On Generalized Derivations in Residuated Lattices

Kuanyun Zhu, Jingru Wang* and Yongwei Yang

***Abstract*—In this paper, we introduce the notion of a generalized derivation determined by a derivation for a residuated lattice. First, we investigate some related properties of isotone (resp. contractive) generalized derivations and ideal generalized derivations. In addition, we get that a good ideal generalized derivation is determined by it's fixed point set. And then, we obtain that the fixed point set of good ideal generalized derivations is still a residuated lattice, which reveals the essence of the fixed point set for ideal generalized derivations. Finally, we show that the relationship between good ideal generalized derivation filters and filters of the fixed point set for good ideal generalized derivations.**

***Index Terms*—Residuated lattice; (Generalized) derivation; Ideal generalized derivation**

I. Introduction

It is well known that non-classical logic has become a formal and useful tool for computer science to deal with uncertain information and fuzzy information. Various logical algebras have been proposed as the semantical systems of non-classical logic systems, for example, BL-algebras [31], MV-algebras [8], MTL-algebras [14], $IMTL$-algebras [14] and DRL-monoids [13] and so on. Among these logical algebras, residuated lattices [33] are very important algebraic structure since the other logical algebras are all special cases of residuated lattices. The origin of residuated lattices lies in mathematical logic without contraction. They have been investigated by Krull [22], Dilworth [12], Ward [32], Ward and Dilworth [33], Balbes and Dwinger [3] and Pavelka [27] and so on. These lattices have been known under many names: BCK-lattices in [18], full BCK-algebras in [22], FL_{ew}-algebras in [24], and integral, residuated, commutative l-monoids in [5], [6]. Apart from their logical interest, residuated lattices have interesting algebraic properties, such as [16], [28], [31], and include two important classes of algebras: BL-algebras [16] and MV-algebras [8].

Derivation is a very interesting and important area of research in the theory of algebraic structures in mathematics. There exist many discussions related to derivation theory, for instance, in 1957, Posner [29] proposed the notion of derivations in a prime ring $(R, +, -)$. In 2004, Jun and Xin [20] applied the notion of derivations to BCI-algebras. In 2005, Zhan and Liu [39] introduced the notion of f-derivations of BCI-algebras. In recent years, Özturk et al. [25] studied some kinds of generalized derivations on BCI-algebras and obtained some important results. Recently, some authors [7], [37], [38] investigated the properties of derivations in lattices. In 2010, Alshehri [2] introduced the concept of derivations in MV-algebra. Using the notion of an isotone derivation, the author gave some characterizations of a derivation of an MV-algebra. In 2016, Xiao and Liu [36] introduced the notion of derivations for a quantale. In the same year, He et al. [17] introduced the concept of derivations in residuated lattices and they characterized some special types of residuated lattices in terms of derivations. In particular, Bawazeer et al. [4] introduced the notion of generalized derivations for a BCC-algebra. Özturk and Çeven [26] introduced the notion of a generalized derivation determined by a derivation for a subtraction algebra and some related properties are investigated. In 2018, Liang et al. [23] introduced the notion of derivations on EQ-algebras. Moreover, Wang et al. [34] introduced the notion of derivations of commutative multiplicative semilattices. In 2019, Wang et al. [35] gave some representations of MV-algebras in terms of derivations. Rasheed and Majeed [30] investigated some results of (α, β)-derivations on prime semirings. Dey et al. [11] considered orthogonal generalized derivations of semiprime-rings. Ciungu [10] studied the properties of implicative derivations in pseudo BCI-algebras. Chaudhuri [9] discussed (σ, τ)-derivations of group rings. In 2020, Guven [15] proposed the notion of generalized (σ, τ)-derivations on rings and considered some related properties of them. Hosseini and Fosner [19] considered the image of Jordan left derivations on algebras. Ali and Rahaman [1] studied on pair of generalized derivations in rings.

Observe that there are no corresponding researches for the generalized derivations in residuated lattices, although He et al. [17] have investigated the derivations on L residuated lattices. Therefore, as a supplement of this topic from the theoretical point of view, in this paper, we consider a generalized derivation determined by a derivation in a residuated lattice L. More precisely, for any $x, y \in L$ and a given derivation d on L, we propose the following formula

$$D(x \otimes y) = (D(x) \otimes y) \vee (x \otimes d(y)),$$

we call D is a generalized derivation on L. Meanwhile, we discuss some related properties of them. As important results of generalized derivations on a residuated lattice, fixed point sets are characterized by them under some conditions.

