Review of the Index Theorem: Applications in Mathematical Physic and Engineering Sciences

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Abstract—**The Index theorem demonstrated at first, in the middle of the last century, has overthrown the world of fundamental mathematics , then that of theoretical physics. The question that mathematicians were asked was, is the analytic index of a Fredholm operator topological? The physicists then discovered the topological nature of quantum field theory, introducing the concept of supersymmetry: The index of the Dirac operator appears in the partition function of a supersymmetric field theory. In engineering science topological theories also have a meaning. Gabriel Kron introduces matrix algebra, then the theory of tensors to the study of networks and the study electrical machines. In his Thesis, A. Kaufmann gives a rigorous mathematical reading of this. The vector spaces of nodes, branches and meshes are defined. In the last part of this paper, we show how the analytical index as well as other invariants (the invariants developed by Kron) can be rigorously found from the properties of linear algebra.**

Index Terms—**Index theorem, Supersymmetry, Heat kernel, K-theory.**

I. INTRODUCTION

PHYSICS has known in history several mutations, which have approached it from the mathematical world. Newton's theory goes hand in hand with the development of the analysis "The calculus of the students." Einstein's theory shows that differential geometry and tensor calculus describe a world of the macroscopic matter very well. General relativity, unifies apparently distinct entities: space-time and matter. At the beginning of the twentieth century, Topology becomes a new branch of mathematics. The most surprising are the fact that it is now a branch of the quantum fields theory (QFT): the topological fields theory (TQFT. At the origin of this is the index theorem of Atiyah-Singer. The first proof of this theorem was given in the early sixties. Others demonstrations will follow usign heat equation. Witten will give another proof, thanks to the concept, for the moment theoretical, of supersymmetry. In this paper, we propose to revisit this theorem as well as some of these many applications to the topological fields theories. We also want to stress that the topology and the search for invariants also takes all sense in engineering science, this will be mentioned in the last part of this article, dedicated to the method of Kron.

II. THE ANALYTIC INDEX OF A FREDHOLM OPERATOR

One problem that the analysts must solve is the follows: Given a Fredholm operator L between two complex vector bundles E and F over a compact manifold X (that mean an operator where kernel and co-kernel of finite dimension), Is there existence and uniqueness of the problem: $L(v) = w$, with $v \in \Gamma(E)$, $w \in \Gamma(F)$ realized. To do this; one is led to transform the Problem by the addition of constraints: part of these constraints, on the side of the kernel (create a **trouble of unicity**) they are injectivity constraints :

$$L(v) = w \Leftrightarrow L(v + \sum_{i=1}^{k} a_i v_k) = w \tag{1}$$

So the unicity is not respected if these constraints not vanish: On the side of the image, we have constraints of surjectivity: Let w_1, ..., w_l l sections of F such as classes $[w_1]$,...$[w_l]$ constitute a basis of $coker(L) = F/Im(L)$ then, if $[w] = \sum_{i=1}^{l} b_i[w_i]$ not vanish, that create a **trouble of existence**

$$\forall w \in F, w \in Im(L) \Leftrightarrow [w] = 0 \Leftrightarrow b_1 = ...b_l = 0 \tag{2}$$

All of This justifies the definition of the analytical index:

$$Ind_a(L) = dim(ker(L) - dim(Coker(L)) \tag{3}$$

Denotes the analytical index of operator L

III. TOPOLOGICAL CHARACTER OF INDEX, INDEX THEOREM

The index theorem states that the analytic index of a Fredholm operator L has a topological nature: this theorem, first demonstrated by Atiyah and Singer [1] in the second half of the twentieth century, bridges between problems of an analytical nature (index of a differential operator) that can be calculated by considering topological invariants. We can define an topological index, $Ind_t(L)$ so

$$Ind_a(L) = Ind_t(L) \tag{4}$$

In the following we will motivate this remark from some simple examples. We will see what this theorem says, in the case where the operator is a linear map between two vector spaces of finite dimension or not.

IV. TOY MODEL: INDEX THEOREM IN LINEAR ALGEBRA

We begin with the more simplest case that is linear algebra in finite dimension then, before addressing the most interesting case, that of compact manifolds, we study what can happen in infinite dimension.

A. Index of linear map between two vector spaces in finite dimension

Let, E and F two vector spaces of finite dimension. then the formula for the index can be found thanks to the formula of the rank:

$$\begin{aligned} Ind_a(L) = dim(ker(L) - dim(Coker(L)) = \\ dim(ker(L) - (dim(F) - dim(Im(L)) = \\ dim(E) - dim(F) \end{aligned} \tag{5}$$

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We find that the index is necessarily constant and reveals in some way the topological nature of this index: the difference in dimension between the initial and target the vector space is a topological invariant. Another property reflects the underlying topological nature of this analytical index. A small deformation of a non-bijective linear mapping can make it bijective without Changing its index. for example, in this simple case, the dimensions of the source spaces and target space are identical: We can, for example, consider the endomorphisms f And f_ε of $\mathbb{R}^2$ whose respective matrices are given by:
$M_f = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ et $M_{f_\varepsilon} = \begin{pmatrix} 1+\varepsilon & 3 \\ 2 & 6+\varepsilon \end{pmatrix}$
It is easy to notice That these two operators have the same index (null).

B. Index of linear map between two vector spaces in infinite dimension

The problem is more omplex, first of all, in infinite dimension, we can construct operators that are not Frehdolm. For this reason, the analytical index cannot always be calculated. For example, the endomorphism of $\mathcal{C}^\infty(\mathbb{R})$ which to f associates $Exp(x).f$ is not Fredholm.(The polynomial space of infinite dimension is not in the image of this endomorphism). Let us give a non-trivial example of a calculable analytic index. We take $\mathcal{C}^\infty(\mathbb{R})$, the infinitely differentiable functions and the derivation operator. Its kernels: the set of constant functions is of dimension 1, let us determine its cokernel: Any continuous function have a primitive, it is easy to deduce that the derivation operator realizes a surjective map on the space of infinitely differentiable functions. In other words, the cokernel is reduced to 0. Then, the analytical index of this operator is 1. With a hindsight, by development on The de Rham complex of the manifold of real numbers, the alternating sum of the Betti numbers in this case is again 1, that is the Euler Poincaré characteristic of $\mathbb{R}$ so: $Ind_a(\frac{d}{dx}) = \chi(\mathbb{R})$=1. Give another example by restricting to periodic 2π functions, and the derivation operator again: the kernel always has one dimension, but now the cokernel has also one dimension (proof left to the reader). ... So, the action of the periodicity on $\mathbb{R}$ amounts to considering the space S^1 and $Ind_a(\frac{d}{d\theta}) = \chi(S^1)$=0

We can conclude this paragraph by insisting that the nature of index operator is not only analytic. Topology has its role to play, the analytical index can be calculated by using topological invariants. One understands why Gelfand asked this question which was first solved by Atiyah and Singer in The early sixties. Taking the example of finite dimensional vector spaces, we can give another face of the index theorem. its link with supersymmetric field theory.

