A Power Mensuration and its Normalization under Multicriteria Situations

Chien-Yuan Cheng, En-Cheng Chi, Kunning Chen, Hsing-Lung Lin and Yu-Hsien Liao

***Abstract*—Here we define a normalization of a power index under multicriteria situations by applying maximal-utilities among decision (level) vectors. We also provide several axiomatic results to analyze the rationality for this normalized index. Based on reduction and excess function respectively, we introduce different formulation and dynamic results for this normalized index.**

***Index Terms*—Multicriteria situation, maximal-utility, reduction; excess function, dynamic process.**

I. Introduction

In the framework of transferable-utility (TU) games, the power indexes have been defined to compute the political power of each agent of a voting system. A agent in a voting system is, e.g., a party in a parliament or a country in a confederation. Each agent will have a certain number of votes, and so its power will be different. Results of the power indexes may be found in, e.g., Dubey and Shapley [4], Haller [5], Lehrer [7], van den Brink and van der Laan [2] and so on. Banzhaf [1] defined a power index in the framework of voting games that was essentially identical to that given by Coleman [3]. This index was later on extended to arbitrary games by Owen [15], [16]. In this paper, we focus on the Banzhaf-Owen index. Briefly speaking, the Banzhaf-Owen index is a rule that gathers each agent's marginal contribution from all coalitions in which he/she/it has participated.

Consistency is an important property among the axiomatic formulations for allocation rules. Consistency states the independence of a value with respect to fixing some agents with their assigned payoffs. It asserts that the recommendation made for any problem should always agree with the recommendation made in the sub-problem that appears when the payoffs of some agents are settled on. This property has been investigated in various problems by applying *reduced games* always. In addition to characterizations for an allocation rule, *dynamic processes* can be described that lead the agents to that allocation rule, starting from an arbitrary efficient payoff vector. The foundation of a dynamic theory was laid by Stearns [18].

In a traditional TU game, each agent is either fully involved or not involved at all in participation with some other agents. However, each agent could be allowed to participate with infinite various activity decisions (levels, strategies) in real situations respectively. A *multi-choice TU game* could be regarded as a natural extension of a traditional TU game in which each agent has various operational decisions (or strategies). By considering overall values for a given agent on multi-choice TU games, Hwang and Liao [6], Liao [8], [9] and Nouweland et al. [14] proposed several extended allocations and related results for the core, the EANSC and the Shapley value [17], respectively.

In different fields, from sciences to industry, engineering and the social sciences, managers face an increasing need to focus on multiple aims efficiently in their operational processes. Related situations include analyzing distribution tradeoffs, selecting optimal decision or process designs, or any other condition where you need an efficient solution with tradeoffs between two or more aims. In many cases these real world efficient situations could be formulated as multicriteria mathematical optimization models. The solutions of such situations requires appropriate techniques to offer optimal results that - unlike traditional viewpoints or methods - take several properties of the aims into account. Here we would like to offer mathematical foundation of multicriteria optimal solutions to analyze situations with multiple aims. Inspired by the result due to Wei et al. [11], Liao [10] defined the *multi-choice Banzhaf-Owen index* and the *multi-choice efficient Banzhaf-Owen index* by focusing on multicriteria situations and multi-choice behavior simultaneously in the framework of *multicriteria multi-choice TU games*. The above pre-existing results raise one motivation:

- whether the power indexes could be extended under multicriteria situations and multi-choice behavior simultaneously.

The paper is devoted to investigate the motivation. The main results of this paper are as follows.

- Different from the pre-existing results in the framework of multicriteria multi-choice TU games, a normalization of the multi-choice Banzhaf-Owen index, the *multi-choice normalized Banzhaf-Owen index* (MNBOI), is proposed in Section 2 by applying maximal-utilities among decision (level) vectors. Based on the notion of the multi-choice normalized Banzhaf-Owen index, all agents allocate the utility of the grand fuzzy coalition proportionally by applying the total marginal contributions related to maximal-utilities of all agents.
- By applying an extended reduction and related consistency, we propose several axiomatic results to analyze the rationality for the MNBOI in Section 3.
- In order to establish the dynamic processes, we present alternative formulation for the MNBOI in terms of *excess functions*. In Section 4, we adopt reduction and excess function to show that the MNBOI can be reached by agents who start from an arbitrary efficient payoff vector.

Y.H. Liao (corresponding author): Department of Applied Mathematics, National Pingtung University, 900 Pingtung, Taiwan. e-mail: twincos@ms25.hinet.net.

E.C. Chi: Office of Physical Activities, National Pingtung University, Pingtung 900, Taiwan

C.Y. Chen, Kunning Chen and H.L. Lin: Department of Physical Educational, National Pingtung University, Pingtung 900, Taiwan

II. Preliminaries

Let U be the universe of agents. For $i \in U$ and $b_i \in \mathbb{N}$, $B_i = \{0, 1, \cdots, b_i\}$ could be treated as the level (decision) space of agent i and $B_i^+ = B_i \setminus \{0\}$, where 0 denotes no participation. Let $B^N = \prod_{i \in N} B_i$ be the product set of the decision (level) spaces of all agents of N. For all $T \subseteq N$, we define $\theta^T \in B^N$ is the vector with $\theta_i^T = 1$ if $i \in T$, and $\theta_i^T = 0$ if $i \in N \setminus T$. Denote 0_N the zero vector in $\mathbb{R}^N$. For $m \in \mathbb{N}$, let 0_m be the zero vector in $\mathbb{R}^m$ and $\mathbb{N}_m = \{1, \cdots, m\}$.

A **multi-choice TU game** is a triple (N, b, v), where N is a non-empty and finite set of agents, $b = (b_i)_{i \in N}$ is the vector that presents the highest levels for each agent, and $v : B^N \to \mathbb{R}$ is a characteristic mapping with $v(0_N) = 0$ which assigns to each $\alpha = (\alpha_i)_{i \in N} \in B^N$ the worth that the agents can gain when each agent i participates at level α_i. Given a multi-choice TU game (N, b, v) and $\alpha \in B^N$, we write $A(\alpha) = \{i \in N | \ \alpha_i \neq 0\}$ and α_T to be the restriction of α at T for each $T \subseteq N$. Further, we define $v_*(T) = \max_{\alpha \in B^N} \{v(\alpha) | A(\alpha) = T\}$ is the **maximal-utility**[1] among all action vector α with $A(\alpha) = T$. A **multicriteria multi-choice TU game** is a triple (N, b, V^m), where $m \in \mathbb{N}$, $V^m = (v^t)_{t \in \mathbb{N}_m}$ and (N, b, v^t) is a multi-choice TU game for all $t \in \mathbb{N}_m$.

