On the First Hitting Time for a Class of Piecewise Deterministic Markov Processes

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***Abstract*—In this paper, we introduce the renewal measure of the defective renewal sequence constituted by the x points of a class of piecewise deterministic Markov processes (PDMPs). And then we give the expression of the renewal measure. By these together with the strong Markov property of the process, the distribution on the first hitting time is explicitly obtained. Finally, two special cases are considered.**

***Index Terms*—First hitting time, Renewal measure, PDMPs, Sequence of x points.**

I. INTRODUCTION

PIECEWISE deterministic Markov processes (PDMPs) was firstly introduced by Davis (1984). Here, we will discuss a class of PDMPs that can be expressed as

$$U(t) = u + \int_0^t g(U(s))ds - \sum_{i=1}^{N(t)} Z_i, \tag{1}$$

where u denotes the initial value, $g : R \to R$ is a continuously differentiable Lipschitz continuous function, $\{N(t), t \geq 0\}$ is a Poisson process with parameter λ, and $\{Z_k, k \geq 1\}$ independent of $\{N(t), t \geq 0\}$, are positive, independent and identically distributed random variables with common density function p. Denote the sequence of the jump times by $\{S_n, n \geq 1\}$ with $S_0 = 0$.

As it can be seen from (1), between jump arrival epochs, the process follows a deterministic path, described by a measurable function $\phi(t, x)$ that satisfies

$$\begin{cases} \dfrac{d\phi(t,x)}{dt} = g(\phi(t,x)), & t > 0, \\ \phi(0,x) = x. \end{cases} \tag{2}$$

Due to these better properties PDMPs has been widely applied in different fields. In insurance literature, Dassios and Embrechts (1989) showed in general how to use the theory of PDMPs (see Davis (1984, 1993)) for solving insurance risk problems. The model (1) was discussed by many authors: such as Asmussen (1995, 2000) and Wang et al. (2003). From Dassios and Embrechts (1989) or Embrechts and Schmidli (1994), we can see that $\{U(t), t \geq 0\}$ is a piecewise deterministic Markov process taking values in R with extended generator A satisfying

$$Af(x) = g(x)\frac{d}{dx}f(x) + \lambda \int_0^\infty (f(x-y) - f(x))p(y)dy,$$

where f belongs to the domain DA of the generator A. Denote by $P(t, x, \Gamma)$ the transition function of the model (1.1). Throughout this paper, it is assumed that $P(t, x, \Gamma)$ has density function $p(t, x, y)$ for $y < \phi(t, x)$.

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Inspired by Wu et al. (2003), we study the first hitting time of the model (1). The organization of this paper is as follows. In section II, we introduce the renewal measure of the defective renewal sequence constituted by the x points. In section III, the expression of the renewal measure is derived. Furthermore, the distribution function on the first hitting time is obtained. In section IV, we consider the two special cases on $g(x)$.

II. THE PRELIMINARY LEMMAS

Let (Ω, F, P) be a complete probability space containing all objects defined at the following. We firstly define the sequence of the epochs that the process $\{U(t), t \geq 0\}$ reaches a fixed level x ($x \in R$) as the following: The first hitting time is defined by

$$T_1^x = \inf\{t > 0 : U(t) = x\} \quad (T_1^x = \infty \text{ if the set is empty}).$$

In general, for $k = 2, 3, ...$, recursively define

$$T_k^x = \inf\{t > T_{k-1}^x : U(t) = x\} \quad (T_k^x = \infty. \text{ if the set is empty})$$

As shown in Fig. 1. By convention, let $T_0^x = 0$.

For every $t > 0$, let

$$N_t^x = \sup\{k > 0, T_k^x \leq t\} \quad (N_t^x = 0 \text{ if the set is empty}).$$

We see that N_t^x is the number of x points before t (and including t). Therefore, $\{N_t^x, t \geq 0\}$ is a counting process and $N_\infty^x = \sup\{k > 0 : T_k^x < +\infty\}$ ($N_\infty^x = 0$ if the set is empty) is the total number of the x points of the process. Denote $F_t = \sigma\{U(s), s \leq t\}$, then each of $\{T_k^x, k \geq 1\}$ is F_t-stopping time. Set $P^x (x \in R)$ denote the probability of $\{U(t), t \geq 0\}$ with initial value x generated on (Ω, F_∞).