C. Index and superspaces

Return to the previous example, E= $\mathbb{C}^n$, F= $\mathbb{C}^m$, we can now build a new space using the tensor product $E \otimes F$. If f denote linear map between us with matrix P and f^* is. We can build a new space using the tensor product, in this, the matrix $\begin{pmatrix} 0_{n,n} & P^* \\ P & 0_{m,m} \end{pmatrix}$ is an endomorphism that exchanges E and F we will say that E is the space of Bosons, F that of fermions. This endomorphism can be seen as a square root of endomorphism with matrix: $\begin{pmatrix} P^*P & 0_{n,m} \\ 0_{m,n} & PP^* \end{pmatrix}$ with, PP^* and P^*P: selfadjoint In physics, the first matrix denote the Dirac operator, and it is the square root of the second matrix: the Laplacian (Hamiltonian) The eigenvalues of PP^* and P^*P are positive (self-adjoint operator). It has two heat equations (bosonic aspect), easily resolvable:

$$\begin{aligned}(\frac{d}{dt} + P^*P)u_1(t) = 0 \\ (\frac{d}{dt} + PP^*)u_2(t) = 0\end{aligned} \tag{6}$$

there solutions are $u_1(t) = e^{-tP^*P}u_1(0)$ and $u_1(t) = e^{-tPP^*}u_2(0)$
The non-zero eigenvalues of PP^* and P^*P are identical. they represent the exited states (in quantum mechanics). We show that the kernel of P^*P (resp. P^*P) are those of P (resp. P^*), They represent non-exited states or ground states. We can then define the "super-trace" of P^2:

$$\begin{aligned}Str(e^{-tP^2}) = Tr(e^{-tP^*P}) - Tr(e^{-tPP^*}) = \\ Tr(e^{-tP)} - Tr(e^{-tP^*}) = \\ dim(ker(f)) - dim(coker(f)) = Ind_a(f)\end{aligned} \tag{7}$$

Solve the heat equation on (E, P^2), gives the spectrum of the selfadjoint operator P^2; Let$P : E \to F$, P^* be adjoint, the embedding of E in the super-space $E \otimes F$ gives a natural self-adjoint: $PP^* + P^*P$ whose in the square root, P and its adjoint exchange E (bosons) and F (fermions). the index of P select the ground states of the supersymmetric system so constructed.

D. Summary:Index equivalences

We have seen a first equivalence: the index theorem connects a problem of analysis to a problem of topology. And now, In a supersymmetric world, only the non-exited states contribute to the super partition function of the quantum theory considered. These ground states are the signatures of the topology, given by Index theorem. We have the "triangle" : Analysis-topology-Physic

ANALYSIS

$\nearrow\swarrow$ **INDEX** $\nwarrow\searrow$

TOPOLOGY $\rightleftarrows$ **PHYSIC**

After this nice introduction, the reader is prepared if he wishes, to read the following developments. We now present in the next paragraph, some indications on the differents proofs of the theorem.

V. Index theorem demonstration

We have seen in the first paragraph how to define the analytical index of a Fredholm operator. In very easy cases we have been able to show that this index has an interpretation Regarding topological invariant. The demonstration in the general case is very complex even for compact manifolds.

We give the general framework in which the index theorem has been demonstrated Let E, F two vector bundle on a compact manifold X. We call order m differential operator P on X the linear application between space of sections $\Gamma(E)$ on $\Gamma(F)$. In local coordinate we have:

$$P = \sum_{|\alpha|\leq m} A^{\alpha}(x)\frac{\partial^{|\alpha|}}{\partial x^{\alpha}} \tag{8}$$

The symbol of a differential operator in a manifold (Riemannian), $S(X)$ being the unit sphere in the cotangent, is the mapping: $\sigma(P) : \pi^*(E) \to \pi^*(F)$, where π is the projection from $S(X)$ to X. in local coordinates:

$$\sigma_{\xi}(P) = \sum_{|\alpha|=m} A^{\alpha}(x)\xi^{|\alpha|} \tag{9}$$

definition 1: analytic index

A differential operator is elliptic if and only if its symbol is a fiber-to-fiber isomorphism. We prove that an elliptic operator is a Fredholm operator, so we can also define its analytic index as before, and it is an integer.
For this same operator, it is possible to define an index which captures only the topological content of the manifold. The symbol is an element of the compact K-theory of T$^*(X)$:$\sigma_{\xi}(P) \in K_{cpt}(T^*X)$

Some elements of K theory

Recall some K-theory tools; We get $\Phi(X)$, the set of isomorphic classes of vectorial bundle on X a compact Riemannian manifold. In order not to weigh down the notation, we confuse a bundle E with its isomorphic classes usually $\dot{E}$). the following construction called symmetrization will generalize the construction of the set of integers from positive integers. We consider the equivalence relation on $\Phi(X)$:

$$\begin{aligned}(E_1, F_1) \sim (E_2, F_2) \Leftrightarrow \exists G, H \in \Phi(X)^2 :\\ (E_1 \oplus G, F_1 \oplus G) = (E_2 \oplus H, F_2 \oplus H)\end{aligned} \tag{10}$$

By definition, the $K-theory$ group of X is the quotient set of $\Phi(X)$ by the equivalence relation defined above:

$$K(X) = \Phi(X)/\sim \tag{11}$$

We denote $[E]-[F]$ or $d([E],[F])$, (d for difference) the class of an element of $K(X)$, we have just defined the "formal difference" of two vector bundles, and a new group the K-theory group of isomorphic classes of Vector bundle on X. We note also that, if the space X is reduced to a point, the difference of the two bundles, element of the K-theory of the point is, in fact, that of two vector spaces: That is to say the **difference of the dimensions** of these vector spaces. The reduced K-theory recalled later, teaches us that the topological index of the linear map between E and F (formal difference between (dimension of) vector spaces) is indeed the analytic index found in **the toy model for vector spaces.**

