Two Proofs and a Corrected Procedure for the Generalized Exponential Function

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***Abstract*—In a previous article we developed the generalized exponential function as a new growth function. Here we complement that article in three ways: It is proven that any three points of quantities $q_3 > q_2 > q_1$ at times $t_3 > t_2 > t_1$ can be connected with a single generalized exponential function. We also prove that any two real quantities of logarithmic relative growth, y_i at time t_i and y_{i+1} at time t_{i+1}, can be connected with the generalized exponential function. Finally, we provide a corrected procedure for the generalized exponential function to convert a sequence of data points $y_i(t_i)$ into a segmented continuous curve $q(t)$.**

I. Introduction

Growth curves are monotonically increasing functions over time. In the case of negative growth, they can also be monotonically decreasing functions. The exponential function models constant instantaneous relative growth. In a previous article [1], we generalized the exponential function: The generalized function is defined by a linear relationship between a continuous quantity (rather than time) and logarithmic relative growth y. The corresponding formula is

$$y(q(t)) = \ln\left[q'(t)/q(t)\right] = a + b \cdot q(t), \qquad (1)$$

where $q'(t)/q(t)$ is instantaneous relative growth of a quantity q, t refers to time, a denotes initial logarithmic relative growth, and b is a shape parameter in terms of its sign, as well as a scaling parameter in terms of its magnitude. With the generalized exponential function, one can model sigmoid growth $(b < 0)$, (standard) exponential growth $(b = 0)$, and explosive growth $(b > 0)$, where the term "explosive growth" refers to a relative growth rate that increases with time. The resulting formula of generalized exponential growth for quantity as a function of time is:

$$q(t) = \begin{cases} -\dfrac{1}{b} \cdot \mathrm{Ei}^{(-1)}_{x>0}\begin{bmatrix} (t-t_C)\cdot\exp[a] \\ +\mathrm{Ei}[-b \cdot q_C] \end{bmatrix} & \text{for } b<0, \\ q_C \cdot \exp\left[(t-t_C)\cdot\exp[a]\right] & \text{for } b=0, \\ -\dfrac{1}{b} \cdot \mathrm{Ei}^{(-1)}_{x<0}\begin{bmatrix} (t-t_C)\cdot\exp[a] \\ +\mathrm{Ei}[-b \cdot q_C] \end{bmatrix} & \text{for } b>0, \end{cases} \qquad (2)$$

where $\mathrm{Ei}[-b \cdot q] = \int \left(\exp[-b \cdot q]/q\right) \mathrm{d}q$, i.e., an exponential integral, and q_C at t_C is a known, calibrating point. We addressed the problem of taking the inverse of $\mathrm{Ei}[x] = z_{Ei}$, which results in two functions that we defined as $\mathrm{Ei}^{(-1)}_{x>0}[z_{Ei}] = x$ or $\mathrm{Ei}^{(-1)}_{x<0}[z_{Ei}] = x$. For its numerical evaluation, we developed an indirect method.

In the case of sigmoid growth, the inflection point quantity is $-1/b$, which depends only on the parameter b. For $b > 0$ (explosive growth), one has the restriction

$$t < t_C - \frac{\mathrm{Ei}[-b \cdot q_C]}{\exp[a]}.$$

Finally, negative growth can be modeled by substituting $t - t_C$ with $t_C - t$.

Here we complement our article:

1) In Section V of [1], the two formulas (3) and (4) below were given for finding the parameters a and b, such that (2) goes through three points with $q_3 > q_2 > q_1$ at times $t_3 > t_2 > t_1$. We did, however, not provide a proof that a unique pair of parameters can always be found. Such a proof is given in Section I of the present article.
2) In Section X of [1], we derived a procedure for the generalized exponential function to convert a sequence of points $y_i(t_i)$ into a segmented continuous curve $q(t)$. In Section II of the present article, we provide a proof that any two real data points of logarithmic relative growth $y_i(t_i)$ and $y_{i+1}(t_{i+1})$ can always be connected uniquely with the generalized exponential function.
3) It turns out that in the procedure of the mentioned Section X, there was an omission that can cause an erroneous result. A corrected procedure (that is also simpler) to connect multiple segments of $y_i(t_i)$ is described in Section III of the present article.