For $k \geq 1$,

$$S_k^x = \begin{cases} T_k^x - T_{k-1}^x, & T_{k-1}^x < \infty, \\ \infty, & \text{otherwise}, \end{cases}$$

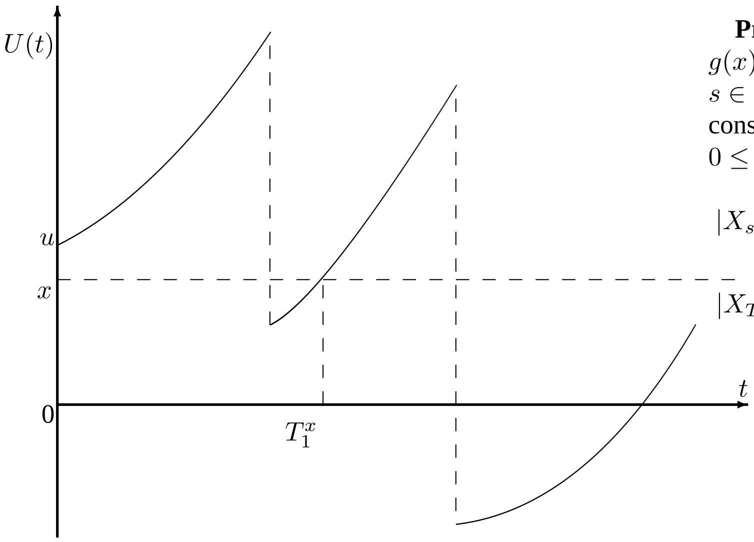

Fig. 1. The Waiting Time

Fig. 1. represents the waiting time from the time that the process reaches the level x for the $(k-1)$th time till the time that it reaches the level x again. For the process $\{U(t), t \geq 0\}$ has strong Markov property, we can prove that$\{S_k^x, k \geq 1\}$ are independent and that $\{S_k^x, k \geq 2\}$ is a sequence of i.i.d. random variables. Therefore, we have that $\{N_t^x, t \geq 0\}$ is a renewal process. The k-th renewal epoch is $T_k^x = \sum_{n=1}^{k} S_n^x$. Let F_x be the common distribution of $\{S_k^x, k \geq 2\}$, and F_u^x be the distribution of S_1^x. The renewal measure G_u^x is then defined by

$$G_u^x(t) = \sum_{k=1}^{\infty} P^u(T_k^x \leq t) = \sum_{k=1}^{\infty} F_u^x * F_x^{(k-1)*}(t). \quad (3)$$

So, we have

$$G_u^x(I) = \sum_{k=0}^{\infty} F_u^x * F_x^{k*}(I), \quad I \subset [0, \infty),$$

where I denotes a general interval. Further set $g_u^x(.)$ and $f_u^x(.)$ be respectively the density function of G_u^x and F_u^x if they exist.

III. Main result

We now indicate in detail how the renewal measure $G_u^x(.)$ can be used to express the first hitting time. The crucial is to obtain $G_u^x(.)$. We first present the following lemma, which can be used to get the expression of $G_u^x(.)$.

Lemma 3.1 Let X_t satisfy the ordinary differential equation

$$\frac{dX_t}{dt} = g(X_t), \text{ for any } t \in [0, T],$$

where $g(\cdot)$ be a continuously differentiable Lipschitz continuous function. If there exists $t^* \in [0, T]$, such that $X_{t^*} = x$, then we have

$$|X_s - X_0| \leq K_1 \cdot s, \quad \text{for any} \quad s \in [0, T].$$
$$|X_T - X_0 - g(x)T| \leq KK_1T^2,$$

where constant K_1 depends only on T and x and constant K depends on $g(\cdot)$.