Properties

1) **P1**: An important result is that any element $[E]-[F] \in K(X)$ can be represented as the difference $[V]-[\theta_N]$ where θ_N represents a trivial bundle. (because by increasing the size of the fibers one can trivialize any vector bundle).
2) **P2**: If θ_n, θ_p are two trivial vector bundle with respective ranks n and p, we have:

$$[E]-[\theta_n] = [E]-[\theta_n] \Leftrightarrow \exists q \in \mathbb{R}, E\oplus\theta_{n+q} \sim F\oplus\theta_{p+q} \tag{12}$$

K theory relative

This theory is crucial to find the situation of elliptic operators, we need to define a **K-theory relative** that is to say groups $K(X,Y)$, $Y \subset X$ defined as follows: Let E, F two bundles and α an isomorphism from $E|_Y$ to $F|_Y$, the equivalence relation is defined by: $(E_1, F_1, \alpha) \sim (E_2, F_2, \alpha')$:

$$\begin{aligned}\exists G, H \in \Phi(X)^2 : (E_1 \oplus G, F_1 \oplus G, \alpha \oplus Id|_G) =\\ (E_2 \oplus H, F_2 \oplus H, \alpha' \oplus Id|_H)\end{aligned} \tag{13}$$

Note $d([E],[F],\ alpha)$ an element of the group of relative $K-theory$. We notice then that this is equivalent to defining a K "pointed" theory denoted $\tilde{K}$, if we divide X by Y because then Y is brought back to a point:

$$K(X,Y) = K(X/Y,*) = \tilde{K}(X/Y) \tag{14}$$

This will be applied to the theory of elliptic operators by taking the group of relative K-theory $K(B(X),S(X))$; if X is a compact manifold, $B(X)$ the sub-bundle in ball units of T^*X, $S(X)$ the sub-bundle into sphere units, it is easy to notice that then:
$K(B(X),S(X)) \approx \mathrm{K}_{cpt}(T^*X) = K(T^*X,\infty)$, Thus for an elliptic operator D with symbol $\sigma(D)$ between two vector bundles E, F
we have: $d([E],[F],\sigma(D))) \in K(B(X),S(X))$

definition 2: topological index

Constructing a topological index [2]: Consider a compact differential manifold X then j an embedding of X into a $\mathbb{R}^n$, this is always possible by a famous Whitney theorem. Let N be a tubular neighborhood of $j(X)$, then TN is a tubular neighborhood of $dj(TX) \in \mathrm{T}\mathbb{R}^n$ we have the diagram:

$$\begin{tikzcd} \pi^*(N\oplus N) \arrow[r] \arrow[d] & N\oplus N \arrow[d] \\ TX \arrow[r, "\pi"] & X \end{tikzcd} \tag{15}$$

The following identifications are then manufactured by: $N \oplus N = N \oplus iN = N \otimes_{\mathbb{R}} \mathbb{C}$
Hence, TN above TX is identified with $\pi^*(N \oplus N) = \pi^*(N \otimes_{\mathbb{R}} \mathbb{C})$
We then have the **Thom homomorphism**:

$$K(TX) \to K(TN) \tag{16}$$

But TN is an open set of $\mathbb{R}^n$, so we have the natural morphism of **excision**:

$$K(TN) \to K(T\mathbb{R}^n) \tag{17}$$

The composition of these two arrows furnishes the morphism:

$$j_! : K(TX) \to K(T\mathbb{R}^n) \tag{18}$$

Finally, since $T(\mathbb{R}^n)$ is of even dimension, one can invoke the isomorphism of Thom by means of a complexification or what amounts to the same **periodicity of Bott** $T\mathbb{R}^n$ compactified is the sphere of dimension $2n$ on then the natural arrow:

$$i_! : K(pt) \to K(T\mathbb{R}^n) \tag{19}$$

Finally, the topological index is the composition:

$$i_!^{-1} \circ j_! \tag{20}$$

Al fine, we show that the topological index can be expressed in terms of **characteristic classes** in particular, the character of Chern

If $\sigma(P)$ denotes the symbol of the operator P, the topological index is the quantity:

$$ind_t(P) = (-1)^n ch([\sigma(P)]).td(TX \otimes \mathbb{C})[TX] \tag{21}$$

Index theorem

$$ind_t(P) = ind_a(P) \tag{22}$$

A. Initial Proof philosophy

The initial proof is based on the construction of two embeddings:
$j_!$ And $i_!$ In a $\mathbb{R}^N$: for N well chosen, then the two arrows:

Arrow 1: $K_{cpt}(T^*X) \xrightarrow{j_!} K_{cpt}(T\mathbb{R}^N)$

Arrow 2: $K_{cpt}(T\mathbb{R}^N) \xrightarrow{i_!^{-1}} K_{cpt}(pt) \simeq \mathbb{Z}$

This demonstration, initial proof given by Atiyah and Singer, is based on the Thom isomorphism and the Bott periodicity in K-theory.

B. Heat kernel approach

Another direction to demonstrate the index theorem is to evoke the heat kernel for the Laplacian operator on a compact Riemannian manifold. Solve the heat equation in euclidian space is easy, however, its resolution in any Riemann manifold is not : it uses the development of the heat kernel. From a mathematical point of view, this method was developed by Bott and Patodi at the end of the sixties. It depicts the square root of classical geometry from the Clifford algebras and the Dirac operator. This method was put back to the day after the introduction in the mid-seventies of supersymmetry. The method developed by Bott and Patodi was taken up by Bismut [3] and Azencott, who demonstrates the theorem by stochastic methods.

1) Heat equation on $\mathbb{R}^n$: Let Δ euclidian Laplacian, the heat equation, is given by:

$$(\partial_t + \Delta_x)k_t(x, y) = 0 \tag{23}$$

solve it his easy, the solution is:

$$k_t(x, y) = \frac{1}{\sqrt[n]{4\pi t}} exp(-\frac{\|x - y\|^2}{4t}) \tag{24}$$

2) Heat equation on a manifold M: Let M a manifold of dimension n, E vectorial bundle on M, Δ, a Laplacian on M, we can consider more generally, P a selfadjoint operator from $\Gamma(E)$ to $\Gamma(E)$ then, heat equation is given by:

$$(\partial_t + \Delta)u_t = 0 \tag{25}$$

solve it is not easy, search solution $u_t(x) = e^{-\Delta t}u(x)$ in integral form:

$$u_t(x) = \int_M K_t(x, y)u(y)dy \tag{26}$$

where K_t is the heat kernel, also solution of (11), little calculation give:

$$K_t(x, y) = \sum_{k=1}^{\infty} e^{-\lambda_k t} u_k(x) \otimes u_k^*(y) \tag{27}$$