Denote the collection of all multicriteria multi-choice TU games by Γ. Let $(N, b, V^m) \in \Gamma$. A **payoff vector** of (N, b, V^m) is a vector $x = (x^t)_{t \in \mathbb{N}_m}$ and $x^t = (x_i^t)_{i \in N} \in \mathbb{R}^N$, where x_i^t denotes the payoff to agent i in (N, b, v^t) for all $t \in \mathbb{N}_m$ and for all $i \in N$. A payoff vector x of (N, b, V^m) is **multicriteria efficient** if $\sum_{i \in N} x_i^t = v_*^t(N)$ for all $t \in \mathbb{N}_m$. The collection of all multicriteria efficient vector of (N, b, V^m) is denoted by $E(N, b, V^m)$. A **solution** is a map σ assigning to each $(N, b, V^m) \in \Gamma$ an element

$$\sigma\big(N, b, V^m\big) = \big(\sigma^t\big(N, b, V^m\big)\big)_{t \in \mathbb{N}_m},$$

where $\sigma^t\big(N, b, V^m\big) = \big(\sigma_i^t\big(N, b, V^m\big)\big)_{i \in N} \in \mathbb{R}^N$ and $\sigma_i^t\big(N, b, V^m\big)$ is the payoff of the agent i assigned by σ in $\big(N, b, v^t\big)$.

Liao [10] provided the multi-choice Banzhaf-Owen index and the multi-choice efficient Banzhaf-Owen index under multicriteria situation.

Definition 1: The **multi-choice Banzhaf-Owen index (MBOI, Liao [10])**, β, is defined by

$$\beta_i^t(N, b, V^m) = \sum_{\substack{S \subseteq N \\ i \in S}} \big[v_*^t(S) - v_*^t(S \setminus \{i\})\big]$$

for all $(N, b, V^m) \in \Gamma$, for all $t \in \mathbb{N}_m$ and for all $i \in N$. Under the solution β, all agents receive their **total marginal contributions** related to maximal-utilities in each $S \subseteq N$ respectively.

A solution σ satisfies **multicriteria efficiency (MEFF)** if for all $(N, b, V^m) \in \Gamma$ and for all $t \in \mathbb{N}_m$, $\sum_{i \in N} \sigma_i(N, b, V^m) = v_*^t(N)$. Property MEFF asserts that all agents allocate all the utility completely. It is easy to check that the MBOI violates MEFF. Therefore, we consider an efficient normalization as follows.

[1]From now on we consider bounded multi-choice TU games, defined as those games (N, b, v) such that, there exists $K_v \in \mathbb{R}$ such that $v(\alpha) \leq K_v$ for all $\alpha \in B^N$. We adopt it to ensure that $v_*(T)$ is well-defined.

Definition 2: The **multi-choice efficient Banzhaf-Owen index (MEBOI, Liao [10])**, $\overline{\beta}$, is defined by

$$\begin{aligned} & \overline{\beta_i^t}(N, b, V^m) \\ = \ & \beta_i^t(N, b, V^m) + \tfrac{1}{|N|} \cdot \big[v_*^t(N) - \sum_{k \in N} \beta_k^t(N, b, V^m)\big] \end{aligned}$$

for all $(N, b, V^m) \in \Gamma$, for all $t \in \mathbb{N}_m$ and for all $i \in N$. Under the solution $\overline{\beta}$, all agents firstly receive their total marginal contributions related to maximal-utilities in each $S \subseteq N$, and further allocate the remaining utility equally.

Here we consider a normalization of the MBOI as follows.

Definition 3: The **multi-choice normalized Banzhaf-Owen index (MNBOI)**, $\overline{\eta}$, is defined by

$$\begin{aligned} & \overline{\eta_i^t}(N, b, V^m) \\ = \ & \frac{v_*^t(N)}{\sum\limits_{k \in N} \beta_k^t(N, b, V^m)} \cdot \beta_i^t(N, b, V^m) \end{aligned}$$

for all $(N, b, V^m) \in \Gamma^*$, for all $t \in \mathbb{N}_m$ and for all $i \in N$, where $\Gamma^* = \{(N, b, V^m) \in \Gamma \mid \sum_{i \in N} \beta_i^t(N, b, V^m) \neq 0 \text{ for all } t \in \mathbb{N}_m\}$. Under the notion of $\overline{\eta}$, all players allocate the maximal-utility of the grand coalition proportionally by applying the MBOI of all players.

Lemma 1: The MNBOI satisfies MEFF on Γ^*.

Proof: For all $(N, b, V^m) \in \Gamma^*$ and for all $t \in \mathbb{N}_m$,

$$\begin{aligned} & \sum_{i \in N} \overline{\eta_i^t}(N, b, V^m) \\ = \ & \sum_{i \in N} \Big[\frac{v_*^t(N)}{\sum\limits_{k \in N} \beta_k^t(N, b, V^m)} \cdot \beta_i^t(N, b, V^m)\Big] \\ = \ & \frac{v_*^t(N)}{\sum\limits_{k \in N} \beta_k^t(N, b, V^m)} \cdot \Big[\sum_{i \in N} \beta_i^t(N, b, V^m)\Big] \\ = \ & v_*^t(N). \end{aligned}$$

Thus, the MNBOI satisfies MEFF on Γ^*. ■

As we mention in Introduction, multicriteria analysis (also known multiattribute analysis, multi-objective analysis, and so on) is a notion of multiple criteria analysis that is concerned with conditions involving more than one aim to be optimized simultaneously. Multicriteria analysis has been applied in many areas, including engineering, politics, economics, logistics where efficient decisions need to be used in the presence of trade-offs among two or more aims. For example, minimizing cost while maximizing comfort while buying a central air conditioning system, and maximizing efficiency whilst minimizing energies consumption and emission of pollutants are examples of multicriteria efficient problems involving two and three aims respectively. In many situations, there can be more than three aims. On the other hand, each agent could be allowed to participate with infinite various activity decisions (levels, strategies) in real situations respectively. Therefore, we consider the framework of multicriteria multi-choice TU games in this paper.

Here we provide a brief application of multicriteria multi-choice TU games in the setting of "management". This kind of problem can be formulated as follows. Let $N = \{1, 2, \cdots, n\}$ be a set of all agents of a grand management system (N, b, V^m). The function v^t could be treated as an utility function which assigns to each level vector $\alpha = (\alpha_i)_{i \in N} \in B^N$ the worth that the agents can obtain when each agent i participates at operation strategy $\alpha_i \in B_i$ in the sub-management system (N, b, v^t). Modeled in this way, the grand management system (N, b, V^m) could be considered as a multicriteria multi-choice TU game, with

v^t being each characteristic function and B_i being the set of all operation strategies of the agent i. In the following sections, we would like to show that the MNBOI could provide "*optimal allocation mechanism*" among all agents, in the sense that this organization can get payoff from each combination of operation strategies of all agents under multicriteria situations and multi-choice behavior.