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I. The Generalized Exponential Function Can Connect Any Three Points That Represent a Growth Phenomenon

Theorem 1. Any three points of quantities $q_3 > q_2 > q_1$ at times $t_3 > t_2 > t_1$ can be connected with a single generalized exponential function. In other words, for function (2) there is a unique combination of a and b, such that the function goes through the three points.

Proof. In Section 5 of [1], we used

$$\frac{\mathrm{Ei}[-b\cdot q_3]-\mathrm{Ei}[-b\cdot q_1]}{\mathrm{Ei}[-b\cdot q_2]-\mathrm{Ei}[-b\cdot q_1]}=\frac{t_3-t_1}{t_2-t_1}, \tag{3}$$

to find a numerical b indirectly. We have to show that a unique solution always exists. Once the parameter b is determined, the parameter a is uniquely determined by the following relationship:

$$a=\ln\left[\frac{\mathrm{Ei}[-b\cdot q_3]-\mathrm{Ei}[-b\cdot q_1]}{t_3-t_1}\right]. \tag{4}$$

Note that one could also use q_2 at t_2, instead of q_3 at t_3, or instead of q_1 at t_1; however, if there is any issue of numerical accuracy, then (4) is the best option. Rearranging (3) yields

$$\begin{aligned}&\left(\mathrm{Ei}[-b\cdot q_3]-\mathrm{Ei}[-b\cdot q_1]\right)\cdot(t_2-t_1)\\&=\left(\mathrm{Ei}[-b\cdot q_2]-\mathrm{Ei}[-b\cdot q_1]\right)\cdot(t_3-t_1)\end{aligned} \tag{5}$$

In accordance with page 176 in [2] and Section II in [1], the series representation of $\mathrm{Ei}(x)$ is:

$$\begin{aligned}&\mathrm{Ei}(x)=\gamma+\ln[|x|]\\&+\frac{x^1}{1!\cdot 1}+\frac{x^2}{2!\cdot 2}+\ldots+\frac{x^k}{k!\cdot k}+R(x,k),\end{aligned} \tag{6a}$$

with the remainder term

$$\begin{aligned}R(x,k)&=\frac{x^{k+1}}{(k+1)!\cdot(k+1)}+\frac{x^{k+2}}{(k+2)!\cdot(k+2)}\\&+\frac{x^{k+3}}{(k+3)!\cdot(k+3)}+\ldots .\end{aligned} \tag{6b}$$

Using the series representation of (6a) in (5), one gets

$$\begin{pmatrix}\gamma+\ln[|-b\cdot q_3|]+\frac{(-b\cdot q_3)^1}{1!\cdot 1}+\frac{(-b\cdot q_3)^2}{2!\cdot 2}+\ldots\\+\frac{(-b\cdot q_3)^k}{k!\cdot k}+R(-b\cdot q_3,k)\\-\gamma-\ln[|-b\cdot q_1|]-\frac{(-b\cdot q_1)^1}{1!\cdot 1}-\frac{(-b\cdot q_1)^2}{2!\cdot 2}-\ldots\\-\frac{(-b\cdot q_1)^k}{k!\cdot k}-R(-b\cdot q_1,k)\end{pmatrix}\cdot(t_2-t_1)=$$

$$=\begin{pmatrix}\gamma+\ln[|-b\cdot q_2|]+\frac{(-b\cdot q_2)^1}{1!\cdot 1}+\frac{(-b\cdot q_2)^2}{2!\cdot 2}+\ldots\\+\frac{(-b\cdot q_2)^k}{k!\cdot k}+R(-b\cdot q_2,k)\\-\gamma-\ln[|-b\cdot q_1|]-\frac{(-b\cdot q_1)^1}{1!\cdot 1}-\frac{(-b\cdot q_1)^2}{2!\cdot 2}-\ldots\\-\frac{(-b\cdot q_1)^k}{k!\cdot k}-R(-b\cdot q_1,k)\end{pmatrix}\cdot(t_3-t_1),$$