$(u_k)_{k=1}\infty$ de $L^2(E)$: orthonormal basis of eigenvector associated to eigenvalues, $0 \leq \lambda_1 \leq .... \leq \lambda_k \to \infty$. we can define his trace:

$$tr(e^{-t\Delta}) = \int_M trace(K_t(x, x)dx = \sum_{k\geq 0} e^{-\lambda_k t} \tag{28}$$

3) supergeometry: Using the supergeometric language, Clifford allows to define a square root of differential form, Dirac operator, the square root of Laplacian, A mathematical decomposition in boson fermions show that Laplacian operator can break in two parts: $DD\dagger$, $D \dagger D$, D is dirac operator and $D\dagger$ its adjoint:

$$\begin{aligned} \Delta &= \begin{pmatrix} 0 & D\dagger \\ D & 0 \end{pmatrix} \begin{pmatrix} 0 & D\dagger \\ D & 0 \end{pmatrix} \\ &= \begin{pmatrix} D \dagger D & 0 \\ 0 & DD\dagger \end{pmatrix} \end{aligned} \tag{29}$$

It is easy to see that the half laplacians: $DD\dagger$ and $D \dagger D$ have the same non-zero eigenvalues then with $ker(DD\dagger) = ker(D\dagger)$ and $ker(D \dagger D) = ker(D)$ we have:

$$Str(e^{t\Delta}) = Tr(e^{-tDD^\dagger}) - Tr(e^{-tD^\dagger D}) = index_a(D) \tag{30}$$

An asymptotic expansion of the heat kernels near 0 leads to a topological index definition:

4) Asymptotic expansion of the heat kernel: we have the asymptotic expansion of the heat kernels for giving small t:

$$K_t(x, x) \to (4\pi t)^{-n/2} \sum_{i=0}^{\infty} t^i a_i(x) \tag{31}$$

by integration we find:

$$\begin{aligned} Ind_a(D) = Tr(e^{-tDD^\dagger}) - Tr(e^{-tD^\dagger D}) = \\ \int_M (a^+_{n/2}(x) - a^-_{n/2}(x))dx \end{aligned} \tag{32}$$

To go further, We have to define the Bochner-Lichnerowicz formula, which makes it possible to enrich the formula of the classical Laplacian, and to exploit the richness of the spinorial geometry. This formula allows several approaches to demonstrate the index theorem. The first is determinist (Berline, Getzler, Vergne) [4] and the second stochastic (Bismuth, Azenkott).

5) Bochner-Licnerowicz formula: Let $\nabla^{\mathcal{E}}$ be a Clifford connection on a Clifford $\mathcal{E}$ module, D the operator of Dirac associated, $\Delta^{\mathcal{E}}$ the Laplacian associated with the Clifford connection,r_M Scalar curvature of M, (it is also possible to add a twisted curvature over the the spinor bundle $\mathcal{S}$, $F_{\mathcal{E}/\mathcal{S}}$, so we have the formula of Lichnerowicz:

$$D^2 = \Delta^{\mathcal{E}} + c(F^{\mathcal{E}/\mathcal{S}}) + \frac{1}{4} r_M \tag{33}$$

6) first strategy to demonstate index theorem: The first strategy to prove the index formula by the asymptotic expansion method is to invoke the Mehler formula:

$$p_t(x, r, f) = (4\pi t)^{-1/2} \\ (\frac{tr/2}{sinh(tr/2)})^{1/2} exp(-tr/2coth(tr/2x^2/4t - tf) \tag{34}$$

This formula is the heat kernel for the Laplacian of the harmonic oscillator:

$$H_x = -\frac{d^2}{dx^2} + \frac{r^2 x^2}{16} + f \tag{35}$$

We note the similarity of this formula with the Bochner-Lichnerowicz formula. We see also on the heat kernel an embryonic character of Todd and an embryonic character of Chern, two topological invariant: the characteristic classes theory(Milnor) allows to go further. The theorem is almost demonstrated. To complete the proof, we use a very technical calculation consisting of rescaling the heat kernel in the Lichnerowicz formula and correctly exploiting the Parallel transport. Then it is possible to find Todd and Chern characters and demonstrate the index theorem.

7) Second strategy to demonstrate index theorem: A second strategy is due to Bismuth. To simplify the exposure, we do not take into account the twisted curvature. We will not get the formula of the complete index, then the Chern character is missing. Let K the scalar curvature; we take again laplacian from Lichnerowicz formula: $D^2 = -\Delta + K/4$, recall that A classical solution of the heat kernel in integral form is given by:

$$exp(-\frac{tD^2}{2}) f(x) = \int_M P_t(x, y) f(y) dy \tag{36}$$

The Bismuth approach consists to substitute left member of previous equation (36) by:

$$exp(-\frac{tD^2}{2}) f(x) \leftrightarrow \mathbb{E}_x(exp(-\int_0^t \frac{K(x_s)ds}{8}) \tau_0^s f(x_s) f(x_s) \tag{37}$$

this formula represents the expectation of a Brownian motion (Brownian bridge) starting from x, it's an adaptation of the **Feynman-Kac formula**. We substitute the right member by

$$\int_M p_t(x, y) \mathbb{E}_{x,y}(exp(-\int_0^t \frac{K(x_s)ds}{8}) \tau_0^t f(y) dy) \tag{38}$$

where $p_t(x, y) = exp(t(-\Delta/2)$ and the second part expectation of a Brownian motion starting to x and ending to y. τ_0^t is an adaptation of parallel transport by substitute ordinary differential equation by a stochastic equation.
To obtain the index, take the supertrace we obtain:

$$In(D) = \int_M Str(P_t(x, x)) dx \tag{39}$$

with:

$$Str(P_t(x, x) = p_t(x, x) \mathbb{E}_{x,x}(exp(-\int_0^t \frac{K(x_s)ds}{8}) Str(\tau_0^t)) \tag{40}$$

It remains to understand the supertrace of the form associated with parallel transport in the language of Clifford's algebra. Let matrix A:

$$A = \begin{pmatrix} \begin{pmatrix} 0 & -\theta_1 \\ \theta_1 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & -\theta_n \\ \theta_n & 0 \end{pmatrix} \end{pmatrix} \tag{41}$$

we have the result:

$$lim_{t \to 0} \frac{Str(EXP(tc(A)}{t^{n/2}} = i^{n/2} Pf(A) \tag{42}$$

Pf(A) is the Pfaffian. Using the formula (42) applied to the supertrace of the right hand side of (40), we express the form of the parallel transport in term of Pfaffian at the end we can, thanks to a formula of Levy, recover the character of Todd.