Proof Without loss of generality we can assume that $g(x) > 0$. Note that $X_{t^*} = x$, then X_s is bounded in $s \in [0, T]$. Since $g(\cdot)$ is a continuous function, there exists a constant K_1 depending only on T and x, such that for any $0 \leq s \leq T$, $|g(X_s)| \leq K_1$. Hence, for any $s \in [0, T]$,

$$|X_s - X_0| = |\int_0^s g(X_l)dl| \leq \int_0^s |g(X_l)|dl \leq K_1 \cdot s.$$

$$\begin{aligned} |X_T - X_0 - g(x)T| &= |X_T - X_0 - g(X_{t^*})T| \\ &= |\int_0^T g(X_l) - g(X_{t^*})dl| \\ &\leq \int_0^T |g(X_l) - g(X_{t^*})|dl \\ &\leq \int_0^T K|X_l - X_{t^*}|dl \\ &\leq K\int_0^T K_1|l - t^*|dl \leq KK_1T^2. \end{aligned}$$

Theorem 3.2 Assume that $g(x) > 0$, we have
(1) When $\phi(s, u) > x$,

$$g_u^x(s) = \begin{cases} g(x)p(s, u, x) & \text{if} \quad s > 0, \\ 0 & \text{if} \quad s = 0; \end{cases} \quad (4)$$

(2) When $\phi(s, u) < x$,

$$g_u^x(s) = 0;$$

(3) When $\phi(s, u) = x$,
(a) If $s > 0$, then $G_u^x(t)$ will jump at time s, that is,

$$G_u^x(t) = 0, \text{ for } 0 \leq t < s, \quad G_u^x(s) = e^{-\lambda s}.$$

(b) If $s = 0$, i.e., $s = 0, u = x$, then

$$g_u^x(s) = 0.$$

proof (1) When $\phi(s, u) > x$, we have

$$\begin{aligned} g_u^x(s)ds &= \sum_{k=1}^{\infty} P^u(T_k^x \in (s, s+ds]) = \sum_{k=1}^{\infty} P^u(T_k^x \in ds) \\ &= \sum_{k=1}^{\infty} P^u(T_k^x \in ds, N(s, s+ds] = 0) \\ &+ \sum_{k=1}^{\infty} P^u(T_k^x \in ds, N(s, s+ds] \geq 1). \end{aligned}$$

By simply calculating, we obtain

$$\begin{aligned} &\sum_{k=1}^{\infty} P^u(T_k^x \in ds, N(s, s+ds] = 0) \\ = \; &P^u(U(s) < x, U(s+ds) \geq x, N(s, s+ds] = 0). \end{aligned}$$

Note that $T^x_{N^x_{s+ds}} \in (s, s+ds]$ and $\frac{dU(t)}{dt} = g(U(t))$, for any $t \in (s, s+ds]$. By Lemma 3.1, we have

$$|U(s+ds) - U(s) - g(x)ds)| \leq KK_1d^2s.$$

Hence, when $s > 0$, we get

$$\begin{aligned} &P^u(U(s) < x, U(s+ds) \geq x, N(s, s+ds] = 0) \\ = \; &P^u(x - g(x)ds + O(d^2s) \leq U(s) < x, N(s, s+ds] = 0) \\ = \; &P^u(x - g(x)ds + O(d^2s) \leq U(s) < x)e^{-\lambda ds} \\ = \; &g(x)p(s, u, x)ds + O(d^2s). \end{aligned}$$

and

$$\sum_{k=1}^{\infty} P^u(T_k^x \in ds, N(s, s+ds] \geq 1) = O(d^2 s).$$

Then,

$$g_u^x(s) = g(x)p(s,u,x). \tag{5}$$

When $s = 0$, we get

$$\begin{aligned} g_u^x(0)ds &= \sum_{k=1}^{\infty} P^u(T_k^x \in (0, ds]) = \sum_{k=1}^{\infty} P^u(T_k^x \in ds) \\ &= \sum_{k=1}^{\infty} P^u(T_k^x \in ds, N(0, ds] = 0) \\ &+ \sum_{k=1}^{\infty} P^u(T_k^x \in ds, N(0, ds] \geq 1) \\ &= O(d^2 s). \end{aligned}$$

Hence,

$$g_u^x(0) = 0. \tag{6}$$

Combining (5) with (6), we obtain (4) immediately.