which can be converted to

$$\begin{aligned}&(t_2-t_1)\cdot\ln\left[\frac{q_3}{q_1}\right]-(t_3-t_1)\cdot\ln\left[\frac{q_2}{q_1}\right]\\&+b^1\cdot\begin{pmatrix}(t_2-t_1)\cdot\frac{(-q_3)^1-(-q_1)^1}{1!\cdot 1}-\\(t_3-t_1)\cdot\frac{(-q_2)^1-(-q_1)^1}{1!\cdot 1}\end{pmatrix}\\&+b^2\cdot\begin{pmatrix}(t_2-t_1)\cdot\frac{(-q_3)^2-(-q_1)^2}{2!\cdot 2}-\\(t_3-t_1)\cdot\frac{(-q_2)^2-(-q_1)^2}{2!\cdot 2}\end{pmatrix}+\ldots\\&+b^k\cdot\begin{pmatrix}(t_2-t_1)\cdot\frac{(-q_3)^k-(-q_1)^k}{k!\cdot k}-\\(t_3-t_1)\cdot\frac{(-q_2)^k-(-q_1)^k}{k!\cdot k}\end{pmatrix}\\&-(t_3-t_2)\cdot R(-b\cdot q_1,k)\\&-(t_3-t_1)\cdot R(-b\cdot q_2,k)\\&+(t_2-t_1)\cdot R(-b\cdot q_3,k)=0.\end{aligned} \tag{7}$$

The three remainder terms $R(-b\cdot q_1,k)$, $R(-b\cdot q_2,k)$, and $R(-b\cdot q_3,k)$ are joined as

$$\begin{aligned}R_{total}&=-(t_3-t_2)\cdot R(-b\cdot q_1,k)\\&-(t_3-t_1)\cdot R(-b\cdot q_2,k)\\&+(t_2-t_1)\cdot R(-b\cdot q_3,k).\end{aligned}$$

According to (6b)

$$\begin{aligned}&R(-b\cdot q_i,k)\\&=\frac{(-b\cdot q_i)^{k+1}}{(k+1)!\cdot(k+1)}+\frac{(-b\cdot q_i)^{k+2}}{(k+2)!\cdot(k+2)}\\&+\frac{(-b\cdot q_i)^{k+3}}{(k+3)!\cdot(k+3)}+\ldots .\end{aligned}$$

Consequently, R_{total} can be expressed as

$$R_{total} = -(t_3 - t_2) \cdot \left(\frac{(-b \cdot q_1)^{k+1}}{(k+1)! \cdot (k+1)} + \frac{(-b \cdot q_1)^{k+2}}{(k+2)! \cdot (k+2)} + \frac{(-b \cdot q_1)^{k+3}}{(k+3)! \cdot (k+3)} + \ldots \right)$$
$$-(t_3 - t_1) \cdot \left(\frac{(-b \cdot q_2)^{k+1}}{(k+1)! \cdot (k+1)} + \frac{(-b \cdot q_2)^{k+2}}{(k+2)! \cdot (k+2)} + \frac{(-b \cdot q_2)^{k+3}}{(k+3)! \cdot (k+3)} + \ldots \right)$$
$$+(t_2 - t_1) \cdot \left(\frac{(-b \cdot q_3)^{k+1}}{(k+1)! \cdot (k+1)} + \frac{(-b \cdot q_3)^{k+2}}{(k+2)! \cdot (k+2)} + \frac{(-b \cdot q_3)^{k+3}}{(k+3)! \cdot (k+3)} + \ldots \right),$$