VI. Index theorem in physic: Supersymmetry

The applications of the index theorem go beyond mathematics. It has applications in theoretical physics and particularly in quantum field theory. We can see, with the following example, how supersymmetry can be introduced easily. We can then show that the partition function in the supersymmetric framework is the index of an operator [5]. On the other hand, the kernel and cokernel in the formula of the analytic index refer to the vacuum states of the quantum system. In other words, the theory of supersymmetric ("topological") fields is a theory with non-exited states.

A. Quantum mechanics on a manifold

1) Bosonic version: If we consider a particle moving on a manifold M, the quantification of this problem consists in solving the Shrdinger equation. Mathematically it is to solve the heat equation on a manifold, the Laplacian $\Delta = -\frac{1}{2}\frac{d^2}{d\theta^2}$ is Hamiltonian H in physic. For simplicity, take the circle S^1 of radius β. The solution of heat equation is here:

$$Tr(e^{-\beta H}) = \sum_{n=-\infty}^{+\infty} exp(\frac{-\beta 2\pi^2 n^2}{R^2}) \tag{43}$$

A physical interpretation of this quantity is the partition function $Z(\beta)$.

2) Supersymmetric version: Physicists in the mid-sixties postulate the existence of a new symmetry that Symmetry of Noether: this is supersymmetry, we shall return in more detail later. Briefly, in the standard model, there are two types of particles: the bosons that carry the interactions, and the fermions that constitute matter. Supersymmetry postulates that each fermion has a super-partner bosonic correspondent (and reciprocally). A boson has the vocation of being a switching variable, that is a pair variable, while a fermion must be anticommutative, we will say that it is an odd variable. That explains why we choose the algebra of the differential forms of degree 1 to construct the fermionic variables. The most suitable framework for introducing supersymmetry is that of Clifford algebras. The idea to add fermions artificially will be to inject algebras functions on a variety In the algebra of differential forms. The differential forms of even degrees will represent the bosons, while the differential forms of odd degrees will be the fermions. We then construct a so-called super-space, we often note ψ, or θ a fermionic variable. We denote by $\mathcal{Q}$ the operator BRST in mathematics is simply the differential of the De Rham complex. One can construct its adjoint noted $\overline{\mathcal{Q}}$. For example, for the circle S^1, to take into account supersymmetry, we replace the Hilbert space $\mathcal{H}(S^1, \mathbb{C})$ by a larger space : $\mathcal{H}(\Omega^*(S^1) \otimes \mathbb{C})$.

$$\mathcal{H}_{\mathcal{B}} = \Omega_0(S^1) \overset{\mathcal{Q},\overline{\mathcal{Q}}}{\longrightarrow} \mathcal{H}_{\mathcal{F}} = \Omega_1(S^1) \tag{44}$$

The Dirac operator D exchanges bosons and fermions and reciprocally in the case of the circle S^1, we simply have:

$$D = \mathcal{Q} + \overline{\mathcal{Q}} \tag{45}$$

This operator is a square root of the Laplacian, the semi-Laplacian which restores the energy H of the system:

$$H = \frac{1}{2}(\mathcal{Q}\overline{\mathcal{Q}} + \overline{\mathcal{Q}}\mathcal{Q}) = \frac{1}{2}\Delta \tag{46}$$

In this context, the partition function on the circle S^1 becomes, the index:

$$\begin{aligned} Z_S(\beta) = Str(e^{-\beta H}) = (-1)^F Tr(e^{-\beta H}) = \\ Tr(e^{-\beta \overline{\mathcal{Q}}\mathcal{Q})}) - Tr(e^{-\beta \mathcal{Q}\overline{\mathcal{Q}})}) \end{aligned} \tag{47}$$

The two operators $\mathcal{Q}\overline{\mathcal{Q}}$, $\overline{\mathcal{Q}}\mathcal{Q}$ have the same eigenvalues and on the other hand: the $\mathcal{Q}\overline{\mathcal{Q}}$ kernels are those of $\overline{\mathcal{Q}}$ and that of $\overline{\mathcal{Q}}\mathcal{Q}$ is the $\mathcal{Q}$; The preceding equality thus becomes:

$$Z_S(\beta) = Str(e^{-\beta H}) = dimKer(\mathcal{Q}) - dimker(\overline{\mathcal{Q}}) \tag{48}$$

In the framework of Riemannian geometry, we know that each class of cohomology has a harmonic representation. Ψ verifying $\mathcal{Q}\psi = \overline{\mathcal{Q}}\psi = 0$ Is in the kernel of the Laplacian, which physically represents the states of the vacuum. In other words, when one is interested in the supersymmetric model, the function of partition depends only on the non-exited states: the state of the vacuum of a theory. That is the topological side of the index of operator (here D): ind (D). In addition, the two kernels: $ker(\overline{\mathcal{Q}}\mathcal{Q})$, $ker(\mathcal{Q}\overline{\mathcal{Q}})$ represents the betti numbers: b_0 and b_1 respectively. So the index of the Dirac operator on S^1 vanish.

The above calculations can be generalized for any compact variety. The de Rham complex is exploited, the even differential forms represent the bosons, the odd forms, the fermions. We find by taking Dirac operator $d + d\dagger$ that the index of this operator on the manifold is the Euler Poincaré characteristic of the manifold.

VII. Other applications in mathematical physic.

The Index theorem is a smooth version of Riemann's Roch theorem [6] [7]. He has greatly advanced the geometry of varieties. In four dimensions, for example Donaldson [8] uses the $Su(2)$ gauge group of physicists to define new invariants for the four dimensional geometry. He defines the Donaldson invariants. To define these invariants, we need spaces of configurations more general than the differentiable manifolds: we speak of moduli spaces (or space of instantons for physicists). The elements intervening there are the self-dual connections defined on the principal bundle of group $Su(2)$. We can introduce a virtual dimension for this moduli space. The dimension is **calculated using the index theorem**. In dimension 4 Seiberg and Witten [9] [10] [11] we also define invariants which carry their names, involving connections on the group of gauge $U(1)$. The index theorem also allows the definition of virtual dimension for the space of the corresponding modules.