(2) when $\phi(s,u) < x$, $P^u(T_1^x > s) = 1$. This follows that $g_u^x(s) = 0$.

(3) when $\phi(s,u) = x$. (a) If $0 \leq t < s$, then $\phi(t,u) < x$, thus $G_u^x(t) = 0$. If $t = s$, then there is no jump that occurs before s, that is

$$G_u^x(s) = P^u(T_1^x = s) = P^u(S_1 > s) = e^{-\lambda s}.$$

(b) When $s = 0, u = x$, we have

$$\begin{aligned} g_u^x(0)ds &= \sum_{k=1}^{\infty} P^u(T_k^x \in (0, ds]) \\ &= \sum_{k=1}^{\infty} P^u(T_k^x \in (0, ds], N(0, ds] \geq 1) = O(d^2 s). \end{aligned}$$

Hence, $g_u^x(0) = 0$. The proof is completed.

For the development of the paper, let us take some Laplace-Stieltjes ($L-S$) transforms, which can be expressed as the following:

$$\begin{aligned} \widehat{G}_u^x(v) &= \int_0^{\infty} e^{-vs} dG_u^x(s), \\ \widehat{G}_x^x(v) &= \int_0^{\infty} e^{-vs} dG_x^x(s), \\ \widehat{F}_u^x(v) &= \int_0^{\infty} e^{-vs} dF_u^x(s). \end{aligned}$$

Lemma 3.3 There exists a constant M, such that

$$\widehat{G}_x^x(v) = \int_0^{\infty} e^{-vs} dG_x^x(s) < 1,$$

for any $v \geq M$.

Proof By Theorem 3.2 and its proof, we have $G_x^x(0) = 0$ and $T_n^x \geq S_n$. Hence,

$$G_x^x(t) = \sum_{k=1}^{\infty} P^x(T_k^x \leq t) \leq \sum_{k=1}^{\infty} P^x(S_k \leq t) = \lambda t.$$

Note that

$$\begin{aligned} \widehat{G}_x^x(v) &= \int_0^{\infty} e^{-vs} dG_x^x(s) \\ &= G_x^x(s) e^{-vs}|_0^{\infty} + v \int_0^{\infty} e^{-vs} G_x^x(s) ds \\ &\leq v \int_0^{\infty} e^{-vs} \lambda s ds = \frac{\lambda}{v}. \end{aligned}$$

So that, when $M > \lambda$, we have

$$\widehat{G}_x^x(v) = \int_0^{\infty} e^{-vs} dG_x^x(s) \leq \frac{\lambda}{v} < 1, \text{ for any } v > M.$$

By Theorem 3.2, we see that the expression of the renewal measure G_u^x can be derived once u and x are fixed under the hypothesis that $g(x) > 0$. Next, we explicitly obtain the distribution on the first hitting time, which is expressed in terms of G_u^x.

Theorem 3.4 Assume that $g(x) > 0$, we have

$$F_u^x(s) = \sum_{n=0}^{\infty} (-1)^n (G_x^x)^{n*} * G_u^x(s).$$

Proof By (3) we have the following defective renewal equation:

$$G_u^x(t) = F_u^x(t) + F_u^x * G_x^x(t). \tag{7}$$

Taking $L-S$ transform on both sides of (7), we get

$$\widehat{F}_u^x(v) = \frac{\widehat{G}_u^x(v)}{1 + \widehat{G}_x^x(v)}.$$

From this together with Lemma 3.3, we obtain

$$\widehat{F}_u^x(v) = \sum_{n=0}^{\infty} (-1)^n \widehat{G}_u^x(v) [\widehat{G}_x^x(v)]^n.$$

Inverting $\widehat{F}_u^x(v)$, we have

$$F_u^x(s) = \sum_{n=0}^{\infty} (-1)^n (G_x^x)^{n*} * G_u^x(s).$$

Using the similar argument as Theorem 3.2, the expression on the first hitting time is explicitly obtained when $g(x) < 0$.