This paper is organized as follows. In Section II, we present some preliminary concepts and results related to residuated lattices. In Section III, we propose the notion of generalized derivations in residuated lattices and investigate some related properties of isotone (resp. contractive) generalized derivations and ideal generalized derivations. In particular, as an application of good ideal generalized derivations, we obtain that the fixed point set of good ideal generalized derivations is still a residuated lattice. Further, we investigate the corresponding relationship between good ideal generalized derivation filters and filters of the fixed point set

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K. Zhu is with the School of Information and Mathematics, Yangtze University, Jingzhou, P.R. China e-mail:kyzhu@yangtzeu.edu.cn

J. Wang is with the School of Business Administration, Hunan University, Changsha, P.R. China e-mail: wangjingru@hnu.edu.cn

Y. Yang is with the School of Mathematics and Statistics, Anyang Normal University, Anyang, P.R. China e-mail: yangyw@aynu.edu.cn

for good ideal generalized derivations. Finally, our researches are concluded in Section IV.

II. Preliminaries

In this section, we recall some fundamental concepts and definitions which shall be needed in the sequel. At first, we give a brief reminder of the definition of residuated lattices.

Definition 2.1: [33] A residuated lattice is an algebraic structure $L = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ of type (2,2,2,2,0,0) satisfying the following conditions:

(1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice;

(2) $(L, \otimes, 1)$ is a commutative monoid;

(3) $(\otimes, \rightarrow)$ forms an adjoint pair, i.e., $x \otimes y \leq z$ if and only if $x \leq y \rightarrow z$ for all $x, y, z \in L$.

In what follows, we denote by L a residuated lattice $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$, unless otherwise specified.

For any $x \in L$ and a natural number n, we define $x' = x \rightarrow 0, x'' = (x')' = (x \rightarrow 0) \rightarrow 0, x^0 = 1$ and $x^n = x^{n-1} \otimes x$ for all $n \geq 1$.

Proposition 2.2: [33] *For all $x, y, z, w \in L$, the following properties hold.*

(1) $1 \rightarrow x = x$, $x \rightarrow 1 = 1$.

(2) $x \leq y$ if and only if $x \rightarrow y = 1$.

(3) If $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$.

(4) If $x \leq z$ and $y \leq w$ then $x \otimes y \leq z \otimes w$.

(5) $x \otimes y \leq x \wedge y$.

(6) $x \rightarrow (y \rightarrow z) = x \otimes y \rightarrow z = y \rightarrow (x \rightarrow z)$.

(7) $0' = 1, 1' = 0, x \leq x''$.

(8) $x \otimes y = 0$ if and only if $x \leq y'$.

(9) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$.

In the following, we recall the notion of filters on residuated lattices.

Definition 2.3: [28] Let $\emptyset \subsetneq F \subseteq L$. Then F is called a filter of L if for any $x, y \in F$,

(1) $x \otimes y \in F$,

(2) $x \leq y$ and $x \in F$ imply $y \in F$.

An equivalent definition for a filter in L is: (1) $1 \in F$, (2) if $x, x \rightarrow y \in F$, then $y \in F$. Hence, a filter is also called a deductive system. We denote by $F(L)$ the set of all filters in $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$.

For a nonempty subset W of L, we denote by $\langle W \rangle$ the filter generated by W. One can check that $\langle W \rangle = \{x \in L | x \geq x_1 \otimes x_2 \otimes \cdots \otimes x_n, x_i \in W, i = 1, 2 \cdot\cdot, n\}$.

We denote by $B(L)$ the set of all complement element of the lattice $(L, \wedge, \vee, 0, 1)$, see [3]. The set $B(L)$ is called the Boolean center of L. For any $t \in L$, $t \in B(L)$ if and only if $t \vee t' = 1$.

Proposition 2.4: [21] For any $t \in B(L)$, the following statements hold.

(1) $t \otimes t = t$.

(2) $t \otimes x = t \wedge x$ for any $x \in L$.

At last, we give the notion of a derivation in L as follows.

Definition 2.5: [17] A mapping $d : L \longrightarrow L$ is called a derivation on L if it satisfies the following conditions: for any $x, y \in L$,

$$d(x \otimes y) = (d(x) \otimes y) \vee (x \otimes d(y)).$$

III. Generalized derivations in residuated lattices

By means of the idea in [4] and [26], in this section, at first, we give the notion of a generalized derivation in a residuated lattice. And then, we study some related vital properties for those generalized derivations. We begin with the definition of the generalized derivation in a residuated lattice.

Definition 3.1: Let $d : L \longrightarrow L$ be a derivation on L. A mapping $D : L \longrightarrow L$ is called a generalized derivation determined by d if

$$D(x \otimes y) = (D(x) \otimes y) \vee (x \otimes d(y))$$

for any $x, y \in L$.

Now, we show an example for a generalized derivation on a residuated lattice.