which can be converted to

$$R_{total} = \frac{(t_2 - t_1) \cdot (-b \cdot q_3)^{k+1}}{(k+1)! \cdot (k+1)} - \frac{(t_3 - t_1) \cdot (-b \cdot q_2)^{k+1}}{(k+1)! \cdot (k+1)}$$
$$- \frac{(t_3 - t_2) \cdot (-b \cdot q_1)^{k+1}}{(k+1)! \cdot (k+1)} + \frac{(t_2 - t_1) \cdot (-b \cdot q_3)^{k+2}}{(k+2)! \cdot (k+2)}$$
$$- \frac{(t_3 - t_1) \cdot (-b \cdot q_2)^{k+2}}{(k+2)! \cdot (k+2)} - \frac{(t_3 - t_2) \cdot (-b \cdot q_1)^{k+2}}{(k+2)! \cdot (k+2)}$$
$$+ \frac{(t_2 - t_1) \cdot (-b \cdot q_3)^{k+3}}{(k+3)! \cdot (k+3)} - \frac{(t_3 - t_1) \cdot (-b \cdot q_2)^{k+3}}{(k+3)! \cdot (k+3)}$$
$$- \frac{(t_3 - t_2) \cdot (-b \cdot q_1)^{k+3}}{(k+3)! \cdot (k+3)} + \ldots,$$

and further to

$$R_{total} = \frac{(-b)^{k+1}}{(k+1)! \cdot (k+1)} \cdot \left((t_2 - t_1) \cdot q_3^{k+1} - (t_3 - t_1) \cdot q_2^{k+1} - (t_3 - t_2) \cdot q_1^{k+1} \right)$$
$$+ \frac{(-b)^{k+2}}{(k+2)! \cdot (k+2)} \cdot \left((t_2 - t_1) \cdot q_3^{k+2} - (t_3 - t_1) \cdot q_2^{k+2} - (t_3 - t_2) \cdot q_1^{k+2} \right)$$
$$+ \frac{(-b)^{k+3}}{(k+3)! \cdot (k+3)} \cdot \left((t_2 - t_1) \cdot q_3^{k+3} - (t_3 - t_1) \cdot q_2^{k+3} - (t_3 - t_2) \cdot q_1^{k+3} \right) + \ldots . \quad (8)$$

Thus, R_{total} is the sum of an infinite number of terms of the form

$$\frac{(-b)^j}{j! \cdot j} \cdot \left((t_2 - t_1) \cdot q_3^j - (t_3 - t_1) \cdot q_2^j - (t_3 - t_2) \cdot q_1^j \right). \quad (9)$$

For $j \to \infty$, (9) converges to zero, because the denominator $j! \cdot j$ increasingly dominates the expression with increasing j. Thus, for increasing k in (8), the remainder sum R_{total} becomes increasingly smaller, and by choosing k sufficiently large, one can achieve any desired accuracy.

Returning to (7), for given sufficiently large k, one calculates now without the remainder sum R_{total}:

$$(t_2 - t_1) \cdot \ln\left[\frac{q_3}{q_1}\right] - (t_3 - t_1) \cdot \ln\left[\frac{q_2}{q_1}\right]$$
$$+ \begin{pmatrix} (t_2 - t_1) \cdot \frac{(-q_3)^1 - (-q_1)^1}{1! \cdot 1} - \\ (t_3 - t_1) \cdot \frac{(-q_2)^1 - (-q_1)^1}{1! \cdot 1} \end{pmatrix} \cdot b^1 + \ldots \quad (10)$$
$$+ \begin{pmatrix} (t_2 - t_1) \cdot \frac{(-q_3)^k - (-q_1)^k}{k! \cdot k} - \\ (t_3 - t_1) \cdot \frac{(-q_2)^k - (-q_1)^k}{k! \cdot k} \end{pmatrix} \cdot b^k = 0,$$

which represents a polynomial in b of degree k. For example, for $k = 3$ (which generally is too low for sufficient accuracy), (10) becomes