Theorem 3.5 Assume that $g(x) < 0$, we have

(1) When $\phi(s,u) > x$,

$$f_u^x(s) = \begin{cases} -g(x)p(s,u,x) & \text{if } s > 0, \\ 0 & \text{if } s = 0; \end{cases} \tag{8}$$

(2) when $\phi(s,u) < x$,

$$f_u^x(s) = 0;$$

(3) when $\phi(s,u) = x$,

(a) If $s > 0$, then $F_u^x(t)$ assigns mass $e^{-\lambda s}$ at point s.

(b) If $s = 0$, i.e., $s = 0, u = x$, then

$$f_u^x(s) = 0.$$

Proof: (1) When $\phi(s,u) > x$. Note that $g(\cdot)$ is a continuous function and $g(x) < 0$. So there exists a positive number $\varepsilon > 0$, such that $g(s) < 0$ for any $s \in (x-\varepsilon, x+\varepsilon)$. Hence, $U(t)$ decreases strictly at the neighborhood of x. The process $\{U(t), t \geq 0\}$ will not recover to the level x once it arrives the level x. So we have

$$\begin{aligned} f_u^x(s)ds &= P^u(T_1^x \in (s, s+ds]) \\ &= P^u(T_1^x \in ds, N(s, s+ds] = 0) \\ &+ P^u(T_1^x \in ds, N(s, s+ds] \geq 1) \\ &= P^u(U(s) > x, U(s+ds) \leq x, \\ &\quad N(s, s+ds] = 0) + O(d^2 s) \\ &= P^u(x < U(s) \leq x - g(x)ds + O(d^2 s), \\ &\quad N(s, s+ds] = 0) + O(d^2 s) \\ &= P^u(x - g(x)ds + O(d^2 s) \leq U(s) < x)e^{-\lambda ds} \\ &+ O(d^2 s) \\ &= -g(x)p(s, u, x)ds + O(d^2 s). \end{aligned}$$

It follows that (8) immediately.

(2) when $\phi(s, u) < x$. It is impossible that the process arrives the level x at time s. So

$$f_u^x(s) = 0;$$

(3) when $\phi(s, u) = x$. (a) If $s > 0$, Since $P^u(T_1^x = s) = P^u(S_1 > s) = e^{-\lambda s}$. Hence $F_u^x(t)$ assigns mass $e^{-\lambda s}$ at point s.
(b) If $s = 0$, i.e., $s = 0, u = x$, then the process is impossible to recover the level x. So $f_u^x(s) = 0$. This completes the proof.

■

In the case where $g(x) = 0$, the distribution on the first hitting time is given in the following Theorem.

Theorem 3.6 Assume that $g(x) = 0$, for any $s \geq 0$, we have

$$F_u^x(s) = \begin{cases} 1, & u = x, \\ 0, & \text{otherwise.} \end{cases}$$

IV. Examples

In this section, we will give two examples and obtain explicitly the solutions to the first hitting time in two special cases. From the theorems of section III, it is clear that the expression for the hitting time can be derived once u and x are fixed, in which the key is to find the transition density function $p(t, x, y)$ for $y < \phi(t, x)$. Therefore, we only need to give the transition function, which are showed in the following.

Example 4.1 Let $g(x) = c > 0$.

In this case, the model (1) reduced to the classical risk model

$$U(t) = u + ct - \sum_{i=1}^{N(t)} Z_i.$$

And the density function of the transition function $P(t, x, y) = P(x + ct - \sum_{i=1}^{N(t)} Z_i \leq y)$ is

$$p(t, x, y) = \begin{cases} \sum_{n=1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} p^{n*}(x + ct - y), & y < x + ct, \\ 0, & y > x + ct, \end{cases}$$

where $p^{n*}(x)$ denotes the n-fold convolution of $p(x)$ with itself. It is obvious that $P(t, x, y)$ assigns mass $e^{-\lambda t}$ at point $y = x + ct$.

Example 4.2 Let $g(x) = \delta x + c$.