Example 3.2: Let $L = \{0, a, b, 1\}$ be a chain and operations $\otimes$ and $\rightarrow$ be defined as follows:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then it is easy to verify that $L = \{0, a, b, 1\}$ is a residuated lattice. We define a mapping $d : L \longrightarrow L$ by

$$d(x) = \begin{cases} 0, & x = 0, a, \\ a, & x = b, 1. \end{cases}$$

Then d is a derivation on L. Based on d, we define a mapping $D : L \rightarrow L$ by

$$D(x) = \begin{cases} 0, & x = 0, \\ a, & x = a, \\ 1, & x = b, 1. \end{cases}$$

It is easy to verify that D is a generalized derivation on L.

Next, we present some properties for generalized derivations in residuated lattices.

Proposition 3.3: Let D be a generalized derivation on L. Then the following statements hold.

(1) $D(0) = 0$.

(2) $D(1) \otimes x \leq D(x)$ and $d(x) \leq D(x)$ for all $x \in L$.

(3) $x^{n-1} \otimes D(x) \leq D(x^n)$ and $x^{n-1} \otimes d(x) \leq D(x^n)$ for all $x, y \in L$ and $n \geq 1$.

(4) If $x \leq y'$, then $D(x) \leq y'$ and $d(y) \leq x'$ for all $x, y \in L$.

(5) $D(x') \leq (D(x))'$ for all $x \in L$.

Proof. (1) It follows from Definition 3.1 that $D(0) = D(0 \otimes 0) = (D(0) \otimes 0) \vee (0 \otimes d(0)) = 0$, i.e., $D(0) = 0$.

(2) Let $x \in L$. Then we have $D(x) = D(1 \otimes x) = (D(1) \otimes x) \vee (1 \otimes d(x))$, which implies $D(1) \otimes x \leq D(x)$ and $d(x) \leq D(x)$.

(3) It follows from Definition 3.1 that $D(x^2) = D(x \otimes x) = (D(x) \otimes x) \vee (x \otimes d(x))$ for all $x \in L$, which implies $D(x) \otimes x \leq D(x^2)$ and $x \otimes d(x) \leq D(x^2)$. By induction, we have $x^{n-1} \otimes D(x) \leq D(x^n)$ and $x^{n-1} \otimes d(x) \leq D(x^n)$ for all $n \geq 1$.

(4) Let $x, y \in L$ and $x \leq y'$. Then it follows from Proposition 2.2 (8) that $x \otimes y = 0$. Thus, $D(x \otimes y) = (D(x) \otimes y) \vee (x \otimes d(y)) = 0$, which implies $D(x) \otimes y = 0$ and $x \otimes d(y) = 0$. Therefore, $D(x) \leq y'$ and $d(y) \leq x'$.

(5) Let $x \in L$. Then it follows from Proposition 2.2 (7) that $x \leq x''$, and from statement (4) that $D(x) \leq x''$.

Thus $D(x') \leq x'''$ and $x''' \leq (D(x))'$. Therefore, $D(x') \leq (D(x))'$. □

In order to characterize some special types of residuated lattices, in what follows, we first introduce some particular generalized derivations and discuss some related properties of them.

Definition 3.4: Let D be a generalized derivation on L. Then, for all $x, y \in L$,

(1) if $x \leq y$ implies $D(x) \leq D(y)$, we call D an isotone generalized derivation,

(2) if $D(x) \leq x$, we call D a contractive generalized derivation.

In particular, if D are both isotone and contractive, then we call D an ideal generalized derivation.

Example 3.5: Define a mapping $d : L \longrightarrow L$ by $d(x) = 0$ for all $x \in L$, then it follows from Definition 2.5 that d is a derivation on L. Let $t \in L$. Then we define a mapping $D : L \longrightarrow L$ by $D(x) = x \otimes t$ for all $x \in L$. It is easy to verify that D is an ideal generalized derivation on L.

Now, some properties of isotone generalized derivations and contractive generalized derivations are investigated, respectively.

Proposition 3.6: Let D be an isotone generalized derivation on L. Then the following statements hold.

(1) If $z \leq x \to y$, then $z \leq D(x) \to D(y)$ and $x \leq d(z) \to D(y)$ for all $x, y, z \in L$.

(2) $x \to y \leq D(x) \to D(y)$ and $d(x \to y) \leq x \to D(y)$ for all $x, y \in L$.

(3) $x \leq d(y) \to D(x)$ for all $x, y \in L$.

Proof. (1) Let $x, y, z \in L$ and $z \leq x \to y$. Then $x \otimes z \leq y$. Since D is an isotone generalized derivation on L, we have $D(x \otimes z) \leq D(y)$. It follows from Definition 3.1 that $D(x \otimes z) = (D(x) \otimes z) \vee (x \otimes d(z))$. Thus, $(D(x) \otimes z) \vee (x \otimes d(z)) \leq D(y)$, which implies $D(x) \otimes z \leq D(y)$ and $x \otimes d(z) \leq D(y)$. Therefore, $z \leq D(x) \to D(y)$ and $x \leq d(z) \to D(y)$.