$$c_0 + c_1 \cdot b + c_2 \cdot b^2 + c_3 \cdot b^3 = 0,$$

where

$$c_0 = (t_2 - t_1) \cdot \ln\left[\frac{q_3}{q_1}\right] - (t_3 - t_1) \cdot \ln\left[\frac{q_2}{q_1}\right],$$
$$c_1 = q_2 \cdot (t_3 - t_1) - q_3 \cdot (t_2 - t_1) - q_1 \cdot (t_3 - t_2),$$
$$c_2 = \frac{1}{4} \cdot \begin{pmatrix} q_3^2 \cdot (t_2 - t_1) - q_2^2 \cdot (t_3 - t_1) \\ +q_1^2 \cdot (t_3 - t_2) \end{pmatrix},$$
$$c_3 = \frac{1}{18} \cdot \begin{pmatrix} q_2^3 \cdot (t_3 - t_1) - q_3^3 \cdot (t_2 - t_1) \\ -q_1^3 \cdot (t_3 - t_2) \end{pmatrix}.$$

According to the “fundamental theorem of algebra”, for uneven k there will necessarily be at least one real solution. Furthermore, as shown already, the accuracy is controlled by the magnitude of k.

To show that there is a single solution, (3) is again re-arranged:

$$\mathrm{Ei}[-b \cdot q_1] \cdot (t_3 - t_2) + \mathrm{Ei}[-b \cdot q_2] \cdot (t_3 - t_1) + \mathrm{Ei}[-b \cdot q_3] \cdot (t_2 - t_1) = 0. \quad (11)$$

All q and t are fixed, and $\mathrm{Ei}[x]$ is a true function with only one function value for any x. Consequently, if there is one b, for which (11) is true, there cannot exist another b, for which it is also true. □

Example:

Choose three arbitrary but increasing quantities, at three arbitrary but increasing time points, such as $q = 0.02 < 0.93 < 5.45$ at $t = -3.6 < 24.1 < 211.4$. According to Theorem 1, there must be a solution for a and b. With (3), one finds $b = -1.7449$, and subsequently with (4) $a = -0.87639$. Using these a and b, with q_C at t_C being any one of the three points, one can verify with function (2) that the other two points lie correctly on the determined generalized exponential function.

II. Any two points of logarithmic relative growth as a function of time can be connected with the generalized exponential function

Theorem 2. Any two real points of logarithmic relative growth $y_1(t_1)$ and $y_2(t_2)$ can be connected with the generalized exponential function non-linearly as $y(t)$.

Proof. The underlying functional relationship of the generalized exponential function according to (1) is

$$y(q) = a + b \cdot q, \tag{12a}$$

so that

$$q(y) = \frac{y-a}{b}. \tag{12b}$$

Any two points $y_1(q_1)$ and $y_2(q_2)$ can be uniquely connected with a straight line, and the coefficients a and b calculated as

$$a = \frac{q_2 \cdot y_1 - q_1 \cdot y_2}{q_2 - q_1}, \quad b = \frac{y_2 - y_1}{q_2 - q_1}.$$

Thus, there is no restriction for calculating $q_2(y_2)$ for given $q_1(y_1)$. Furthermore, as derived in Section II in [1], the basic formula of the generalized exponential function

$$q'(t)/q(t) = \exp\left[a + b \cdot q(t)\right]$$

converts to

$$t_2 = \begin{cases} t_1 + \dfrac{\mathrm{Ei}[-b \cdot q_2] - \mathrm{Ei}[-b \cdot q_1]}{\exp[a]} & \text{for } b \neq 0, \\ t_1 + \dfrac{\ln[q_2] - \ln[q_1]}{\exp[a]} & \text{for } b = 0. \end{cases} \tag{13}$$

For $q_1 \neq 0$, $q_2 \neq 0$, and $q_2 > q_1$, there are no restrictions for the functional relationship between q_2 and t_2. Thus, there is no restriction for calculating $t_2(q_2)$ for given $t_1(q_1)$. As there was also no restriction to calculate $q_2(y_2)$ for given $q_1(y_1)$, there is no restriction for calculating directly $t_2(y_2)$ for given $t_1(y_1)$.