III. Reduction and axiomatic results

In this section, we show that there exists corresponding reduced game that could be adopted to analyze the MNBOI.

Liao [10] introduced reduced games and related consistency properties as follows. Let ψ be a solution, $(N, b, V^m) \in \Gamma$ and $S \subseteq N$. The **1-reduced game** $(S, b_S, V^{1,m}_{S,\psi})$ is defined by $V^{1,m}_{S,\psi} = (v^{1,t}_{S,\psi})_{t\in\mathbb{N}_m}$ and

$$v^{1,t}_{S,\psi}(\alpha) = \begin{cases} 0 & , \quad \alpha = 0_S, \\ \sum\limits_{Q\subseteq N\setminus S} \Big[v^t_*\big(A(\alpha)\cup Q\big) - \sum\limits_{i\in Q}\psi^t_i(N,b,V^m)\Big] & , \quad \text{otherwise.} \end{cases}$$

The 1-reduced game asserts that given a proposed payoff vector $\psi(N, b, V^m)$, the worth of a level vector α in the 1-reduced game $(S, b_S, V^{1,m}_{S,\psi})$ with respect to ψ and S is computed under the assumption that α can secure the cooperation of any subgroup Q of $N \setminus S$, provided each member of Q receives his component of $\psi(N, b, V^m)$. After these payments are made, what remains for α is the difference $v^t_*\big(A(\alpha) \cup Q\big) - -\sum\limits_{i\in Q} \psi^t_i(N, b, V^m)$ for all $t \in \mathbb{N}_m$. Summing behavior on the part of α involves finding the sum of the differences $v^t_*\big(A(\alpha)\cup Q\big) - -\sum\limits_{i\in Q}\psi^t_i(N, b, V^m)$ over all $Q \subseteq N \setminus S$. ψ satisfies **1-consistency (1CON)** if $\psi^t_i(S, b_S, V^{1,m}_{S,\psi}) = \psi^t_i(N, b, V^m)$ for all $(N, b, V^m) \in \Gamma$, for all $S \subseteq N$, for all $t \in \mathbb{N}_m$ and for all $i \in S$. Further, ψ satisfies **1-standard for games (1SG)** if $\psi(N, b, V^m) = \beta(N, b, V^m)$ for all $(N, b, V^m) \in \Gamma$ with $|N| \leq 2$.

The statement of the consistency could be found in Introduction. Property 1SG is a generalization of the two-person standardness axiom of Wei et al. [11].

It is easy to check that the index $\overline{\beta}$ and $\overline{\eta}$ violates 1CON. Therefore, Liao [10] introduced the *2-reduced game* as follows. Let ψ be a solution, $(N, b, V^m) \in \Gamma$ and $S \subseteq N$. The **2-reduced game** $(S, b_S, V^{2,m}_{S,\psi})$ is defined by $V^{2,m}_{S,\psi} = (v^{2,t}_{S,\psi})_{t\in\mathbb{N}_m}$ and

$$v^{2,t}_{S,\psi}(\alpha) = \begin{cases} 0 & , \quad \alpha = 0_S, \\ v^t_*(N) - \sum\limits_{i\in N\setminus S}\psi^t_i(N,b,V^m) & , \quad A(\alpha) = S, \\ \sum\limits_{Q\subseteq N\setminus S} \Big[v^t_*\big(A(\alpha)\cup Q\big) - \sum\limits_{i\in Q}\psi^t_i(N,b,V^m)\Big] & , \quad \text{otherwise.} \end{cases}$$

ψ satisfies **2-consistency (2CON)** if $\psi^t_i(S, b_S, V^{2,m}_{S,\psi}) = \psi^t_i(N, b, V^m)$ for all $(N, b, V^m) \in \Gamma$, for all $S \subseteq N$, for all $t \in \mathbb{N}_m$ and for all $i \in S$. Further, ψ satisfies **2-standard for games (2SG)** if $\psi(N, b, V^m) = \overline{\beta}(N, b, V^m)$ for all $(N, b, V^m) \in \Gamma$ with $|N| \leq 2$.

Remark 1: Liao [10] provided several axiomatic results as follows.

1) The MBOI and MEBOI satisfy 1CON and 2CON on Γ respectively.
2) On Γ, the MBOI is the only solution satisfying 1SG and 1CON.
3) On Γ, the MEBOI is the only solution satisfying 2SG and 2CON.

Unfortunately, it is also easy to see that $(S, b_S, V^m_{S,\psi})$ does not exist if $\sum_{i\in S}\beta^t_i(N, b, V^m) = 0$. Thus, we consider the **3-consistency** as follows. ψ satisfies **3-consistency (3CON)** if $(S, b_S, V^m_{S,\psi}) \in \Gamma^*$ for some $(N, b, V^m) \in \Gamma$ and for some $S \subseteq N$, it holds that $\psi^t_i(S, b_S, V^m_{S,\psi}) = \psi^t_i(N, b, V^m)$ for all $t \in \mathbb{N}_m$ and for all $i \in S$. Further, ψ satisfies **3-standard for games (3SG)** if $\psi(N, b, V^m) = \overline{\eta}(N, b, V^m)$ for all $(N, b, V^m) \in \Gamma$ with $|N| \leq 2$.

Next, we characterize the MNBOI by applying the properties of 3CON and 3SG.

Theorem 1:

1) The MNBOI satisfies 3CON on Γ^*.
2) If ψ satisfies 3SG and 3CON, then it also satisfies MEFF on Γ^*.
3) On Γ^*, the MNBOI is the only solution satisfying 3SG and 3CON.