(2) Since $x \otimes (x \to y) \leq y$ for all $x, y \in L$, we have $D(x \otimes (x \to y)) \leq D(y)$. It follows from Definition 3.1 that $D(x \otimes (x \to y)) = (D(x) \otimes (x \to y)) \vee (x \otimes d(x \to y))$, which implies $D(x) \otimes (x \to y) \leq D(y)$ and $x \otimes d(x \to y) \leq D(y)$. Therefore, $x \to y \leq D(x) \to D(y)$ and $d(x \to y) \leq x \to D(y)$ for all $x, y \in L$.

(3) Since $x \otimes y \leq x$ for all $x, y \in L$, we have $D(x \otimes y) \leq D(x)$. It follows from Definition 3.1 that $D(x \otimes y) = (D(x) \otimes y) \vee (x \otimes d(y))$. Thus, $x \otimes d(y) \leq D(x)$. Therefore, $x \leq d(y) \to D(x)$. □

Proposition 3.7: Let D be a contractive generalized derivation on L. Then the following statements hold.

(1) $D(x) \otimes d(y) \leq D(x \otimes y) \leq D(x) \vee D(y)$ for all $x, y \in L$.

(2) If D is isotone, then $D(x \to y) \leq d(x) \to D(y) \leq d(x) \to y$ for all $x, y \in L$.

(3) If $D(1) = 1$, then D is an identity generalized derivation.

Proof. (1) Since D is a contractive derivation on L, we have $D(x) \otimes d(y) \leq D(x) \otimes D(y) \leq D(x) \otimes y$ and $D(x) \otimes d(y) \leq x \otimes d(y)$ for all $x, y \in L$. Thus, $D(x) \otimes d(y) \leq (D(x) \otimes y) \vee (x \otimes d(y)) = D(x \otimes y)$, which implies $D(x) \otimes d(y) \leq D(x \otimes y)$ for all $x, y \in L$. On the other hand, since $D(x) \otimes y \leq D(x)$ and $x \otimes d(y) \leq d(y) \leq D(y)$, we have $D(x \otimes y) = (D(x) \otimes y) \vee (x \otimes d(y)) \leq D(x) \vee D(y)$. Therefore, $D(x) \otimes d(y) \leq D(x \otimes y) \leq D(x) \vee D(y)$ for all $x, y \in L$.

(2) Since $x \otimes (x \to y) \leq y$ for all $x, y \in L$, we have $D(x \otimes (x \to y)) \leq D(y)$. It follows from the statement (1) that $D(x \to y) \otimes d(x) \leq D(x \otimes (x \to y))$, which implies $D(x \to y) \leq d(x) \to D(y)$. On the other hand, since $D(y) \leq y$, we have $d(x) \to D(y) \leq d(x) \to y$. Therefore, $D(x \to y) \leq d(x) \to D(y) \leq d(x) \to y$.

(3) It follows from Proposition 3.3 (2) that $D(1) \otimes x \leq D(x)$ for all $x \in L$. Let $D(1) = 1$. Then we have $x = x \otimes D(1) \leq D(x) \leq x$, which implies $D(x) = x$ for all $x \in L$. Therefore, D is an identity generalized derivation. □

Theorem 3.8: Let D be a generalized derivation on L. Then the following statements are equivalent.

(1) D is an ideal generalized derivation on L and $D^2 = D$, where $D^2(x) = D(D(x))$ for all $x \in L$.

(2) $D(x) \to D(y) = D(x) \to y$ for all $x, y \in L$.

Proof. (1) $\Rightarrow$ (2) Let D be an ideal generalized derivation on L and $D^2 = D$. Since $D(y) \leq y$ for all $y \in L$, we have $D(x) \to D(y) \leq D(x) \to y$. On the other hand, let $m \leq D(x) \to y$ for all $m \in L$. Then we have $D(x) \otimes m \leq y$. Since D is isotone, we have $D(D(x) \otimes m) \leq D(y)$ for all $x, y, m \in L$. It follows from Definition 3.1 that $D(x) \otimes y \leq D(x \otimes y)$, which implies $D(D(x)) \otimes m \leq D(D(x) \otimes m)$. Since $D^2 = D$, we have $D(x) \otimes m \leq D(D(x) \otimes m) \leq D(y)$. Then $m \leq D(x) \to D(y)$ for all $m \in L$, which implies $D(x) \to y \leq D(x) \to D(y)$. Therefore, $D(x) \to D(y) = D(x) \to y$.