Furthermore, each relationship is given with a true function (i.e., one dependent value for any independent value), so that there can only be a single result. □

Example:

We choose two arbitrary points of y_1 at t_1, and y_2 at t_2, such as 0.74 at -3.1, and -0.67 at 0.8. According to Theorem 2, there must be a solution for a and b that lets us determine the non-linear $y(t)$. For numerical calculation, one can use the procedure described in the next section (which works also for a single segment). It requires a calibrating quantity q_C at either t_1 or t_2. If $q_C = 55.2$ at $t_1 = -3.1$, then one gets $a = 0.743229$, $b = -5.85005 \cdot 10^{-5}$, and $q_2 = 24{,}156$. For verification, one can use (12a) to calculate $y_1 = 0.743229 - 5.85005 \cdot 10^{-5} \cdot 55.2 = 0.74$, and $y_2 = 0.743229 - 5.85005 \cdot 10^{-5} \cdot 24{,}156 = -0.67$. Using next (13) for $b \neq 0$:

$$t_2 = -3.1 + \frac{1}{\exp[0.743229]} \cdot \begin{pmatrix} \mathrm{Ei}\left[5.85005 \cdot 10^{-5} \cdot 24{,}156\right] - \\ \mathrm{Ei}\left[5.85005 \cdot 10^{-5} \cdot 55.2\right] \end{pmatrix} = 0.8.$$

To calculate the whole function of $y(t)$, the recommended procedure is to calculate numerically the inverse function $t(y)$ (see Section IV in [1]), and subsequently invert the values. Inserting (12b) into (13) for $b \neq 0$:

$$t(y) = t_1 + \frac{\mathrm{Ei}[a - y] - \mathrm{Ei}[-b \cdot q_1]}{\exp[a]}.$$

Inserting the parameters from the example and simplifying results in $t(y) = -0.648376 + 0.475576 \cdot \mathrm{Ei}[0.743229 - y]$.

III. Corrected procedure to convert a sequence of points $y_i(t_i)$ into a segmented, continuous growth curve $q(t)$ for the generalized exponential function

A procedure for the generalized exponential function to convert a sequence of points $y_i(t_i)$ into a segmented continuous growth curve $q(t)$ was first derived in Section X of [1]. There is, however, an omission in the described procedure that may cause an erroneous result, causing the function of $y(t)$ to become discontinuous between some segments (in the function $q(t)$ the discontinuity may not even be obvious): It turns out that (28) in [1], which is (15) below, has two solutions, and one has to choose the right one. Furthermore, subsequently one can simplify the procedure. Thus, we describe a corrected algorithm. It works also for a single segment.

Assume that there are two known points of logarithmic relative growth, $y_i(t_i)$ and $y_{i+1}(t_{i+1})$, where i is an index for the segment. The solution is easy for standard exponential growth: one has $a = y_i = y_{i+1}$. Given known q_C, one uses directly (2) for $b = 0$ to calculate any q.

For generalized exponential growth, excluding standard exponential growth $\left(y_{i+1} \neq y_i\right)$, corresponding to (13), one has

$$t_{i+1} = t_{C,i} + \frac{\mathrm{Ei}\left[-b \cdot q_{i+1}\right] - \mathrm{Ei}\left[-b \cdot q_{C,i}\right]}{\exp[a]}.$$