Proof: To verify result 1, let $(N, b, V^m) \in \Gamma^*$ and $S \subseteq N$. If $|N| = 1$, then the proof is completed. Assume that $|N| \geq 2$, $S = \{i, j\}$ for some $i, j \in N$ and $(S, b_S, V^m_{S,\psi}) \in \Gamma^*$. Let $a^t = \frac{v^t(N)}{\sum_{k\in N}\beta^t_k(N,b,V^m)}$ and $a^t_S = \frac{(v^{2,t}_{S,x})_*(S)}{\sum_{k\in S}\beta_k(S,b_S,V^m_{S,\overline{\eta}})}$ for all $t \in \mathbb{N}_m$. So we have that

$$\begin{aligned} a^t_S &= \frac{(v^{2,t}_{S,\overline{\eta}})_*(S)}{\sum\limits_{k\in S}\beta_k(S,b_S,V^m_{S,\overline{\eta}})} \\ &= \frac{v^t_*(N) - \sum\limits_{i\in N\setminus S}\overline{\eta}^t_i(N,b,V^m)}{\sum\limits_{k\in S}\beta_k(S,b_S,V^m_{S,\overline{\eta}})} \\ &= \frac{\sum\limits_{i\in S}\overline{\eta}^t_i(N,b,V^m)}{\sum\limits_{k\in S}\beta_k(S,b_S,V^m_{S,\overline{\eta}})} \\ &= \frac{v^t(N)}{\sum\limits_{k\in N}\beta^t_k(N,b,V^m)} \\ &= a^t \end{aligned} \tag{1}$$

for all $t \in \mathbb{N}_m$. By applying the proof of Remark 1,

$$\beta^t_l(S, b_S, V^m_{S,\overline{\eta}}) = \beta^t_l(N, b, V^m) \tag{2}$$

for all $t \in \mathbb{N}_m$ and for all $l \in S$. By equations (1), (2) and definitions of $\overline{\eta}$ and $(S, b_S, V^{2,m}_{S,\overline{\beta}})$,

$$\begin{aligned} \overline{\beta^t_l}(S, b_S, V^{2,m}_{S,\overline{\beta}}) &= a^t_S \cdot \beta^t_l(S, b_S, V^m_{S,\overline{\eta}}) \\ &= a^t \cdot \beta^t_l(N, b, V^m) \\ &= \overline{\beta^t_l}(N, b, V^m). \end{aligned}$$

for all $t \in \mathbb{N}_m$ and for all $l \in S$. That is, $\overline{\beta}$ satisfies 3CON.

To prove result 2, suppose ψ satisfies 3SG and 3CON. Let $(N, b, V^m) \in \Gamma^*$ and $t \in \mathbb{N}_m$. If $|N| \leq 2$, it is trivial that ψ satisfies MEFF by 3SG. The case $|N| > 2$: Assume, on the contrary, that there exists $(N, b, V^m) \in \Gamma^*$ such that $\sum_{i\in N}\psi^t_i(N, b, V^m) \neq v^t_*(N)$. This means that there exist $i \in N$ and $j \in N$ such that $[v^t_*(N) - \sum_{k\in N\setminus\{i,j\}}\psi^t_k(N, b, V^m)] \neq [\psi^t_i(N, b, V^m) + \psi^t_j(N, b, V^m)]$. By 3CON and ψ satisfies MEFF for two-person games, this contradicts with

$$\begin{array}{rl} & \psi_i^t(N,b,V^m)+\psi_j^t(N,b,V^m) \\ = & \psi_i^t(\{i,j\},b_{\{i,j\}},v^{2,t}_{\{i,j\},\psi})+\psi_j^t(\{i,j\},b_{\{i,j\}},v^{2,t}_{\{i,j\},\psi}) \\ = & v_*^t(N)-\sum_{k\in N\setminus\{i,j\}}\psi_k^t(N,b,V^m). \end{array}$$

Hence ψ satisfies MEFF.

Next, we prove result 3. By result 1, the MNBOI satisfies 3CON. Clearly, the MNBOI satisfies 3SG. To prove uniqueness, suppose ψ satisfies 3CON and 3SG on Γ^*. By result 2, ψ satisfies MEFF on Γ^*. Let $(N,b,V^m)\in\Gamma^*$. We will complete the proof by induction on $|N|$. If $|N|\le 2$, it is trivial that $\psi(N,b,V^m)=\overline{\eta}(N,b,V^m)$ by 3SG. Assume that it holds if $|N|\le r-1$, $r\ge 3$. The case $|N|=r$: Let $i,j\in N$ with $i\ne j$ and $t\in\mathbb{N}_m$. By Definition 3, $\overline{\eta_k^t}(N,b,V^m)=\frac{v_*^t(N)}{\sum_{h\in N}\beta_h^t(N,b,V^m)}\cdot\beta_k^t(N,b,V^m)$ for all $k\in N$. Assume that $\alpha_k^t=\frac{\beta_k^t(N,b,V^m)}{\sum_{h\in N}\beta_h^t(N,b,V^m)}$ for all $k\in N$ and for all $t\in\mathbb{N}_m$. Therefore,

$$\begin{array}{rl} & \psi_i^t(N,b,V^m) \\ = & \psi_i^t\big(N\setminus\{j\},b_{N\setminus\{j\}},v^{2,t}_{N\setminus\{j\},\psi}\big) \\ & \textbf{(by 3CON of } \psi\textbf{)} \\ = & \overline{\eta_i^t}\big(N\setminus\{j\},b_{N\setminus\{j\}},v^{2,t}_{N\setminus\{j\},\psi}\big) \\ & \textbf{(by 3SG of } \psi\textbf{)} \\ = & \frac{\left(v^{2,t}_{N\setminus\{j\},\psi}\right)_*\left(N\setminus\{j\}\right)}{\sum_{k\in N\setminus\{j\}}\beta_k^t\left(N\setminus\{j\},b_{N\setminus\{j\}},v^{2,t}_{N\setminus\{j\},\psi}\right)} \\ & \times\,\beta_i^t\Big(N\setminus\{j\},b_{N\setminus\{j\}},v^{2,t}_{N\setminus\{j\},\psi}\Big) \\ = & \frac{v_*^t(N)-\psi_j^t(N,b,V^m)}{\sum_{k\in N\setminus\{j\}}\beta_k^t(N,b,V^m)}\cdot\eta_i^t(N,b,V^m) \\ & \textbf{(by equation (2))} \\ = & \frac{v_*^t(N)-\psi_j^t(N,b,V^m)}{-\beta_j^t(N,b,V^m)+\sum_{k\in N}\beta_k^t(N,b,V^m)}\cdot\beta_i^t(N,b,V^m). \end{array} \quad (3)$$

By equation (3),

$$\begin{array}{l} \psi_i^t(N,b,V^m)\cdot[1-\alpha_j^t]=[v_*^t(N)-\psi_j^t(N,b,V^m)]\cdot\alpha_j^t \\ \Longrightarrow \\ \sum_{i\in N}\psi_i^t(N,b,V^m)\cdot[1-\alpha_j^t] \\ =[v_*^t(N)-\psi_j^t(N,b,V^m)]\cdot\sum_{i\in N}\alpha_j^t \\ \Longrightarrow \\ v_*^t(N)\cdot[1-\alpha_j^t]=[v_*^t(N)-\psi_j^t(N,b,V^m)]\cdot 1 \\ \textbf{(by MEFF of } \psi\textbf{)} \\ \Longrightarrow \\ v_*^t(N)-v_*^t(N)\cdot\alpha_j^t=v_*^t(N)-\psi_j^t(N,b,V^m) \\ \Longrightarrow \\ \overline{\eta_j^t}(N,b,V^m)=\psi_j^t(N,b,V^m). \end{array}$$

The proof is completed. ■

The following examples are to show that each of the axioms adopted in Theorem 1 is logically independent of the remaining axioms.