(2) $\Rightarrow$ (1) Let $D(x) \to D(y) = D(x) \to y$ for all $x, y \in L$. Since $D(x) \otimes 1 \leq D(x)$, we have $1 \leq D(x) \to D(x) = D(x) \to x$. Thus $D(x) \leq x$ for all $x \in L$, which implies D is contractive. On the other hand, let $x \leq y, x, y \in L$. Then we have $D(x) \otimes 1 = D(x) \leq x \leq y$, i.e., $1 \leq D(x) \to y = D(x) \to D(y)$, which implies $D(x) \leq D(y)$. Thus, D is isotone. Therefore, D is an ideal generalized derivation on L. Finally, it follows from $D(x) \otimes 1 \leq D(x)$ that $1 \leq D(x) \to D(x) = D(x) \to D(D(x))$. Then $D(x) \leq D(D(x))$. Since $D(D(x)) \leq D(x)$, we have $D(D(x)) = D(x)$ for all $x \in L$, i.e., $D^2 = D$. □

Theorem 3.9: Let D be a contractive generalized derivation on L. If $D(1) \in B(L)$, then the following statements are equivalent.

(1) D is an ideal generalized derivation on L.

(2) $D(x) \leq D(1)$ for all $x \in L$.

(3) $D(x) = D(1) \otimes x$ for all $x \in L$.

(4) $D(x \wedge y) = D(x) \wedge D(y)$ for all $x, y \in L$.

(5) $D(x \vee y) = D(x) \vee D(y)$ for all $x, y \in L$.

(6) $D(x \otimes y) = D(x) \otimes D(y)$ for all $x, y \in L$.

Proof. (1) $\Rightarrow$ (2) It is straightforward.

(2) $\Rightarrow$ (3) Let $D(x) \leq D(1)$ for all $x \in L$. If $D(1) \in B(L)$, then we have

$$\begin{aligned} D(x) &= D(1) \wedge D(x) \\ &= D(1) \otimes D(x) \\ &= D(1) \otimes x. \end{aligned}$$

On the other hand, it follows from Proposition 3.3 (2) that $D(1) \otimes x \leq D(x)$. Therefore, $D(x) = D(1) \otimes x$.

(3) $\Rightarrow$ (4) Let $D(x) = D(1) \otimes x$ for all $x \in L$. Then for all $x, y \in L$

$$\begin{aligned} D(x \wedge y) &= D(1) \otimes (x \wedge y) \\ &= D(1) \wedge (x \wedge y) \\ &= (D(1) \wedge x) \wedge (D(1) \wedge y) \\ &= (D(1) \otimes x) \wedge (D(1) \otimes y) \\ &= D(x) \wedge D(y). \end{aligned}$$

(4) $\Rightarrow$ (1) Let $x \leq y, x, y \in L$. It follows from (4) that $D(x) = D(x \wedge y) = D(x) \wedge D(y)$, which implies $D(x) \leq D(y)$ for all $x, y \in L$. Therefore, D is an ideal generalized derivation on L.

(3) $\Rightarrow$ (5) For all $x, y \in L$, it follows from (3) and Proposition 2.2 (9) that

$$\begin{aligned} D(x \vee y) &= D(1) \otimes (x \vee y) \\ &= (D(1) \otimes x) \vee (D(1) \otimes y) \\ &= D(x) \vee D(y). \end{aligned}$$

(5) $\Rightarrow$ (1) Let $x, y \in L$ and $x \leq y$. It follows from (5) that $D(y) = D(x \vee y) = D(x) \vee D(y)$, which implies $D(x) \leq D(y)$, i.e., D is isotone. Since D is a contractive derivation on L, we have D is an ideal generalized derivation on L.

(3) $\Rightarrow$ (6) For all $x, y \in L$, it follows from (3) that

$$\begin{aligned} D(x \otimes y) &= D(1) \otimes (x \otimes y) \\ &= (D(1) \otimes x) \otimes (D(1) \otimes y) \\ &= D(x) \otimes D(y). \end{aligned}$$

(6) $\Rightarrow$ (2) For all $x \in L$, it follows from (6) that

$$\begin{aligned} D(x) &= D(x \otimes 1) \\ &= D(x) \otimes D(1) \\ &= D(x) \wedge D(1). \end{aligned}$$

Thus, $D(x) \leq D(1)$ for all $x \in L$. $\square$

An ideal generalized derivation is said to be good if $D(1) \in B(L)$.

Example 3.10: Define a mapping $d : L \longrightarrow L$ by $d(x) = 0$ for all $x \in L$, then d is a derivation on L. Moreover, we define a mapping $D : L \longrightarrow L$ by $D(x) = x$ for all $x \in L$. It is easy to verify that D is a good ideal generalized derivation on L.