In this case, the model (1) can be expressed as

$$U(t) = ue^{\delta t} + \frac{c(e^{\delta t} - 1)}{\delta} - \sum_{i=1}^{N(t)} Z_i e^{\delta(t - S_i)},$$

it is the classical risk model with constant interest. And the transition function is

$$\begin{aligned} P(t, x, y) &= P(xe^{\delta t} + \frac{c(e^{\delta t} - 1)}{\delta} - \sum_{i=1}^{N(t)} Z_i e^{\delta(t - S_i)} \leq y) \\ &= P(\sum_{i=1}^{N(t)} Z_i e^{\delta(t - S_i)} \geq xe^{\delta t} + \frac{c(e^{\delta t} - 1)}{\delta} - y) \\ &= 1 - P(\sum_{i=1}^{N(t)} Z_i e^{\delta(t - S_i)} < xe^{\delta t} + \frac{c(e^{\delta t} - 1)}{\delta} - y). \end{aligned}$$

Since, for any $x > 0$, we have

$$\begin{aligned} P & \quad (\sum_{i=1}^{N(t)} Z_i e^{\delta(t - S_i)} < x) \\ &= \sum_{n=0}^{\infty} P(\sum_{i=1}^{n} e^{\delta(t - S_i)} Z_i < x | N(t) = n) P(N(t) = n) \\ &= e^{-\lambda t} + \sum_{n=1}^{\infty} P(\sum_{i=1}^{n} e^{\delta(t - S_i)} Z_i < x | N(t) = n) \frac{e^{-\lambda t}(\lambda t)^n}{n!} \\ &= e^{-\lambda t} + \sum_{n=1}^{\infty} P(\sum_{i=1}^{n} e^{\delta(t - U_{i:n})} Z_i < x) \frac{e^{-\lambda t}(\lambda t)^n}{n!} \\ &= e^{-\lambda t} + \sum_{n=1}^{\infty} P(\sum_{i=1}^{n} e^{\delta(t - U_i)} Z_i < x) \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \end{aligned}$$

where $\{U_1, ..., U_n\}$ independent of $\{Z_i, i \geq 1\}$, are independent and identically distributed random variables with Uniform distribution $U(0, t)$, and $U_{1:n} \leq U_{2:n} \leq ... \leq U_{n:n}$ are the order statistics of $\{U_i : 1 \leq i \leq n\}$. Note that $\{e^{(t - U_k)} Z_k, k \geq 1\}$ are i.i.d., then

$$P(\sum_{i=1}^{n} e^{\delta(t - U_i)} Z_i < x) = F_t^{n*}(x),$$

where $F_t(x) = P(e^{\delta(t - U_1)} Z_1 \leq x) = \frac{1}{t} \int_0^t \int_0^{xe^{-\delta(t - v)}} p(y) dy dv$ is the common distribution function of $\{e^{\delta(t - U_k)} Z_k, k \geq 1\}$. Therefore

$$P(\sum_{i=1}^{N(t)} Z_i e^{\delta(t - S_i)} < x) = e^{-\lambda t} + \sum_{n=1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} F_t^{n*}(x).$$

It is obvious that random variable $\sum_{i=1}^{N(t)} Z_i e^{\delta(t - S_i)}$ assigns mass $e^{-\lambda t}$ at point zero and when $t > 0, x > 0$, it has density function $\sum_{n=1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} f_t^{n*}(x)$, where $f_t(x)$ is the density function of $F_t(x)$.

Hence, $P(t, x, y)$ assigns mass $e^{-\lambda t}$ at point $y = xe^{\delta t} + \frac{c(e^{\delta t} - 1)}{\delta}$, and its transition density function is given

$$p(t, x, y) = \begin{cases} \sum_{n=1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} f_t^{n*}(xe^{\delta t} + \frac{c(e^{\delta t} - 1)}{\delta} - y), \\ \qquad\qquad y < xe^{\delta t} + \frac{c(e^{\delta t} - 1)}{\delta}, \\ 0, \\ \qquad\qquad y > xe^{\delta t} + \frac{c(e^{\delta t} - 1)}{\delta}. \end{cases}$$

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