Example 1: Define a solution ψ by for all $(N,b,V^m)\in\Gamma^*$, for all $t\in\mathbb{N}_m$ and for all $i\in N$, $\psi_i^t(N,b,V^m)=0$. On Γ^*, ψ satisfies 3CON, but it violates 3SG.

Example 2: Define a solution ψ by for all $(N,b,V^m)\in\Gamma^*$, for all $t\in\mathbb{N}_m$ and for all $i\in N$,

$$\psi_i^t(N,b,V^m)=\begin{cases}\overline{\eta_i^t}(N,b,V^m) & \text{if } |N|\le 2,\\ 0 & \text{otherwise.}\end{cases}$$

On Γ^*, ψ satisfies 3SG, but it violates 3CON.

IV. EXCESS FORMULATION AND DYNAMIC RESULTS

In this section, we adopt excess functions and reductions to propose dynamic processes for the MNBOI.

In order to establish consistency of the MNBOI, it will be useful to present alternative formulation for the MNBOI in terms of *excess*. Let $(N,b,V^m)\in\Gamma$, $S\subseteq N$ and x be a payoff vector in (N,b,V^m). Define that $x^t(S)=\sum_{i\in S}x_i^t$ for all $t\in\mathbb{N}_m$. The **excess** of a coalition $S\subseteq N$ at x is the real number

$$\begin{array}{l} e(S,V^m,x)=(e(S,v^t,x^t))_{t\in\mathbb{N}_m} \text{ and} \\ e(S,v^t,x^t)=v_*^t(S)-x^t(S). \end{array} \quad (4)$$

The value $e(S,v^t,x^t)$ can be treated as the **complaint** of coalition S when all agents receive their payoffs from x^t in (N,b,v^t).

Lemma 2: Let $(N,b,V^m)\in\Gamma^*$ and $x\in E(N,b,V^m)$. Then for all $i,j\in N$ and $t\in\mathbb{N}_m$,

$$\begin{array}{rl} & \sum_{S\subseteq N\setminus\{i,j\}}e(S\cup\{i\},v^t,\frac{x^t}{2^{|N|-1}a^t}) \\ = & \sum_{S\subseteq N\setminus\{i,j\}}e(S\cup\{j\},v^t,\frac{x^t}{2^{|N|-1}a^t}) \\ \Longleftrightarrow & \\ & x=\overline{\eta}(N,b,V^m), \end{array}$$

where $a^t=\frac{v^t(N)}{\sum_{k\in N}\beta_k^t(N,b,V^m)}$.

Proof: Let $(N,b,V^m)\in\Gamma^*$ and $x\in E(N,b,V^m)$. For all pairs $i,j\in N$ and $t\in\mathbb{N}_m$,

$$\begin{array}{rl} & \sum_{S\subseteq N\setminus\{i,j\}}e(S\cup\{i\},v^t,\frac{x^t}{2^{|N|-1}a^t}) \\ = & \sum_{S\subseteq N\setminus\{i,j\}}e(S\cup\{j\},v^t,\frac{x^t}{2^{|N|-1}a^t}) \\ \Longleftrightarrow & \\ & \sum_{S\subseteq N\setminus\{i,j\}}\left[v_*^t(S\cup\{i\})-\frac{x^t(S\cup\{i\})}{2^{|N|-1}a^t}\right] \\ = & \sum_{S\subseteq N\setminus\{i,j\}}\left[v_*^t(S\cup\{j\})-\frac{x^t(S\cup\{j\})}{2^{|N|-1}a^t}\right] \\ \Longleftrightarrow & \\ & \left[\sum_{S\subseteq N\setminus\{i,j\}}v_*^t(S\cup\{i\})\right]-\frac{2^{|N|-2}\cdot x_i^t}{2^{|N|-1}a^t} \\ = & \left[\sum_{S\subseteq N\setminus\{i,j\}}v_*^t(S\cup\{j\})\right]-\frac{2^{|N|-2}\cdot x_j^t}{2^{|N|-1}a^t} \\ \Longleftrightarrow & \\ & x_i^t-x_j^t \\ = & 2a^t\cdot\sum_{S\subseteq N\setminus\{i,j\}}\Big[v_*^t(S\cup\{i\})-v_*^t(S\cup\{j\})\Big]. \end{array} \quad (5)$$

By definition of $\overline{\eta}$,

$$\begin{array}{rl} & \overline{\eta_i^t}(N,b,V^m)-\overline{\eta_j^t}(N,b,V^m) \\ = & 2a^t\cdot\sum_{S\subseteq N\setminus\{i,j\}}\Big[v^t(S\cup\{i\})-v^t(S\cup\{j\})\Big] \end{array} \quad (6)$$

By equations (5) and (6),

$$x_i^t-x_j^t=\overline{\eta_i^t}(N,b,V^m)-\overline{\eta_j^t}(N,b,V^m).$$

Hence,

$$\sum_{j\ne i}\Big[x_i^t-x_j^t\Big]=\sum_{j\ne i}\Big[\overline{\eta_i^t}(N,b,V^m)-\overline{\eta_j^t}(N,b,V^m)\Big].$$

That is, $\Big(|N|-1\Big)\cdot x_i^t-\sum_{j\neq i}x_j^t=\Big(|N|-1\Big)\cdot\overline{\eta_i^t}(N,b,V^m)-\sum_{j\neq i}\overline{\eta_j^t}(N,b,V^m)$. Since $x\in E(N,b,V^m)$ and $\overline{\eta}$ satisfies MEFF, $|N|\cdot x_i^t-v_*^t(N)=|N|\cdot\overline{\eta_i^t}(N,b,V^m)-v_*^t(N)$. Therefore, $x_i^t=\overline{\eta_i^t}(N,b,V^m)$ for all $(N,b,V^m)\in\Gamma^*$, for all $i\in N$ and for all $t\in\mathbb{N}_m$. ■

In order to establish the dynamic processes of the MNBOI, we firstly define correction functions by means of excess functions. The correction functions are based on the idea that, each agent shortens the complaint relating to his own and others' non-participation, and adopts these regulations to correct the original payoff.