In what follows, we discuss the properties of good ideal generalized derivations.

Proposition 3.11: Let D be a good ideal generalized derivation on L. Then the following statements are equivalent.

(1) $D^2(x) = D(x)$ for all $x \in L$.

(2) $D(x) \leq y$ if and only if $D(x) \leq D(y)$ for all $x, y \in L$.

(3) $D(x \wedge D(y)) = D(x \wedge y) = D(D(x) \wedge y)$ for all $x, y \in L$.

Proof. (1) Since D is a good ideal generalized derivation on L, we have $D(1) \in B(L)$. It follows from Theorem 3.9 (3) that $D(x) = D(1) \otimes x$. Then $D(D(x)) = D(1) \otimes D(x) = D(1) \otimes (D(1) \otimes x) = D(1) \otimes x = D(x)$, which implies $D^2(x) = D(x)$ for all $x \in L$.

(2) Let $D(x) \leq y, x, y \in L$. Then $D(D(x)) \leq D(y)$. It follows from (1) that $D(D(x)) = D(x)$. Thus $D(x) \leq D(y)$. Conversely, if $D(x) \leq D(y)$, since D is a good ideal generalized derivation on L, we have $D(x) \leq D(y) \leq y$, i.e., $D(x) \leq D(y) \leq y$.

(3) It follows from (1) that $D(D(x)) = D(x)$ for all $x \in L$. Since $D(y) \leq y$ for all $y \in L$, we have $x \wedge D(y) \leq x \vee y$. Then $D(x \wedge D(y)) \leq D(x \wedge y)$. On the other hand, since $D(x \wedge y) \leq D(y)$ and $D(x \wedge y) \leq D(x) \leq x$, we have $D(x \wedge y) \leq x \wedge D(y)$. Then $D(x \wedge y) = D(D(x \wedge y)) \leq D(x \wedge D(y))$. Thus $D(x \wedge y) = D(x \wedge D(y))$. In a similar way, we can obtain that $D(x \wedge y) = D(D(x) \wedge y)$. Therefore, $D(x \wedge D(y)) = D(x \wedge y) = D(D(x) \wedge y)$. $\square$

Next, we discuss the structures and properties of the fixed point set of ideal generalized derivations. Firstly, we give the concept of the fixed set of a generalized derivation in residuated lattices as follows.

Definition 3.12: Let D be a generalized derivation on L. Define a set $Fix_D(L) = \{x \in L | D(x) = x\}$, $Fix_D(L)$ is called the set of all fixed elements of L for D.

Now, we investigate some operations on $Fix_D(L)$.

Theorem 3.13: Let D be an ideal generalized derivation on L.

(1) $x \otimes y \in Fix_D(L)$, $x \vee y \in Fix_D(L)$ for all $x, y \in Fix_D(L)$.

(2) If D_1, D_2 are good, then $D_1 = D_2$ if and only if $Fix_{D_1}(L) = Fix_{D_2}(L)$.

Proof. (1) Let $x, y \in Fix_D(L)$. Then $D(x) = x$, $D(y) = y$. It follows from Definition 3.1 that $x \otimes y \leq D(x) \otimes y \leq D(x \otimes y)$. On the other hand, since D is an ideal generalized derivation on L, we have $D(x \otimes y) \leq x \otimes y$, which implies $D(x \otimes y) = x \otimes y$. Therefore, $x \otimes y \in Fix_D(L)$. Moreover, on one hand, it is easy to know that $x \vee y = D(x) \vee D(y) \leq D(x \vee y)$. On the other hand, since $D(x \vee y) \leq x \vee y$, we have $D(x \vee y) = x \vee y$, which implies $x \vee y \in Fix_D(L)$.

(2) Let $D_1 = D_2$. Then it is easy to see that $Fix_{D_1}(L) = Fix_{D_2}(L)$. Conversely, let $Fix_{D_1}(L) = Fix_{D_2}(L)$. It follows from Proposition 3.11 (1) that $D_1(D_1(x)) = D_1(x)$ for all $x \in L$, which implies $D_1(x) \in Fix_{D_1}(L) = Fix_{D_2}(L)$. Hence, $D_2(D_1(x)) = D_1(x)$ for all $x \in L$. In a similar way, we have $D_1(D_2(x)) = D_2(x)$ for all $x \in L$. On the other hand, since D_1, D_2 are two good ideal generalized derivations on L, we have $D_1(D_2(x)) \leq D_1(x) = D_2(D_1(x))$ for all $x \in L$. In a similar way, we have $D_2(D_1(x)) \leq D_1(D_2(x))$, which implies $D_1(D_2(x)) = D_2(D_1(x))$. Thus, $D_2(x) = D_1(D_2(x)) = D_2(D_1(x)) = D_1(x)$ for all $x \in L$, i.e., $D_1 = D_2$. $\square$

Remark 3.14: It follows from Theorem 3.13 that a good ideal generalized derivation D is determined by it's fixed point set $Fix_D(L)$.