Rearranging, with $b_i \cdot q_i = a_i - y_i$ at t_i, and $b_i \cdot q_{i+1} = a_i - y_{i+1}$ at t_{i+1}, there are two equations for y_i at t_i, and y_{i+1} at t_{i+1}, respectively:

$$\mathrm{Ei}\left[a_i - y_i\right] = \left(t_i - t_{C,i}\right) \cdot \exp\left[a_i\right] + \mathrm{Ei}\left[-b_i \cdot q_{C,i}\right],$$
$$\mathrm{Ei}\left[a_i - y_{i+1}\right] = \left(t_{i+1} - t_{C,i}\right) \cdot \exp\left[a_i\right] + \mathrm{Ei}\left[-b_i \cdot q_{C,i}\right].$$

With $t_{C,i} = t_i$, the first equation simplifies to $a_i = y_i - b_i \cdot q_{C,i}$. Substituting a_i in the second equation with $y_i - b_i \cdot q_{C,i}$ yields for $b_i \neq 0$:

$$\begin{aligned} &\mathrm{Ei}\left[y_i - y_{i+1} - b_i \cdot q_{C,i}\right] - \mathrm{Ei}\left[-b_i \cdot q_{C,i}\right] \\ &-\left(t_{i+1} - t_i\right) \cdot \exp\left[y_i - b_i \cdot q_{C,i}\right] = 0. \end{aligned} \tag{14}$$

The variables b_i and $q_{C,i}$ occur always together in (14) as $-b_i \cdot q_{C,i}$. Thus, we substitute in (14)

$$r_i = -b_i \cdot q_{C,i},$$

and rearrange into a form that works better to find r_i numerically with a root-finding algorithm (chapter 9 in [3]):

$$\frac{\mathrm{Ei}\left[y_i - y_{i+1} + r_i\right] - \mathrm{Ei}\left[r_i\right]}{\exp\left[y_i + r_i\right]} - \left(t_{i+1} - t_i\right) = 0, \tag{15}$$

where $y_{i+1} \neq y_i$, $y_i - y_{i+1} + r_i \neq 0$, and $r_i \neq 0$. With (15), one can determine r_i indirectly, for example with the "FindRoot" function in *Mathematica* 12 (https://www.wolfram.com/mathematica/). There are always two solutions, a negative and a positive one. This is because the exponential integral function $\mathrm{Ei}[x]$ goes to minus infinity at $x = 0$ from the left, as well as from the right (see Figure 1 in [1]); consequently, one has $\mathrm{Ei}\left[y_i - y_{i+1} + r_i\right] > \mathrm{Ei}\left[r_i\right]$ on the left function arm with $r_i < 0$, and again on the right function arm with $r_i > 0$. The distinction for finding the right solution for all $q > 0$ is as follows:

$$\begin{aligned} &\text{for } y_{i+1}\left(t_{i+1}\right) < y_i\left(t_i\right) \\ &\Rightarrow y_{i+1}\left(q_{i+1}\right) < y_i\left(q_i\right) \Rightarrow b_i < 0 \text{ (sigmoid)} \\ &\Rightarrow r_i > 0, \\ \\ &\text{for } y_{i+1}\left(t_{i+1}\right) > y_i\left(t_i\right) \\ &\Rightarrow y_{i+1}\left(q_{i+1}\right) > y_i\left(q_i\right) \Rightarrow b_i > 0 \text{ (explosive)} \\ &\Rightarrow r_i < 0. \end{aligned} \tag{16}$$

The first inequality can be observed from the data. The second inequality is implicit from the first one in a continuously increasing (or decreasing) growth curve, with $q_{i+1} > q_i$ (or $q_{i+1} < q_i$) for $t_{i+1} > t_i$. The third inequality follows from (1), and the last inequality from $r_i = -b_i \cdot q_{C,i}$, with $q_{C,i}$ always being positive (using the wrong solution for r_i would imply that one can work with negative quantities). Consequently, if $y_{i+1} < y_i$ one has to search for $r_i > 0$, and if $y_{i+1} > y_i$ for $r_i < 0$. Recommended starting values for the "FindRoot" procedure are 5 for $r_i > 0$, and -5 for $r_i < 0$.