A. Supersymmetry and string theory

At the same time, Gromov [12] [13] defined the pseudoholomorphic curves. In four dimension, Taube, shows that the invariants of Seiberg-Witten can be defined from these complex curves. In supersymmetric field theories, Witten defines new invariants, the Gromov-Witten invariants. By defining supersymmetric fields theories, he shows that the path integral, by definition mathematically incalculable is in these cases. For this, he uses a localization [14] technique. The theories of fields are localized around space of instantons (moduli space). The virtually finite dimension of these moduli spaces makes it possible to make calculations, in particular, to make enumerative geometry [15] on the correlation functions. This technique of localization allows him to find in four dimensions, the instanton modules of the Seiberg-Witten theory. In the superstring theory (ten-dimensional spacetime), the instantons discovered by Witten on the A side of mirror symmetry, are precisely the pseudoholomorphic curves. These considerations greatly advance the symplectic geometry [16] [17] .

VIII. Exemple: Symplectic geometry and correlation function for strings

A. A toy model

In symplectic geometry, there are very few local invariants. This is due to Darboux's theorem which assumes that locally all symplectic manifolds are similar, unlike the Riemanian varieties that can be separated locally by the curvature. A strategy, due to M. Gromov, for constructing invariants is to consider sub varieties such as, for example, holomorphic curves (function from Riemann surface to a symplectic manifold); There are parameterized curves: $u : (\Sigma, j) \to (Y, J)$,checking the conditions of Cauchy Riemann: $du \circ j = J \circ du$, where j and J are almost complex structures respectively on Σ and Y, and modelized a sigma-model in quantum field theory. Counting the holomorphic functions passing through marked pointson a Riemann surface, makes it possible to determine the correlation functions in superstring theory, the so-called invariants of Gromov Witten. Indeed E. Witten showed that a holomorphic function represents an instanton among all complex parametric curves. these parametric curves represent the evolution of a strings in space-time, in theoretical physics. A toy model, consists in defining the moduli space of the planar curves: (function $:\mathbf{P}^1(\mathbf{C}) \to \mathbf{P}^2(\mathbf{C})$ of given degree (this degree corresponds to a class of cohomology in $H_2(Y, \mathbf{Z})$. For example, for degree one:

$$\mathcal{M} = \{u/u : \mathbf{P}^1(\mathbf{C}) \to \mathbf{P}^2(\mathbf{C})/PGL(2, \mathbf{C}) \tag{49}$$

$PGL(2, \mathbf{C})$ represents automorphism group of $\mathbf{P}^2(\mathbf{C})$, his dimension is three; the space of the applications u is of complex dimension 5, therefore, one finds again that the space of the complex lines has complex dimension 2 For example, for lines passing through two fixed points, we have another moduli space:

$$\mathcal{M}' = \{(u, z_1, z_2)/u : \mathbf{P}^1(\mathbf{C}) \to \mathbf{P}^2(\mathbf{C}), z_1 \neq z_2\}/PGL(2, \mathbf{C}) \tag{50}$$

with the constraint of passing through two points, we find that this space has the dimension 4 Because you add, two parameters (two points) each of them is an element of $\mathbf{P}^2(\mathbf{C})$. It is possible now to evaluate (u, z_1, z_2) in other words, construct:

$$ev : (u, z_1, z_2) \in \mathcal{M}' \to (u(z_1), u(z_2)) \in \mathbf{P}^2(\mathbf{C}) \times \mathbf{P}^2(\mathbf{C}) \tag{51}$$

Here we have the simplest example of what is called a Gromov-Witten invariant: the evaluation from a degree one map u through two points give only one line... In physics this correlation function is called a propagator.

If we now choose a complex curve of degree 2, we define a conic, we can show that the moduli space considered has dimension 5: five points determine only one conic. In this case, the moduli space must be compactified: there is a sequence of conics which converges towards a couple of line for example...

Kontsevich [15] has demonstrated a recurrence formula for counting all planar complex curves of given degree and thereby solved an enumerative geometry conjecture. The consideration of mirror symmetry in string theory has made it possible to demonstrate other conjectures in simple cases.

B. Theorical model

we are now considering an application of a Riemann surface in any complex manifold. let ϕ an application of a Riemann surface in any complex manifold. Note respectively $\mathcal{M}_g, \mathcal{M}_{g,n}$ the space of the curves modules (actually riemann surfaces), and the space of curve with n marked points. Thee Riemann-Roch formula for Curve give:

$$\begin{aligned} dim_{\mathbf{C}}H^0(T\Sigma) - dim_{\mathbf{C}}H^1(T\Sigma) &= \textstyle\int_\Sigma ch(T\Sigma)td(T\Sigma) \\ &= 3 - 3g \end{aligned} \tag{52}$$

If $\phi : \Sigma \to X$ is a map from Σ to X The Riemann Roch formula give:

$$\begin{aligned} & dim_{\mathbf{C}}H^0(\phi^*TX) - dim_{\mathbf{C}}H^1(\phi^*TX) \\ &= \textstyle\int_\Sigma ch(\phi^*TX)td(\Sigma) \\ &= n(1-g) + \textstyle\int_\Sigma \phi^* c_1(TX) \end{aligned} \tag{53}$$

The deformation invariant of the problem are obtained thanks to the short exact sequence.

$$0 \to T_\Sigma \to \phi^* T_X \to N_{\Sigma/X} \to 0 \tag{54}$$

The long exact sequence associated, gives the index of the complex: the dimension of the moduli space of the applications $\mathcal{M}_g(X, \beta, n)$, β degree of the map, n number of marked point on Σ :

Roughly, the first term manages the deformation of the Riemann surface, the second the deformation of the ϕ the Riemann surface being fixed, and the third term the deformations of the application. The long exact sequence associated,combines the two previous formula [9] and [10] and compute the index of the complex: the dimension of the compactified moduli space of the applications $\overline{\mathcal{M}}_{g,n}(X, \beta)$ degree of the map, n number of marked point on Σ :

$$\begin{aligned} & dim_{virt}\overline{\mathcal{M}}_{g,n}(X, \beta) = \\ & (dimX)(1-g) + \textstyle\int_{f_*(\Sigma)} c_1(TX) + 3g - 3 + n \end{aligned} \tag{55}$$

Taking care not to confuse real and complex dimensions, in the case of the plane curves of degree one (the straight lines), we retrieve the dimension of the space of module $\mathcal{M}'$ seen previously.

IX. Tensorial analysis of networks: Kron method

Gabriel Kron, inspired by Einstein's work on general relativity, proposes to study electrical machines from the angle of tensor analysis [18]. An electrical circuit can then be decomposed into nodes (vertices of a graph), edges then meshes.