Further, as an application of the above propositions and theorems, we have the following result.

Theorem 3.15: Let D be a good ideal generalized derivations on L. Then $(Fix_D(L), \sqcap, \vee, \otimes, \mapsto, 0, \overline{1})$ is a residiated lattice, where $x \sqcap y = D(x \wedge y)$, $x \mapsto y = D(x \to y)$ and $\overline{1} = D(1)$ for all $x, y \in Fix_D(L)$.

Proof. We complete the proof by three steps.

(1) First of all, we show that $(Fix_D(L), \sqcap, \vee, 0, \overline{1})$ is a bounded lattice with 0 as the smallest element and $\overline{1} = D(1)$ as the greatest element. It follows from Theorem 3.13 (1) that $Fix_D(L)$ is closed under $\vee$. Since D is a good ideal generalized derivations on L, we have $D(x \wedge y) \leq D(x) = x$ and $D(x \wedge y) \leq D(y) = y$ for all $x, y \in Fix_D(L)$. Hence, $D(x \wedge y)$ is a lower bound of x and y in $Fix_D(L)$. Let $m = D(m) \in Fix_D(L)$ is any lower bound of x, y in $Fix_D(L)$. Then we have $m \leq x \wedge y$. Thus $m = D(m) \leq D(x \wedge y)$, which implies the infimum of x and y exists in $Fix_D(L)$ and is $D(x \wedge y)$, denoted by $x \sqcap y = D(x \wedge$

y). Therefore, $(Fix_D(L), \sqcap, \vee, 0, \overline{1})$ is a bounded lattice. Let $x \in Fix_D(L)$. Then it follows from Theorem 3.9 that

$$\begin{aligned} x \sqcap 0 &= D(x \wedge 0) \\ &= D(0) \\ &= 0, \end{aligned}$$

$$\begin{aligned} x \vee D(1) &= D(x) \vee D(1) \\ &= D(x \vee 1) \\ &= D(1). \end{aligned}$$

Therefore, 0 is the smallest element and $\overline{1} = D(1)$ is the greatest element in $Fix_D(L)$, respectively.

(2) Next, we show that $(Fix_D(L), \otimes, \overline{1})$ is a commutative monoid with $\overline{1} = D(1)$ as a neutral element. It follows from Theorem 3.13 (1) that $Fix_D(L)$ is closed under $\otimes$. On the other hand, it is easy to see that $Fix_D(L)$ satisfies associative law. Thus $(Fix_D(L), \otimes)$ is a commutative semigroup. Let $x \in Fix_D(L)$. Then it follows from Theorem 3.9 that

$$\begin{aligned} x \otimes \overline{1} &= D(x) \otimes D(1) \\ &= D(x \otimes 1) \\ &= D(x) \\ &= x, \end{aligned}$$

which implies $\overline{1} = D(1)$ is a neutral element.

(3) Finally, we show that $x \otimes y \leq z$ if and only if $y \leq x \mapsto z$ for all $x, y, z \in Fix_D(L)$. Let $x, y \in Fix_D(L)$. We define $x \mapsto y = D(x \to y)$. Then it follows from Proposition 3.11 (2) that $D(x) \leq y$ if and only if $D(x) \leq D(y)$ for all $x, y \in L$. Thus for all $x, y, z \in Fix_D(L)$

$$\begin{aligned} x \otimes y \leq z &\Leftrightarrow y \leq x \to z \\ &\Leftrightarrow D(y) \leq x \to z \\ &\Leftrightarrow D(y) \leq D(x \to z) \\ &\Leftrightarrow D(y) \leq x \mapsto z \\ &\Leftrightarrow y \leq x \mapsto z. \end{aligned}$$

Therefore, $(Fix_D(L), \sqcap, \vee, \otimes, \mapsto, 0, \overline{1})$ is a residiated lattice. □

Remark 3.16: It follows from Theorem 3.15 that the fixed point set $Fix_D(L)$ of good ideal generalized derivations in a reisduated lattice L has the same structure as L, which reveals the essence of the fixed point set for ideal generalized derivations.

In what follows, we introduce the corresponding filters of residuated lattices with ideal generalized derivations, which are called ideal generalized derivation filters. In particular, we investigate the corresponding relationship between ideal generalized derivation filters and filters of the fixed point set $Fix_D(L)$ for good ideal generalized derivations.