Definition 4: Let $(N,b,V^m)\in\Gamma$ and $i\in N$. The **correction function** is defined to be $f=(f^t)_{t\in\mathbb{N}_m}$, where $f^t=(f_i^t)_{i\in N}$ and $f_i^t:E(N,b,V^m)\to\mathbb{R}$ is define by

$$\begin{aligned}&f_i^t(x)\\ =\;&x_i^t+w\sum_{j\in N\setminus\{i\}}a^t\sum_{S\subseteq N\setminus\{i,j\}}\Big[e(S\cup\{i\},v^t,\tfrac{x^t}{2^{|N|-1}a^t})\\&\qquad-e(S\cup\{j\},v^t,\tfrac{x^t}{2^{|N|-1}a^t})\Big],\end{aligned}$$

where $a^t=\frac{v_*^t(N)}{\sum_{k\in N}\beta_k^t(N,b,V^m)}$ and $w\in\mathbb{R}$ is a fixed positive number, which reflects the assumption that agent i does not ask for full correction (when $w=1$) but only (usually) a fraction of it. Define $[x]^0=x,\ [x]^1=f([x]^0),\cdots,\ [x]^q=f([x]^{q-1})$ for all $q\in\mathbb{N}$.

Lemma 3: $f(x)\in E(N,b,V^m)$ for all $(N,b,V^m)\in\Gamma$ and for all $x\in E(N,b,V^m)$.

Proof: Let $(N,b,V^m)\in\Gamma$, $t\in\mathbb{N}_m$, $i,j\in N$, $x\in E(N,b,V^m)$ and $a^t=\frac{v_*^t(N)}{\sum_{k\in N}\beta_k^t(N,b,V^m)}$. Similar to equation (5),

$$\begin{aligned}&\sum_{j\in N\setminus\{i\}}a^t\sum_{Q\subseteq N\setminus\{i,j\}}\Big[e(Q\setminus\{j\},v^t,\tfrac{x^t}{2^{|N|-1}a^t})\\&\qquad-e(Q\setminus\{i\},v^t,\tfrac{x^t}{2^{|N|-1}a^t})\Big]\\ =\;&\sum_{j\in N\setminus\{i\}}\Big[\Big(a^t\sum_{Q\subseteq N\setminus\{i,j\}}\big[v^t(Q\setminus\{j\})-v^t(Q\setminus\{i\})\big]\Big)\\&\qquad-\tfrac{x_i^t}{2}+\tfrac{x_j^t}{2}\Big].\end{aligned}\tag{7}$$

By equations (6) and (7),

$$\begin{aligned}&\sum_{j\in N\setminus\{i\}}a^t\sum_{Q\subseteq N\setminus\{i,j\}}\Big[e(Q\setminus\{j\},v^t,\tfrac{x^t}{2^{|N|-1}a^t})\\&\qquad-e(Q\setminus\{i\},v^t,\tfrac{x^t}{2^{|N|-1}a^t})\Big]\\ =\;&\tfrac{1}{2}\sum_{j\in N\setminus\{i\}}\Big(\overline{\eta_i^t}(N,b,V^m)-\overline{\eta_j^t}(N,b,V^m)-x_i^t+x_j^t\Big)\\ =\;&\tfrac{1}{2}\Big((|N|-1)\big(\overline{\eta_i^t}(N,b,V^m)-x_i^t\big)\\&\qquad-\sum_{j\in N\setminus\{i\}}\overline{\eta_j^t}(N,b,V^m)+\sum_{j\in N\setminus\{i\}}x_j^t\Big)\\ =\;&\tfrac{1}{2}\Big(|N|\big(\overline{\eta_i^t}(N,b,V^m)-x_i^t\big)\\&\qquad-\sum_{j\in N}\overline{\eta_j^t}(N,b,V^m)+\sum_{j\in N}x_j^t\Big)\\ =\;&\tfrac{1}{2}\Big(|N|\big(\overline{\eta_i^t}(N,b,V^m)-x_i^t\big)-v_*^t(N)+v_*^t(N)\Big)\\&\Big(\textbf{by MEFF of }\overline{\eta}\textbf{,}\ x\in E(N,b,V^m)\Big)\\ =\;&\tfrac{|N|}{2}\Big(\overline{\eta_i^t}(N,b,V^m)-x_i^t\Big).\end{aligned}\tag{8}$$

Moreover,

$$\begin{aligned}&\sum_{i\in N}\sum_{j\in N\setminus\{i\}}a^t\sum_{Q\subseteq N\setminus\{i,j\}}\Big[e(Q\setminus\{j\},v^t,\tfrac{x^t}{2^{|N|-1}a^t})\\&\qquad-e(Q\setminus\{i\},v^t,\tfrac{x^t}{2^{|N|-1}a^t})\Big]\\ =\;&\sum_{i\in N}\tfrac{|N|}{2}\Big(\overline{\eta_i^t}(N,b,V^m)-x_i^t\Big)\\ =\;&\tfrac{|N|}{2}\Big(\sum_{i\in N}\overline{\eta_i^t}(N,b,V^m)-\sum_{i\in N}x_i^t\Big)\\ =\;&\tfrac{|N|}{2}\big(v_*^t(N)-v_*^t(N)\big)\\&\Big(\textbf{by MEFF of }\overline{\eta}\textbf{,}\ x\in E(N,b,V^m)\Big)\\ =\;&0.\end{aligned}\tag{9}$$

So we have that

$$\begin{aligned}&\sum_{i\in N}f_i^t(x)\\ =\;&\sum_{i\in N}x_i^t+w\sum_{i\in N}\sum_{j\in N\setminus\{i\}}a^t\\&\times\sum_{Q\subseteq N\setminus\{i,j\}}\Big[e(Q\setminus\{j\},v^t,\tfrac{x^t}{2^{|N|-1}a^t})\\&\qquad-e(Q\setminus\{i\},v^t,\tfrac{x^t}{2^{|N|-1}a^t})\Big]\\ =\;&v_*^t(N)+0\quad\Big(\textbf{by equation (9) and }x\in E(N,b,V^m)\Big)\\ =\;&v_*^t(N).\end{aligned}$$

Hence, $f(x)\in E(N,b,V^m)$ if $x\in E(N,b,V^m)$. ■

Theorem 2: Let $(N,b,V^m)\in\Gamma$. If $0<t<\frac{4}{|N|}$, then $\{[x]^q\}_{q=1}^{\infty}$ converges geometrically to $\overline{\eta}(N,b,V^m)$ for each $x\in E(N,b,V^m)$.