The most classical invariant to which we think in graph theory is the characteristic of Euler Poincaré. For a graph, we can consider the number of cycles decreased by the number of vertices and increased by a number of edges. This invariant is not interesting for the study of electrical circuits because it does not allow to distinguish in the spaces of vertices, branches and cycles how many are independent. Only should be taken into account, the linearly independent edges to transform the currents in the vector space of the meshes.

A. Kron Invariant

We consider the vector space of the formal chains constituted by the nodes: $n^1, n^2, ...n^N$.

Similarly, we consider the vector space of the branches generated by $\mathcal{B}$: $b_1, b_2..., b_B$.

we consider linear map: δ de $\mathcal{B}$ dans $\mathcal{N}$ define by:

$\delta(b_i) = \varepsilon_j.n^j$ with $\varepsilon_j = 1$ if the end of b_i is n^j

$\varepsilon_j = -1$ the origin of b_i is n^j

$\varepsilon_j = 0$ if the end of b_i is origin of b_i

For example for the graph whose branches are:

b_1: origin n^1 and the end n^2

b_2: origin n^2 and the end n^3

b_3: origin n^1 and the end n^3

b_4: origin n^2 and the end n^2

The matrix of linear map is given by:

$$G = \begin{pmatrix} -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

The fundamental relation of linear algebra gives:

$dim(\mathcal{B}) = dimker(\delta) + dim(Im(\delta))$

This relation is the first relation of Kron, in fact the kernels of δ is the vector space $\mathcal{M}$ of meshes of dimension M. The image of δ, the vector space $\mathcal{P}$ of pairs of nodes, of dimension P of or the dimensional relation:

$$B = M + P \tag{56}$$

We also have the relation:
$dimIm(\delta) = dim(\mathcal{N}) - dim(\mathcal{N}/Im(\delta))$,

the last part of this equality is the quotient of the set of nodes, by the nodes that go in pairs. This gives the number of connected components of the graph: the number of subnetworks. This is the second relation of Kron

$$P = N - S \tag{57}$$

these two quantities are topological invariants because it depends only on the dimension of the spaces and sub-spaces vector considered. The previous relationships make it possible to define the right spaces to represent the electric currents in stationary regime. In particular the Kron method shows that it is useful to represent the currents of branches in currents of meshes. We thus have tensorial quantities (vertices, chains, cycle ...) that we can dualize .... In a previous paper, we use, starting from the singular homology, finer topological invariants, to find topologically the law of the mesh and that of the nodes [19],[20]
The expression of the analytical index of the delta operator, also makes it possible to find the first topological relation of Kron:
Let: $\delta : \mathcal{B} \rightarrow \mathcal{N}$, we have

$$Ind_a(\delta) = dim(ker(\delta)) - dim(coker(\delta)) = M - (N - P) \tag{58}$$

Thus in linear algebra Ind_a is just given by:

$$Ind_a(\delta) = dim(\mathcal{B}) - dim(\mathcal{N}) = B - N \tag{59}$$

these two relations allow to find the relation (56) In the next section, we show how Kron's topological relationships are used to define the adequate representation of an electrical circuit in the mesh space.

B. Example

Figure 1 shows an example of circuit :two networks such that each one is controlled by the other. The second network is powered by the voltage Vdc(t) reported from the first network, and the load current of the second network i_s is injected in the first network depending on a command law.

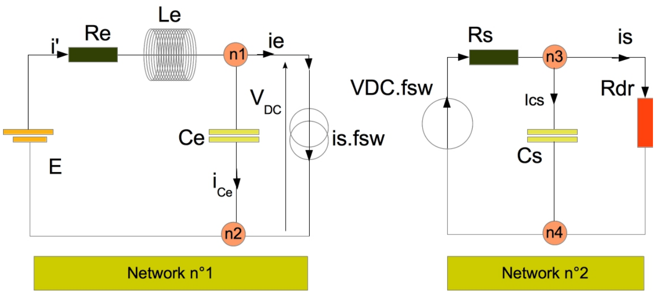

Figure 1. Network with two connected components

The second network includes a generator E_2, given by: $E_2 = Vdc * fsw$. The visible elements in the graph given Figure 5 are the topological following character :

- 4 physical nodes n1,..,n4 ($\rightarrow N = 4$)
- 5 branches b1,..,b5 ($\rightarrow B = 5$)
- 3 meshes m1,m2,m3 ($\rightarrow M = 3$)
- 2 networks R1, R2 ($\rightarrow R = 2$)
- 2 nodes pair ($\rightarrow P = 2$)

1) Choice of a topology:

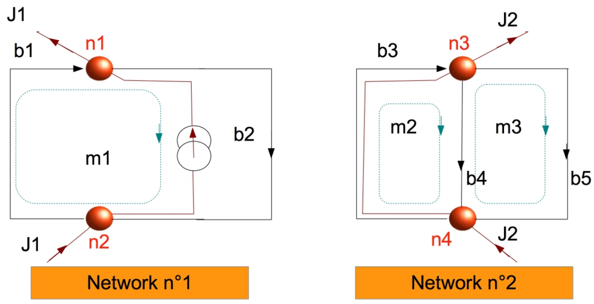

Figure 2. Topology of previous network

we choosing arbitrarily the initial node N_1 on our first network, , we start of this Node worm node N_2, we have an return of node N_2 to N_1. We construct by this return , the first couple P_1 who will wear the current source J_1, and will be in final, our current injected in the first network coming from the second network. We verify the relationship for node pair: $P = N - S = 4 - 2 = 2$ and meshes: $M = B - N + S = 5 - 4 + 2 = 3$. As in our first Network, we choosing arbitrarily on our second network the node N_3, as reference from depart. We depart of this node worm node N_4, we have an return from node N_4 to node N_3 , we construct with this return, the second couple P_2, who will wear normally the current source J_2, but all along our study we assume that J_2 is null, because it is rattached to a branch which comported not a current source. The good number of nodes, edges, pairs of nodes and mesh provided by the invariant of Kron makes it possible to transpose the electrical study of the circuit in the space of the meshes. It is one of the main objectives of the analysis tensorial network (TAN)
Remark: another Choice of topolgy is possible:
For the second network, we can choose another topology, different than that above, we not change the choice of current and couple, but we can change the choise of the meshes. But the computations give the same results in both case due to the invariant theorem. In all our calculs, we choose the topology presented figure 3.