Definition 3.17: Let (L, D) be a residuated lattice with good ideal generalized derivations and F be a filter of L. Then F is called a good generalized derivations filter of (L, D) if $x \in F$ implies $D(x) \in F$.

We denote the set of all good ideal generalized derivation filters of a residuated lattice with good ideal generalized derivations (L, D) by $GDF(L, D)$.

Theorem 3.18: Let (L, D) be a residuated lattice with good ideal generalized derivations and F be a filter of L. If F is a good ideal generalized derivation filter of (L, D), then $F = \langle F \cap Fix_D(L) \rangle$.

Proof. Let F be a good ideal generalized derivation filter of (L, D). If $x \in F$, then $D(x) \in F$. It follows from Proposition 3.11 (1) that $D(x) \in Fix_D(L)$. So we have $D(x) \in \langle F \cap Fix_D(L) \rangle$. Since $D(x) \leq x$, we have $x \in \langle F \cap Fix_D(L) \rangle$, which implies $F \subseteq \langle F \cap Fix_D(L) \rangle$. On the other hand, let $x \in \langle F \cap Fix_D(L) \rangle$. Then there exists $y \in F \cap Fix_D(L)$ such that $y \leq x$, which implies $x \in F$. Thus, $\langle F \cap Fix_D(L) \rangle \subseteq F$. Therefore, $F = \langle F \cap Fix_D(L) \rangle$. □

Finally, we point out that the corresponding relationship between good ideal generalized derivation filters and filters of the fixed point set $Fix_D(L)$ for good ideal generalized derivations.

Theorem 3.19: Let (L, D) be a residuated lattice with good ideal generalized derivations. Then the lattice $GDF(L, D)$ is isomorphic to the lattice $F(Fix_D(L))$ of all filters of the residuated lattice $(Fix_D(L), \sqcap, \vee, \otimes, \mapsto, 0, \overline{1})$.

Proof. Define a mapping $\eta : F(Fix_D(L)) \to GDF(L, D)$ by $\eta(F) = \langle F \rangle$ for all $F \in F(Fix_D(L))$.

We complete the proof by three steps.

(1) First of all, we show that η is well defined. If $x \in \eta(F) = \langle F \rangle$, then there exists $m \in F$ such that $m \leq x$. Thus, $m = D(m) \leq D(x)$, which implies $D(x) \in \langle F \rangle = \eta(F)$. Hence, $\eta(F)$ is a good ideal generalized derivation filter of (L, D). Therefore, η is a mapping from the lattice $F(Fix_D(L))$ to the lattice $GDF(L, D)$, i.e., η is well defined.

(2) Next, it is easy to show that $\eta(F) \cap (Fix_D(L)) = F$ for all $F \in F(Fix_D(L))$. Thus, η is injective. Moreover, let $J \in GDF(L, D)$. Then it follows from Theorem 3.18 that $J = \langle J \cap Fix_D(L) \rangle$. Since $J \cap Fix_D(L) \in F(Fix_D(L))$, we have $J = \eta(J \cap Fix_D(L))$. Therefore, η is surjective.

(3) At last, for any $F_1, F_2 \in F(Fix_D(L))$, one can easily check that $F_1 \subseteq F_2$ if and only if $\eta(F_1) \subseteq \eta(F_2)$.

Thus, η is an isomorphism from the lattice $F(Fix_D(L))$ to the lattice $GDF(L, D)$. Moreover, for any $F \in GDF(L, D)$, we have $\eta^{-1}(F) = F \cap Fix_D(L)$. Therefore, $F(Fix_D(L))$ and $GDF(L, D)$ are isomorphic. □

Remark 3.20: It follows from Theorem 3.19 that good ideal generalized derivation filters and filters of the fixed point set $Fix_D(L)$ for good ideal generalized derivations are isomorphic, which reveals the essence of good ideal generalized derivation filters.

IV. CONCLUSIONS

The notion of derivations is helpful for studying structures and properties in algebraic systems. In this paper, a generalized derivation on a residuated lattice is introduced, some properties of isotone generalized derivations and contractive generalized derivations are discussed. We investigated isotone (resp. contractive) generalized derivations and good ideal generalized derivations. By means of fixed point set, we obtained that the fixed point set of good ideal generalized derivations in a residuated lattice L has the same structure as L, which revealed the essence of the fixed point set for ideal generalized derivations. In particular, we obtained that the set of all good ideal generalized derivation filters on a residuated lattice is isomorphic to the lattice of all filters in the fixed point set for good ideal generalized derivations.

As an extension of this work, the following topics may be considered:

(1) Constructing generalized derivations to other algebras, such as EQ-algebras, non-commutative residuated lattices and so on;

(2) Investigating generalized derivations in quantales;

(3) Studying some other generalized derivations in residuated lattices.

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