Proof: Let $(N,b,V^m)\in\Gamma$, $t\in\mathbb{N}_m$, $i\in N$ and $x\in E(N,b,V^m)$. By equation (8) and definition of f,

$$f_i^t(x)-x_i^t=w\cdot\frac{|N|}{2}\cdot\Big(\overline{\beta_i^t}(N,b,V^m)-x_i^t\Big).$$

Hence,

$$\begin{aligned}&\overline{\beta_i^t}(N,b,V^m)-f_i^t(x)\\ =\;&\overline{\beta_i^t}(N,b,V^m)-x_i^t+x_i^t-f_i^t(x)\\ =\;&\overline{\beta_i^t}(N,b,V^m)-x_i^t-w\cdot\tfrac{|N|}{2}\cdot(\overline{\beta_i^t}(N,b,V^m)-x_i^t)\\ =\;&\Big(1-w\cdot\tfrac{|N|}{2}\Big)\Big[\overline{\beta_i^t}(N,b,V^m)-x_i^t\Big].\end{aligned}$$

So, for all $q\in\mathbb{N}$,

$$\overline{\beta}(N,b,V^m)-[x]^q=\Big(1-w\cdot\frac{|N|}{2}\Big)^q\Big[\overline{\beta}(N,b,V^m)-x\Big].$$

If $0<w<\frac{4}{|N|}$, then $-1<\Big(1-w\cdot\frac{|N|}{2}\Big)<1$ and $\{[x]^q\}_{q=1}^{\infty}$ converges geometrically to $\overline{\beta}(N,b,V^m)$. ■

Inspired by Maschler and Owen [12], we will find a dynamic process under reductions.

Definition 5: Let ψ be a solution, $(N,b,V^m)\in\Gamma$, $S\subseteq N$ and $x\in E(N,b,V^m)$. The (x,ψ)**-reduced game** $(S,b_S,V_{\psi,S,x}^{r,m})$ is given by $V_{\psi,S,x}^{r,m}=(v_{\psi,S,x}^{r,t})_{t\in\mathbb{N}_m}$ and for all $T\subseteq S$,

$$v_{\psi,S,x}^{r,t}(\alpha)=\begin{cases}v_*^t(N)-\sum_{i\in N\setminus S}x_i^t & ,A(\alpha)=S,\\ v_{S,\psi}^{2,t}(\alpha) & ,\text{otherwise.}\end{cases}$$

Inspired by Maschler and Owen [12], we also define different correction function as follow. The **R-correction function** to be $g=(g^t)_{t\in\mathbb{N}_m}$, where $g^t=(g_i^t)_{i\in N}$ and $g_i^t:E(N,b,V^m)\to\mathbb{R}$ is define by

$$g_i^t(x)=x_i^t+w\sum_{k\in N\setminus\{i\}}\Big(\overline{\eta_i^t}(\{i,k\},v_{\overline{\eta},\{i,k\},x}^t)-x_i^t\Big).$$

Define $[\theta]^0 = x,\ [\theta]^1 = g([\theta]^0), \cdots,\ [\theta]^q = g([\theta]^{q-1})$ for all $q \in \mathbb{N}$.

Lemma 4: $g(x) \in E(N, b, V^m)$ for all $(N, b, V^m) \in \Gamma$ and for all $x \in E(N, b, V^m)$.

Proof: Let $(N, b, V^m) \in \Gamma$, $t \in \mathbb{N}_m$, $i, k \in N$ and $x \in E(N, b, V^m)$. Let $S = \{i, k\}$, by MEFF of $\overline{\eta}$ and Definition 5,

$$\overline{\eta_i^t}(S, b_S, V^{r,m}_{\overline{\eta},S,x}) + \overline{\eta_k^t}(S, b_S, V^{r,m}_{\overline{\eta},S,x}) = x_i^t + x_k^t.$$

By 3CON and 3SG of $\overline{\eta}$,

$$\begin{array}{rl} & \overline{\eta_i^t}(S, b_S, V^{r,m}_{\overline{\eta},S,x}) - \overline{\eta_k^t}(S, b_S, V^{r,m}_{\overline{\eta},S,x}) \\ = & \overline{\eta_i^t}(S, b_S, V^{2,m}_{S,\overline{\eta}}) - \overline{\eta_k^t}(S, b_S, V^{2,m}_{S,\overline{\eta}}) \\ = & \overline{\eta_i^t}(N, b, V^m) - \overline{\eta_k^t}(N, b, V^m). \end{array}$$

Therefore,

$$\begin{array}{rl} & 2 \cdot \Big[\overline{\eta_i^t}(S, b_S, V^{r,m}_{\overline{\eta},S,x}) - x_i^t\Big] \\ = & \overline{\eta_i^t}(N, b, V^m) - \overline{\eta_k^t}(N, b, V^m) - x_i^t + x_k^t. \end{array} \tag{10}$$

By definition of g and equation (10),

$$\begin{array}{rl} & g_i^t(x) \\ = & x_i^t + \frac{w}{2} \cdot \Big[\sum\limits_{k \in N \setminus \{i\}} \overline{\eta_i^t}(N, b, V^m) - \sum\limits_{k \in N \setminus \{i\}} x_i^t \\ & \quad - \sum\limits_{k \in N \setminus \{i\}} \overline{\eta_k^t}(N, b, V^m) + \sum\limits_{k \in N \setminus \{i\}} x_k^t \Big] \\ = & x_i^t + \frac{w}{2} \cdot \Big[\sum\limits_{k \in N \setminus \{i\}} \overline{\eta_i^t}(N, b, V^m) - \big(|N| - 1\big) x_i^t \\ & \quad - \sum\limits_{k \in N \setminus \{i\}} \overline{\eta_k^t}(N, b, v) + \big(v_*^t(N) - x_i^t\big) \Big] \\ = & x_i^t + \frac{w}{2} \cdot \Big[\big(|N| - 1\big) \overline{\eta_i^t}(N, b, V^m) - \big(|N| - 1\big) x_i^t \\ & \quad - \big(v_*^t(N) - \overline{\eta_i^t}(N, b, V^m)\big) + \big(v_*^t(N) - x_i^t\big) \Big] \\ = & x_i^t + \frac{|N| \cdot w}{2} \cdot \Big[\overline{\eta_i^t}(N, b, V^m) - x_i^t \Big]. \end{array} \tag{11}$$

