12|c|^2.
\]

Since \(x(t) = 5, y(t) = 0, z(t) = 12, t \in (0, 1)\), then

\[
\phi_3(2)(|x|^{3/2} - 1 + |y|^{3/2} - 1 + |z|^{3/2} - 1) = 2^{-3/2}|5 - \frac{2}{5} + 12 - \frac{1}{2}| = 0.6934(0.5848 + 0.4368) = 0.7083 < 1.
\]

Therefore, condition \((E_1)\) is satisfied.

Next we show that condition \((E_2)\) holds. Let \(D = 3\). and \(u \in \text{dom} M\). if \(|u(t)| > D, t \in (0, 1)\), then either \(u(t) > D\) or \(u(t) < -D\).

For \(u(t) > D\), we have

\[
\sum_{i=1}^{m} \alpha_i \int_0^x \int_0^x w(v, u, u', u''') dv dt
= 4 \int_0^t \int_0^t \left(v + 5u^2 + 12(\cos(u')^2) + 12(u'')^2\right) dv dt
- 2 \int_0^t \int_0^t \left(v + 5u^2 + 12\cos(u')^2 + 12(u'')^2\right) dv dt
> 4 \int_0^t \int_0^t \left(v + 5D^2 - 12 + 12D^2\right) dv dt
- 2 \int_0^t \int_0^t \left(v + 5D^2 - 12 + 12D^2\right) dv dt
> \frac{17}{24} D^2 - \frac{47}{24} > 0.
\]
Similarly, if \( u(t) < -D \), then

\[
\sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^{t} w(v, u, u', u'') dv dt \\
= 4 \int_0^{\frac{1}{2}} \int_0^{t} \left( v + 5u^2 + 12 \cos(u')^2 + 12(u'')^2 \right) dv dt \\
- 2 \int_0^{\frac{1}{2}} \int_0^{t} \left( v + 5u^2 + 12 \cos(u')^2 + 12(u'')^2 \right) dv dt \\
< 4 \int_0^{\frac{1}{2}} \int_0^{t} \left( v - 5D^2 + 12 - 12D^2 \right) dv dt \\
- 2 \int_0^{\frac{1}{2}} \int_0^{t} \left( v - 5D^2 + 12 - 12D^2 \right) dv dt \\
< \frac{73}{24} - \frac{17}{4} D^2 < 0
\]

Therefore, condition \((E_2)\) holds.

Finally, we will show that condition \((E_3)\) holds. Here,

\[
d \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^{t} w(v, d, 0, 0) dv dt \\
= d \left[ 4 \int_0^{\frac{1}{2}} \int_0^{t} \left( v + \frac{1}{5} d \right) dv dt - 2 \int_0^{\frac{1}{2}} \int_0^{t} \left( v + \frac{1}{5} d \right) dv dt \right] \\
= d \left[ \frac{1}{20} d + \frac{1}{24} \right]
\]

Let \( F = \frac{1}{5} > 0 \), then for \( c \in \mathbb{R} \), such that \( |d| > F \), then either \( d > F \) or \( d < -F \). For \( d > F \), we have

\[
d \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^{t} w(v, d, 0, 0) dv dt > 0,
\]

while for \( d < F \),

\[
d \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^{t} w(v, d, 0, 0) dv dt < 0.
\]

Thus, Condition \((E_3)\) is holds. The \( p \)-Laplacian problem (13) - (14) has at least one solution in \( C^2[0,1] \) since it satisfies Theorem 2.

References