Next, the initial logarithmic relative growth for each segment i is calculated as

$$a_i = y_i + r_i, \tag{17}$$

where y_i for the last i is not used (there is one less parameter of a_i and r_i than of y_i). Now, all r_i and a_i are known. The remaining unknown variables are linearly related in the following way:

$$\begin{aligned} y_i &= a_i + b_i \cdot q_i, \\ y_{i+1} &= a_i + b_i \cdot q_{i+1}, \end{aligned} \tag{18}$$

for i from 1 to $n-1$, where n refers to the number of data points. In this way, (18) represents a system of $2 \cdot n - 2$ linear equations, where all y_i, y_{i+1}, and a_i are known. If in addition a single q_i (or q_{i+1}) at t_i (or t_{i+1}) is known, then there are also $2 \cdot n - 2$ unknown variables of b and the remaining q, and the system can easily be solved (in *Mathematica* with the "NSolve" function).

Given all parameters a_i, b_i, and $q_{C,i}$, one can calculate the continuous curve with (2). According to Theorem 2, there is always a continuous function of $q(t)$ for all segments. Note, however, that while $y(t)$ is continuous at the limits between segments, $y'(t)$ is generally not (i.e., there is a "knot" between segments).

Example:

Using the example from Figure 5 in [1], there are four data points of $y_i(t_i)$ for three segments: $-2.76(9)$, $-2.77(10)$, $-2.86(11)$, and $-3.20(12)$. Furthermore, one known time-quantity point is $q_{C,2} = 1.41$ at $t_{C,2} = 10$. Applying (14), one gets:

$$\frac{\mathrm{Ei}\left[-2.76 - (-2.77) + r_1\right] - \mathrm{Ei}\left[r_1\right]}{\exp\left[-2.76 + r_1\right]} - (10 - 9) = 0$$
$$\Rightarrow \frac{\mathrm{Ei}\left[r_1 + 0.01\right] - \mathrm{Ei}\left[r_1\right]}{\exp\left[r_1 - 2.76\right]} = 1,$$

which results in either $r_1 = 0.1538$ or $r_1 = -0.004868$ for the first segment with "FindRoot" in *Mathematica* 12. According to (16), one has $y_{i+1} = -2.77 < y_i = -2.76$, and consequently one chooses the positive $r_1 = 0.1538$ as the correct solution. Using the subsequent data points, one calculates $r_2 = 1.458$, and $r_3 = 6.893$. Next, one calculates with (17): $a_1 = -2.76 + 0.1538 = -2.606$, and in the same way $a_2 = -1.312$, and $a_3 = 4.033$. The linear system of equations, according to (18), becomes

$$
\begin{aligned}
-2.76 &= -2.606 + b_1 \cdot q_1, \\
-2.77 &= -1.312 + b_2 \cdot 1.41, \\
-2.86 &= 4.033 + b_3 \cdot q_3, \\
-2.77 &= -2.606 + b_1 \cdot 1.41, \\
-2.86 &= -1.312 + b_2 \cdot q_3, \\
-3.20 &= 4.033 + b_3 \cdot q_4,
\end{aligned}
$$

which results in $b_1 = 0.116$, $b_2 = -1.034$, $b_3 = -4.604$, $q_1 = 1.32$, $q_3 = 1.50$, and $q_4 = 1.57$. All necessary parameters are known now, and the whole growth curve $q(t)$ can be calculated with (2).

References

[1] M. Ricker, D. von Rosen, "A generalization of the exponential function to model growth", *IAENG International Journal of Applied Mathematics*, vol. 48(2), pp. 152-167, 2018.

[2] A. Jeffrey, H.H. Dai, *Handbook of Mathematical Formulas and Integrals*, 4th edition. Amsterdam: Elsevier, 2008.

[3] W.H. Press, S.A. Teukolsky, W.T. Vetterling, B.P. Flannery, *Numerical Recipes in C++: The Art of Scientific Computing*, 2nd edition. Cambridge: Cambridge University Press, 2005.