So we have that

$$\begin{array}{rl} & \sum\limits_{i \in N} g_i^t(x) \\ = & \sum\limits_{i \in N} \Big[x_i^t + \frac{|N| \cdot w}{2} \cdot \big[\overline{\eta_i^t}(N, b, V^m) - x_i^t \big] \Big] \\ = & \sum\limits_{i \in N} x_i^t + \frac{|N| \cdot w}{2} \cdot \big[\sum\limits_{i \in N} \overline{\eta_i^t}(N, b, V^m) - \sum\limits_{i \in N} x_i^t \big] \\ = & v_*^t(N) + \frac{|N| \cdot w}{2} \cdot \big[v_*^t(N) - v_*^t(N) \big] \\ = & v_*^t(N). \end{array}$$

Thus, $g(x) \in E(N, b, V^m)$ for all $x \in E(N, b, V^m)$. ■

Theorem 3: Let $(N, b, V^m) \in \Gamma$. If $0 < \alpha < \frac{4}{|N|}$, then $\{[\theta]^q\}_{q=1}^{\infty}$ converges to $\overline{\eta}(N, b, V^m)$ for each $x \in E(N, b, V^m)$.

Proof: Let $(N, b, V^m) \in \Gamma$, $t \in \mathbb{N}_m$ and $x \in E(N, b, V^m)$. By equation (11), $g_i^t(x) = x_i^t + \frac{|N| \cdot w}{2} \cdot \Big[\overline{\eta_i^t}(N, b, V^m) - x_i^t\Big]$ for all $i \in N$. Therefore,

$$\big(1 - \frac{|N| \cdot w}{2}\big) \cdot \big[\overline{\eta_i^t}(N, b, V^m) - x_i^t\big] = \big[\overline{\eta_i^t}(N, b, V^m) - g_i^t(x)\big]$$

So, for all $q \in \mathbb{N}$,

$$\overline{\eta}(N, b, V^m) - [\theta]^q = \Big(1 - \frac{|N| \cdot w}{2}\Big)^q \Big[\overline{\eta}(N, b, V^m) - x\Big].$$

If $0 < w < \frac{4}{|N|}$, then $-1 < \Big(1 - \frac{|N| \cdot w}{2}\Big) < 1$ and $\{[\theta]^q\}_{q=1}^{\infty}$ converges to $\overline{\eta}(N, b, v)$ for all $(N, b, V^m) \in \Gamma$, for all $t \in \mathbb{N}_m$ and for all $i \in N$. ■

V. CONCLUSIONS

In this paper, we investigate the multi-choice normalized Banzhaf-Coleman index. Based on reduced game, several axiomatic results for the multi-choice normalized Banzhaf-Coleman index are proposed. By applying reductions and excess functions respectively, we also introduce alternative formulation and related dynamic processes for the multi-choice normalized Banzhaf-Coleman index. One should compare our results with related pre-existing results:

- In the frameworks of traditional TU games and multi-criteria multi-choice TU games, the multi-choice normalized Banzhaf-Coleman index is introduced initially.
- The idea of our correction functions in Definitions 4, 5 and related dynamic processes are based on that of Maschler and Owen's [12] dynamic results for the Shapley value [17]. The major difference is that our correction functions in Definition 4 is based on "excess functions", and Maschler and Owen's [12] correction function is based on "reductions".

These mentioned above raise one question:

- Whether there exist some more solutions, its extensions and related results in the framework of multicriteria multi-choice TU games.

To our knowledge, these issues are still open questions.

REFERENCES

[1] J.F. Banzhaf, "Weighted voting doesn't work: A mathematical analysis," *Rutgers Law Review*, vol. 19, pp. 317-343, 1965.
[2] R van den. Brink and G van der. Lann, "Axiomatizations of the normalized banzhaf value and the Shapley value," *Social Choice and Welfare*, vol. 15, pp. 567-582, 1998.
[3] J.S. Coleman, "Control of collectivities and the power of a collectivity to act," B.Lieberman, Ed., *Social Choice, Gordon and Breach*, London, U.K., 1971, pp. 269-300.
[4] P. Dubey and L.S. Shapley, "Mathematical properties of the Banzhaf power index," *Mathematics of Operations Research*, vol. 4, pp. 99-131, 1979.
[5] H. Haller, "Collusion properties of values," *International Journal of Game Theory*, vol. 23, pp. 261-281, 1994.
[6] Y.A. Hwang and Y.H. Liao, "The unit-level-core for multi-choice games: the replicated core for TU games," *Journal of Global Optimization*, vol. 47, pp. 161-171, 2010.
[7] E. Lehrer, "An axiomatization of the Banzhaf value," *International Journal of Game Theory*, vol. 17, pp. 89-99, 1988.
[8] Y.H. Liao, "The maximal equal allocation of nonseparable costs on multi-choice games," *Economics Bulletin*, vol. 3, pp. 1-8, 2008.
[9] Y.H. Liao, "The duplicate extension for the equal allocation of nonseparable costs," *Operational Research: An International Journal*, vol. 13, pp. 385-397, 2012.
[10] Y.H. Liao, "Power allocation rules under multicriteria situation," *Yugoslav Journal of Operations Research*, vol. 28, pp. 170-183, 2018.
[11] H.C. Wei, A.T. Li, W.N. Wang and Y.H. Liao, "Solutions and its axiomatic results under fuzzy behavior and multicriteria situations," *IAENG International Journal of Applied Mathematics*, vol. 49, pp. 612-617, 2019.
[12] M. Maschler and G. Owen, "The consistent Shapley value for hyperplane games," *International Journal of Game Theory*, vol. 18, pp. 389-407, 1989.
[13] H. Moulin, "The separability axiom and equal-sharing methods," *Journal of Economic Theory*, vol. 36, pp. 120-148, 1985.
[14] A van den. Nouweland, J. Potters, S. Tijs and J.M. Zarzuelo, "Core and related solution concepts for multi-choice games," *ZOR-Mathematical Methods of Operations Research*, vol. 41, pp. 289-311, 1995.
[15] G. Owen, "Multilinear extensions and the Banzhaf value," *Naval Research Logistics Quart*, vol. 22, pp. 741-750, 1975.
[16] G. Owen, "Characterization of the Banzhaf-Coleman index," *SIAM Journal on Applied Mathematics*, vol. 35, pp. 315-327, 1978.
[17] L.S Shapley, "A value for n-person game," in: Kuhn, H.W., Tucker, A.W.(Eds.), *Contributions to the Theory of Games* **II**, Princeton, 1953, pp. 307-317.
[18] R.E. Stearns, "Convergent transfer schemes for n-person games," *Trans. Amer. Math. Soc.*, vol. 134, pp. 449-459, 1968.