2) Connection matrix: According to the complete space of first and second network look at figure 6 and figure 7, according to the topological character determined before, we can determined the connection matrix C linking meshes,

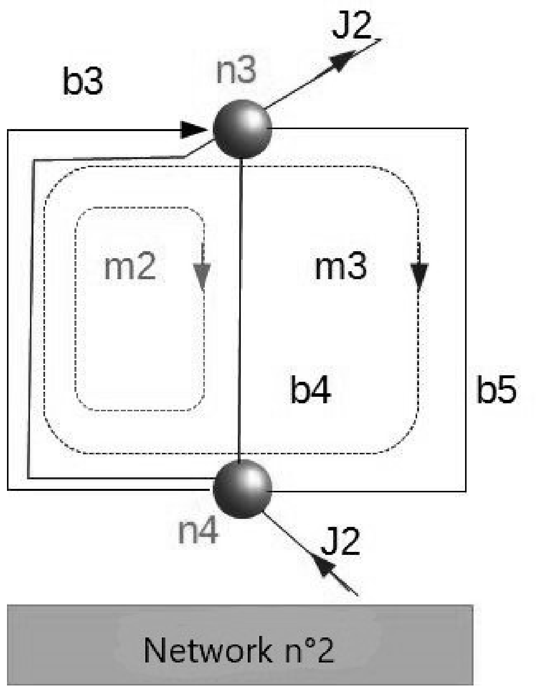

Figure 3. Second network possible topology

nodes pair and branches currents:

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_{m1} \\ i_{m2} \\ i_3 \\ J_1 \end{bmatrix} \tag{60}$$

3) Impedance tensor: The impedances tensor in the space of the branches is:

$$z = \begin{bmatrix} Re + Le * p & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{Ce*p} & 0 & 0 & 0 \\ 0 & 0 & Rs + Ls * p & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{Cs*p} & 0 \\ 0 & 0 & 0 & 0 & R_{dr} \end{bmatrix} \tag{61}$$

Applying the bilinear transformation given by: $Z = C^t.z.C$, we obtain the impedance tensor in the mesh space. The source covector in the branch space is given by:

$$s = \begin{bmatrix} E & 0 & fsw * V_{dc} & 0 & 0 \end{bmatrix} \tag{62}$$

After transformation we obtain the source covector in the mesh space:

$$S = C^t * e^t = \begin{bmatrix} E \\ -fsw * V_{dc} \\ -fsw * V_{dc} \\ 0 \end{bmatrix} \tag{63}$$

The voltage covector in the branches space is given by:

$$v = \begin{bmatrix} 0 & V_{dc} & 0 & 0 & 0 \end{bmatrix} \tag{64}$$

Following similar transformation (in the mesh space) gives:

$$V = C^t * v^t = \begin{bmatrix} V_{dc} \\ 0 \\ 0 \\ -V_{dc} \end{bmatrix} \tag{65}$$

Here Kron method consist to solve the integrodifferential equation given by:

$$\begin{bmatrix} E \end{bmatrix} + \begin{bmatrix} V \end{bmatrix} = \begin{bmatrix} Z \end{bmatrix} \begin{bmatrix} I \end{bmatrix} \tag{66}$$

Finally we obtain the two matrices W (impedance tensor matrix in the space of meshes) and T (sources covector in the space of meshes)
T is given by:

$$\begin{bmatrix} E(t) + \frac{L_e}{\delta t} i_{m1}(t-1) - \frac{\delta t}{Ce} \int_0^{t-1} i_{m1}(t)dt \\ + \frac{\delta t}{Ce} \int_0^{t-1} J_1(t)dt \\ \\ -fsw(t)V_{dc}(t) - \frac{\delta t}{Cs} \int_0^{t-1} i_{m1}(t)dt - \frac{\delta t}{Cs} \int_0^{t-1} i_{m2}(t)dt \\ + \frac{L_s}{\delta} i_{m2}(t-1) \\ \\ -fsw(t)V_{dc}(t) - \frac{\delta t}{Ce} \int_0^{t-1} i_{m2}(t)dt - \frac{\delta t}{Ce} \int_0^{t-1} i_{m3}(t)dt \\ \\ -V_{dc}(t) + \frac{\delta t}{Ce} * \int_0^{t-1} i_{m1}(t)dt - \frac{\delta t}{Ce} * \int_0^{t-1} J_1(t)dt \end{bmatrix}$$

(67)

W is given by:

$$\begin{bmatrix} \frac{L_e}{\delta} + \frac{\delta t}{Ce} + Re & 0 & 0 & -\frac{\delta}{Ce} \\ 0 & \frac{L_s}{\delta} + \frac{\delta}{Cs} + Rs & \frac{\delta}{Cs} & 0 \\ 0 & \frac{\delta}{Cs} & \frac{\delta}{Cs} + R_{dr} & 0 \\ -\frac{\delta}{Ce} & 0 & 0 & \frac{\delta}{Ce} \end{bmatrix} \tag{68}$$

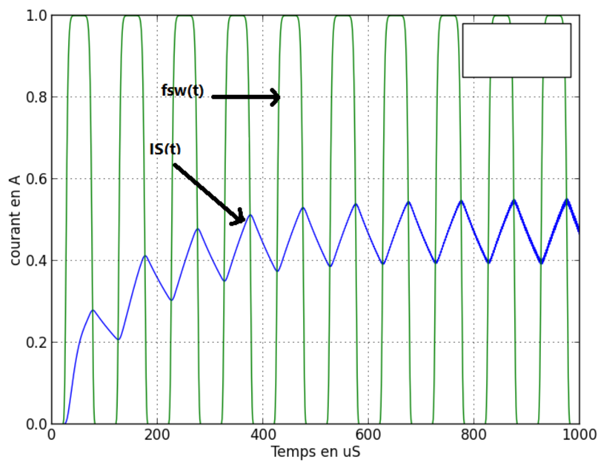

Figure 4. Second network possible topology

The equation was implemented under a python program. It takes about two minutes without any problem of convergence, despite the fact that the numerical schematic used is here the simplest one. For example, figure 8 shows the load current depending on the command law fsw(t).

X. Conclusion

In conclusion, we can say that the index theorem, discovered more than fifty years ago, has been decisive in many branches of mathematics and physics. In physics a new branch of mathematics, after analysis, the probabilities and geometry done are made an entry: This is the topology. Supersymmetry is at the origin of a new discipline: the topological field theory . Supersymmetric particles have not been discovered by physicists. As always mathematical concepts are advanced, and even if supersymmetry does not exist, we think that it remains a theoretical tool allowing us to do mathematical physics properly. The engineering sciences, also exploits the notion of topological invariant. Practically invariants can be built from rigorous theoretical concepts, linear algebra and topology and can provide valuable assistance in the study of electromagnetic compatibility that uses the contribution of tensor analysis from the Kron method.

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